A COMPARISON THEOREM FOR SELFADJOINT OPERATORS

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Abstract. In this work we shall establish a result concerning the spectral theory of differential operators. Let $L_1$ and $L_2$ be two self-adjoint operators acting in two different Hilbert spaces. Then under some conditions we shall prove that

$$(d\Gamma_1/d\Gamma_2)(L_2) = V V',$$

where $\Gamma_1(\lambda)$ and $\Gamma_2(\lambda)$ are the spectral functions associated with $L_1$ and $L_2$ respectively. $V$ is the shift operator mapping the set of generalized eigenfunctions of $L_1$ into the set of generalized eigenfunctions of $L_2$, that is

$$y = V\varphi,$$

where $L_2y = \lambda y$ and $L_1\varphi = \lambda \varphi$.

1. Introduction

As we shall be manipulating eigenfunctions we need to recall the theory of operators in rigged Hilbert spaces. Let $\Phi$ be a nuclear space, $N$-space, that is a countably normed space $\Phi = \bigcap_{n>1} \Phi_n$, such that, for any $p$, there exists $n > p$ so that the embedding $\Phi_n \hookrightarrow \Phi_p$ is a Hilbert–Schmidt operator, (see [7]). We recall that an $N$-space is a perfect space, and so each bounded set is relatively compact.

We now come to some interesting applications of the above idea. Suppose that an operator $L$ is symmetric in a Hilbert space $H$. Assume that there exists an $N$-space $\Phi$ (perfect) invariant under the operator $L$ and such that $H$ can be obtained as a completion of $\Phi$ under the inner product of $H$. We shall assume that the embedding $\Phi \hookrightarrow H$ is the identity. Since

$$\Phi \overset{L}{\hookrightarrow} \Phi,$$

we have

$$\Phi' \overset{L^*}{\hookrightarrow} \Phi'.$$

Then, using the symmetry of $L$, i.e., $L \subset L^*$, and the fact that

$$\Phi \hookrightarrow H \hookrightarrow \Phi',$$

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we obtain that \( L^* = L \) on \( \Phi \), and \( L^* \) is seen as an extension of \( L \) to \( \Phi' \). Thus we shall agree to denote by \( L \) the operator \( L^* \), and so the definition:

**Definition 1.** We shall say that a linear functional \( \phi \in \Phi' \) is a generalized eigenfunction or eigenfunctional if

\[
L\phi = \lambda \phi \quad \text{in} \quad \Phi'.
\]

Below we recall a well known result, see [5, vol. 3].

**Result.** Let \( L \) be a symmetric linear operator which is defined on \( \Phi \) and maps \( \Phi \) into itself. Assume that \( \Phi \) is an \( N \)-space and that \( L \) admits a self-adjoint extension to the Hilbert space \( H \). Then \( L \) possesses a complete system of eigenfunctionals in the space \( \Phi' \).

Let \( L \) be a self-adjoint operator with simple spectrum acting in a separable Hilbert space \( H \). If \( \phi(\lambda) \) are the eigenfunctionals, that is \( L\phi(\lambda) = \lambda \phi(\lambda) \) in \( \Phi' \), then the associated isometry or \( \phi \)-Fourier transform is given by

\[
\hat{\phi}(\lambda) = (f, \phi(\lambda))_{\Phi'\times\Phi'} \quad \forall f \in \Phi
\]

and the inverse is

\[
f = \int \frac{\hat{\phi}(\lambda)}{\sqrt{\Gamma(\lambda)}} d\Gamma(\lambda) \in H.
\]

\( \Gamma(\lambda) \) is a nondecreasing function and is called the spectral function. The Parseval equality reads

\[
\forall f, \forall \psi \in \Phi \quad (f(x), \psi(x))_H = \int \hat{\phi}(\lambda)\overline{\hat{\psi}(\lambda)} d\Gamma(\lambda).
\]

Let us agree on some notations. Let \( L_1 \) and \( L_2 \) be two self-adjoint operators with simple spectrum and acting in two separable Hilbert spaces \( H_1 \) and \( H_2 \) respectively.

We assume the existence of two (perfect) \( N \)-spaces \( \Phi_1 \) and \( \Phi_2 \) such that

\[
\Phi_i \hookrightarrow \Phi_i', \quad i = 1, 2.
\]

In all that follows, \( \{ \Phi_i, H_i, \Phi_i' \} \) and \( \Gamma_i(\lambda) \) will denote, respectively, the rigged spaces and the spectral functions associated with the self-adjoint operator \( L_i \), where \( i = 1, 2 \). Denote by \( \phi(\lambda) \) and \( y(\lambda) \) the generalized eigenfunctionals defined by

\[
L_i\phi(\lambda) = \lambda \phi(\lambda) \quad \text{in} \quad \Phi_i'
\]

\[
L_2y(\lambda) = \lambda y(\lambda) \quad \text{in} \quad \Phi_2'.
\]

\( \sigma_i \) denotes the spectrum of \( L_i \), for \( i = 1, 2 \). The Fourier transform in this case is given by

\[
f \in \Phi_1 \quad \hat{f}^1(\lambda) \equiv (f, \phi(\lambda))_{\Phi_1 \times \Phi_1'},
\]

\[
\psi \in \Phi_2 \quad \hat{\psi}^2(\lambda) \equiv (\psi, y(\lambda))_{\Phi_2 \times \Phi_2'}.
\]
In order to compare operators, we shall need to establish a correspondence between the two sets of eigenfunctionals. Assume the existence of a one-to-one mapping between the real sets $\sigma_1$ and $\sigma_2$, namely, 

$$T: \sigma_2 \rightarrow \sigma_1.$$  

**Definition 2.** Let $\phi(\lambda)$ and $\psi(\lambda)$ be the eigenfunctionals defined by (1.1), and let $T: \sigma_2 \rightarrow \sigma_1$ be a one-to-one mapping. Then $V$ is said to be a shift operator if

$$V \phi(T(\lambda)) = \psi(\lambda), \quad \forall \lambda \in \sigma_2.$$  

**Remark.** It is clear that the shift operator $V$ is a one-to-one mapping between the sets $\{\phi(\lambda)\}_{\sigma_1}$ and $\{\psi(\lambda)\}_{\sigma_2}$. Hence it is defined on $\{\phi(\lambda)\}_{\sigma_1}$, a subset of $\Phi'$. We next extend $V$ to the algebraic span of $\{\phi(\lambda)\}_{\sigma_1}$. For our immediate use, we shall only need the fact that $V$ is densely defined. Indeed, from the reflexivity of $\Phi$ and the completeness of $\{\phi(\lambda)\}_{\sigma_1}$, the space spanned by $\{\phi(\lambda)\}_{\sigma_1}$ is dense in $\Phi'$. Therefore $V$ is densely defined. This enables us to define the adjoint operator $V'$. By definition we have

$$\langle \psi, Vf \rangle_{\Phi'_{2} \times \Phi'_{1}} = \langle V'\psi, f \rangle_{\Phi'_{2} \times \Phi'_{1}}.$$ 

Since the spaces are reflexive,

$$\langle \psi, Vf \rangle_{\Phi_2 \times \Phi_1} = \langle V'\psi, f \rangle_{\Phi_2 \times \Phi_1}.$$ 

The domain of $V'$ is defined by

$$D_{V'} = \{ \psi \in \Phi_2 | f \rightarrow \langle \psi, Vf \rangle \text{ is continuous} \}.$$ 

However there is a simple connection between $D_{V'}$ and $\Phi_2$, indeed, we have the well-known result that, for example see [6, Chapter 2],

$$V \text{ admits closure} \Leftrightarrow D_{V'} \text{ is dense in } \Phi_2.$$ 

### 2. The factorization theorem

Let us start with some notations. We know that $V'$ acts between $\Phi_2$ and $\Phi_1$, which are imbedded in $H_2$ and $H_1$ respectively. Hence $V'$ has a natural extension as an operator from $H_2$ into $H_1$, which we shall denote by $\tilde{V}'$. Thus

$$H_2 \overset{\tilde{V}'}{\rightarrow} H_1$$

and, for $f \in D_{V'}$, $\tilde{V}'f = V'f$ in $H_1$.

Define an operator $G$ in $\Phi_2$ by

$$Gf = \int \overline{\tilde{f}^2(\lambda)} y(\lambda) \, d\Gamma_1(T(\lambda)),$$

where $\Gamma_1(\lambda)$ is the spectral function of $L_1$, and $T(\cdot)$ is the mapping used in the definition of the shift operator. The domain of $G$ is $D_{G} = \{ f \in \Phi_2 | Gf \in \Phi'_{2} \}$. Clearly if $\tilde{f}^2(\lambda)$ has a compact support then $Gf$ is defined. It is possible to represent $G$ if we knew the behaviour of $\Gamma_1(T(\lambda))$. 


**Theorem 3.** Let $L_1$ and $L_2$ be two self-adjoint operators with simple spectrum, acting in the rigged Hilbert spaces $\Phi_1 \hookrightarrow H_1 \hookrightarrow \Phi'_1$ and $\Phi_2 \hookrightarrow H_2 \hookrightarrow \Phi'_2$ respectively. If the shift operator $V$ admits closure, then

$$G = \overline{VV'},$$

where $\overline{V}$ denotes the closure of $V$.

**Proof.** Let us give the diagram of the operator $V$

$$\begin{array}{c}
\Phi_1 \hookrightarrow H_1 \hookrightarrow \Phi'_1 \\
\downarrow V' \quad \downarrow V \\
\Phi_2 \hookrightarrow H_2 \hookrightarrow \Phi'_2
\end{array}$$

where $V\phi(T(\lambda)) = y(\lambda)$.

Let $f$ and $\psi$ be two arbitrary elements of $D_{V'}$.

$$f^2(\lambda) = \langle f, y(\lambda) \rangle_{\Phi_1 \times \Phi'_1}$$

(2.1)

$$= \langle f, V\phi(T(\lambda)) \rangle_{\Phi_1 \times \Phi'_1} = \langle V'f, \phi(T(\lambda)) \rangle_{\Phi_1 \times \Phi'_1} = \widehat{V'f}^1(T(\lambda))$$

and, similarly,

$$\psi^2(\lambda) = \widehat{V'\psi}^1(T(\lambda)).$$

(2.2)

Observing that the right-hand side of equations (2.1), (2.2) are the $\varphi$-Fourier transform of $V'f$ and $V'\psi$, respectively, we obtain by using the Parseval equality,

$$\int \overline{V'f}^1(\lambda)\overline{V'\psi}^1(\lambda) d\Gamma_1(\lambda) = \int f^2(\lambda)\overline{\psi^2(\lambda)} d\Gamma_1(\lambda).$$

(2.3)

Clearly,

$$\langle f, G\psi \rangle_{\Phi_1 \times \Phi'_1} = \left\langle f, \int \overline{\psi^2(\lambda)} y(\lambda) d\Gamma_1(\lambda) \right\rangle_{\Phi_1 \times \Phi'_1}$$

$$= \int \overline{\psi^2(\lambda)} f^2(\lambda) d\Gamma_1(\lambda).$$

Hence $(V'f, V'\psi)_{H_1} = \langle f, G\psi \rangle_{\Phi_1 \times \Phi'_1}$ which implies that $D_{V'} \subset D_G$. Since the imbedding $\Phi_1 \hookrightarrow H_1 \hookrightarrow \Phi'_1$ is the identity,

$$(V'f, V'\psi)_{H_1} = (V'f, V'\psi)_{\Phi_1 \times \Phi'_1}.$$

Therefore

$$\langle f, G\psi \rangle_{\Phi_1 \times \Phi'_1} = (V'f, V'\psi)_{\Phi_1 \times \Phi'_1},$$

and, since $V$ admits closure, we have

$$(V'f, V'\psi)_{\Phi_1 \times \Phi'_1} = (f, \overline{V'\psi})_{\Phi_1 \times \Phi'_1},$$
and so $\langle f, G\psi \rangle_{\Phi_2 \times \Phi_2'} = \langle f, \overline{V'}\psi \rangle_{\Phi_2 \times \Phi_2'}$. Therefore for $\psi \in D_{V'}$, $G\psi = \overline{V'}\psi$ in $\Phi_2'$, and so $D_G = D_{\overline{V}'}$. □

**Remark.** We had to use the fact that $V$ admitted closure. We shall see that we do not need such an assumption if we took $\overline{V}'$ instead of $V'$. For that define

$$
\tilde{G} : H_2 \to H_2 \\
\tilde{G} f = \int \hat{f}^2(\lambda)\overline{\psi^2(\lambda)} d\Gamma_1(T(\lambda)),
$$

$D_{\tilde{G}} = \{ f \in H_2 | \tilde{G} f \in H_2 \}$. Here the Fourier transform is extended to $H_2$ by taking the closure of the Fourier transform in $H_2$. It is clear that $D_{\tilde{G}}$ is dense in $H_2$. To see that, take the dense set of smooth compactly supported functions in $L^2_{\Gamma_1(T(\lambda))}$. Then, using the inverse $\psi$-Fourier transform, we shall obtain a dense set in $H_2$, which is also contained in $D_{\tilde{G}}$. Hence $D_{\tilde{G}}$ is dense in $H_2$, and so $\tilde{G}$ is densely defined. From (2.3),

$$
(V'f, V'\psi)_{H_1} = \int \hat{f}^2(\lambda)\overline{\psi^2(\lambda)} d\Gamma_1(T(\lambda)).
$$

Since $f$ and $\psi$ are also in $H_2$, $(V'f, V'\psi)_{H_1} = (\tilde{V}'f, \tilde{V}'\psi)_{H_1}$ and

$$
\int \hat{f}^2(\lambda)\overline{\psi^2(\lambda)} d\Gamma_1(T(\lambda)) = (f, \tilde{G}\psi)_{H_2}.
$$

Therefore

$$
(2.4) \quad (\tilde{V}'f, \tilde{V}'\psi)_{H_1} = (f, \tilde{G}\psi)_{H_2}.
$$

It is readily seen that $\tilde{V}'$ is densely defined. Indeed, since

$$
\|\tilde{V}'f\|^2 = \int |\hat{f}^2(\lambda)|^2 d\Gamma_1(T(\lambda)),
$$

the argument used for $D_{\tilde{G}}$ will go through. $\tilde{V}'$ densely defined means that the adjoint operator is well defined and, by (2.4), we deduce that

$$
(f, [\tilde{V}'\dagger] \tilde{V}'\psi)_{H_2} = (f, \tilde{G}\psi)_{H_2}.
$$

Therefore $\tilde{G} = [\tilde{V}'\dagger] \tilde{V}'$, and we have just proved

**Theorem 4.** Let $L_1$ and $L_2$ be two self-adjoint operators with simple spectrum, acting in the rigged Hilbert spaces $\Phi_1 \hookrightarrow H_1 \hookrightarrow \Phi_1'$ and $\Phi_2 \hookrightarrow H_2 \hookrightarrow \Phi_2'$, respectively. Then

$$
\tilde{G} = [\tilde{V}'\dagger] \tilde{V}'.
$$
There exists a particular case where it is possible to obtain a simple representation of the operator $G$. For that we need

**Definition 5.** $\Gamma_1(T(\lambda))$ is said to be absolutely continuous with respect to $\Gamma_2(\lambda)$ (denoted by $\text{ABS}-d\Gamma_2(\lambda)$) if there exists a $d\Gamma_2(\lambda)$-summable function $g(\lambda)$ such that

$$\Gamma_1(T(\lambda)) = \int_{-\infty}^{\lambda} g(\lambda) \, d\Gamma_2(\lambda).$$

(Notation: $g(\lambda) = (d\Gamma_1(T)/d\Gamma_2)(\lambda)$.)

Let $g(L_2)$ be the operator defined by

$$g(L_2) = \frac{d\Gamma_2(\lambda)}{\Gamma_1(T(\lambda))} f_2(X),$$

or

$$g(L_2)f(x) = \int g(\lambda)f^2(\lambda)\, y(\lambda) \, d\Gamma_2(\lambda).$$

Its domain is given by

$$D_{g(L_2)} = \{ f \in \Phi_2 \mid g(\lambda)f^2(\lambda) \in L^2 \}.$$

Therefore $Gf = I_{\Phi_2}g(L_2)f$, for any $f \in D_{g(L_2)}$, and where $I_{\Phi_2}$ is the imbedding from $H_2 \hookrightarrow \Phi_2$, the identity. So we can write, for any $f \in D_{g(L_2)}$,

$$Gf = g(L_2)f \quad \text{in } \Phi_2.'$$

In this way $G$ is an extension of $g(L_2)$ to $\Phi_2'$. We shall agree to write $G \equiv g(L_2)$ in $\Phi_2'$. Thus

**Corollary 6.** Let $L_1$ and $L_2$ be two self-adjoint operators with simple spectrum, acting in the rigged Hilbert spaces $\Phi_1 \hookrightarrow H_1 \hookrightarrow \Phi_1'$ and $\Phi_2 \hookrightarrow H_2 \hookrightarrow \Phi_2'$, respectively. Assume that $V$ admits closure. If the function $\Gamma_1(T(\lambda))$ is $\text{ABS}-d\Gamma_2(\lambda)$, i.e., $d\Gamma_1(T(\lambda)) = g(\lambda) \, d\Gamma_2(\lambda)$. Then, for any $f \in D_{g(L_2)}$,

$$g(L_2)f = \overline{V}V'f \quad \text{in } \Phi_2',$$

where $\overline{V}$ denotes the closure of $V$.

If $\tilde{g}(L_2)$ denotes the extension of $g(L_2)$ to $H_2 \rightarrow H_2$, then $\tilde{g}(L_2) = \tilde{G}$ in $H_2$, and so from Theorem 4,

**Corollary 7.** Let $L_1$ and $L_2$ be two self-adjoint operators with simple spectrum, acting in the rigged Hilbert spaces $\Phi_1 \hookrightarrow H_1 \hookrightarrow \Phi_1'$ and $\Phi_2 \hookrightarrow H_2 \hookrightarrow \Phi_2'$, respectively. If the function $\Gamma_1(T(\lambda))$ is $\text{ABS}-d\Gamma_2(\lambda)$, i.e., $d\Gamma_1(T(\lambda)) = g(\lambda) \, d\Gamma_2(\lambda)$. Then

$$\tilde{g}(L_2) = [\tilde{V}']\tilde{V}'$$. 

We have defined $g(L_2)$ through the Fourier transform and claimed that it was the usual function of the operator $L_2$. Let us briefly show that the two
definitions coincide. If \( E_\lambda \) is the spectral family associated with \( L_2 \), then since the operator \( L_2 \) has a simple spectrum,

\[
dE_\lambda f = \overline{\jmath^2(\lambda)} y(x, \lambda) d\Gamma_2(\lambda).
\]

Therefore

\[
g(L_2) f = \int g(\lambda) dE_\lambda f = \int g(\lambda) \overline{\jmath^2(\lambda)} y(x, \lambda) d\Gamma_2(\lambda),
\]

and so the two definitions are in fact identical.

3. General results

We have shown that \( g(L_2) = [\tilde{V}']' \tilde{V}' \) on \( D_{\nu'} \). Clearly the boundedness of \( g(L_2) \), which depends on the behaviour of \( g(\lambda) \), must be related to the boundedness of \( \tilde{V}' \).

**Theorem 8.** Assume that conditions in Corollary 7 hold; then \( \tilde{V}' \) is a bounded operator \( H_2 \rightarrow H_1 \) if and only if \( \sqrt{g(\lambda)} \) is bounded \( d\Gamma_2(\lambda) \) a.e.

**Proof.** Assume that \( \sqrt{g(\lambda)} \) is bounded \( d\Gamma_2(\lambda) \) a.e. Then there exists \( M \) such that \( |\sqrt{g(\lambda)}| \leq M d\Gamma_2(\lambda) \) a.e. From (2.4) we obtain

\[
\| \tilde{V}' f \|_{H_1} = \| \sqrt{g(\lambda)} \overline{\jmath^2(\lambda)} \|_{d\Gamma_2} \quad \text{for } f \in \Phi_2 \hookrightarrow H_2,
\]

but \( \| \sqrt{g(\lambda)} \overline{\jmath^2(\lambda)} \|_{d\Gamma_2} \leq M \| \overline{\jmath^2(\lambda)} \|_{d\Gamma_2} \leq M \| f \|_{H_2} \). Hence \( \| \tilde{V}' f \|_{H_1} \leq M \| f \|_{H_2} \), which shows that \( \tilde{V}' \) is a bounded operator from \( H_2 \) to \( H_1 \). Conversely, if \( \tilde{V}' \) is bounded then, for any \( f \in \Phi_2 \), we do have

\[
\| \sqrt{g(\lambda)} \overline{\jmath^2(\lambda)} \|_{d\Gamma_2} = \| \tilde{V}' f \|_{H_1} \leq M \| f \|_{H_2} \leq M \| \overline{\jmath^2(\lambda)} \|_{d\Gamma_2}.
\]

From the above inequality it is readily seen that \( \sqrt{g(\lambda)} \) is \( d\Gamma_2 \) bounded. \( \square \)

**Theorem 9.** \( \tilde{V}' : H_2 \rightarrow H_1 \) is invertible if and only if

\[
\int_{k_g} d\Gamma_2(\lambda) = 0,
\]

where \( k_g \equiv \{ \lambda | g(\lambda) = 0 \} \) and

\[
\| \tilde{V}' \| = \text{ess sup}_{\lambda \in \sigma_2} \sqrt{g(\lambda)}.
\]

From (2.4) we have that \( \| \tilde{V}' f \|_{H_1} = \| \sqrt{g(\lambda)} \overline{\jmath^2(\lambda)} \|_{d\Gamma_2} \). So the operator \( \tilde{V} : H_2 \rightarrow H_1 \) is invertible if and only if \( \sqrt{g(L_2)} \) is invertible. Thus we should have

\[
\sqrt{g(L_2)} f = 0 \Rightarrow f = 0.
\]

Using the Fourier transform,

\[
(3.1) \quad \sqrt{g(\lambda)} \overline{\jmath^2(\lambda)} = 0 \Rightarrow \overline{\jmath^2(\lambda)} = 0 \quad d\Gamma_2(\lambda) \text{ a.e.}
\]
As \( \| f^2 \| \) depends on the support of \( \Gamma_2 \), (3.1) will have to be verified only on the support of \( \Gamma_2 \), or, in other words, on \( \sigma_2 \). Thus (3.1) means that there is no set of nonzero measure, where \( g(\lambda) \) vanishes. \( \Box \)

Suppose that we need to find \( L_1 \) from its spectral function \( \Gamma_1 \), i.e., the inverse spectral problem. Let \( L_2 \) be given with its spectral function \( \Gamma_2 \), and form the operator \( G \). If we can solve \( G = V V' \), then we claim that the inverse spectral problem is solved. Indeed, if we regard the eigenfunctions as a basis for the differential operator then the result is immediate,

\[
L_2 y = \lambda y \quad \forall \lambda \in \sigma_2.
\]

So, by using the shift operator,

\[
y = V \varphi \quad \text{or} \quad \varphi = V^{-1} y
\]

\[
L_2 V \varphi = \lambda V \varphi,
\]

and clearly

\[
V^{-1} L_2 \varphi = \lambda \varphi
\]

so that

\[
L_1 = V^{-1} L_2 V \quad \text{in } \Phi_1',
\]

which is exactly the formula for the change of basis. From (3.2) we can see that we can recover \( L_1 \) if \( V^{-1} \) exists.

4. **The second factorization**

Notice that the function \( g(\lambda) \) in Definition 5 might not exist. In this section we shall give another way of relating the spectral functions. Let \( T \) be a non-decreasing one-to-one one mapping between \( \sigma_1 \) and \( \sigma_2 \). As usual the shift operator is defined by \( y(\lambda) = V \varphi(T(\lambda)) \). Since \( \Gamma_1(T(\lambda)) \) and \( \Gamma_2(\lambda) \) are non-decreasing functions we can assume the existence of an increasing function \( s(\lambda) \) such that

\[
\Gamma_1(T(s(\lambda))) = \Gamma_2(\lambda).
\]

With the help of \( s(\lambda) \) we can define the following operator:

\[
\Phi_2 \xrightarrow{A_s} H_2
\]

\[
\hat{f} \rightarrow A_s(\hat{f})
\]

\[
\tilde{2} \downarrow \quad 1 \downarrow 2^{-1}
\]

\[
\hat{f}^2(\lambda) \rightarrow \hat{f}^2(s(\lambda))
\]

The domain of \( A_s \) is \( D_{A_s} = \{ f \in \Phi_2 | \hat{f}^2(s(\lambda)) \in L_{\Gamma_2}^2 \} \). Clearly \( A_s = \hat{2}^{-1} s \hat{2} \), where \( s \circ \hat{2} \) denotes the composition with the function \( s(\lambda) \).

Denote by \( \tilde{A}_s \) the closure of \( A_s \) in \( H_2 \).
Theorem 10. Let $L_1$ and $L_2$ be two self-adjoint operators having their spectral functions such that $\Gamma_1(T(s(\lambda))) = \Gamma_2(\lambda)$. Then

\begin{equation}
[\tilde{V}' \tilde{V}]' = \tilde{A}_{T_5}^{-1} \tilde{A}_{T_5}.
\end{equation}

Proof. Let $f$ and $\psi$ be two elements of $D_{\tilde{V}'}$. As usual we shall work with the Fourier transform. By (2.3),

\begin{equation}
(V'f, \tilde{V}'\psi)_{H_1} = \int f^2(\lambda)\tilde{\psi}^2(\lambda) d\Gamma_1(T(\lambda))
\end{equation}

\begin{equation}
= \int f^2(T(s(\lambda)))\tilde{\psi}^2(T(s(\lambda))) d\Gamma_1(T(s(\lambda)))
\end{equation}

\begin{equation}
= \int \tilde{A}_{T_5} f^2(\lambda)\tilde{A}_{T_5}\tilde{\psi}^2(\lambda) d\Gamma_2(\lambda) = (\tilde{A}_{T_5}f, \tilde{A}_{T_5}\psi)_{H_2}.
\end{equation}

From (4.2), we deduce that $D_{\tilde{A}_{T_5}} = D_{\tilde{V}'}$, and $\tilde{A}_{T_5}$ is densely defined in $H_2$, so

\begin{equation}
(f, [\tilde{V}' \tilde{V}]' \psi)_{H_1} = (f, \tilde{A}_{T_5}^2 \tilde{A}_{T_5}\psi(x))_{H_2}.
\end{equation}

Hence

\begin{equation}
[\tilde{V}' \tilde{V}]' = \tilde{A}_{T_5}^{-1} \tilde{A}_{T_5}.
\end{equation}

Let us illustrate the next idea by an example. Let $s(\lambda)$ be an increasing function and define

\begin{equation}
L_x = s(L_2).
\end{equation}

It is clear that $L_x \varphi(\lambda) = \lambda \varphi(\lambda)$, where $\varphi(\lambda) = y(s^{-1}(\lambda))$. Indeed

\begin{equation}
s(L_2)y(s^{-1}(\lambda)) = s(s^{-1}(\lambda))y(s^{-1}(\lambda)) = \lambda y(s^{-1}(\lambda)).
\end{equation}

Therefore

\begin{equation}
V'y(s^{-1}(\lambda)) = y(\lambda).
\end{equation}

For any $f \in D_{\tilde{V}'}$ we have

\begin{equation}
(f, y(\lambda))_{\Phi_2 \times \Phi_2} = (f, V'y(s^{-1}(\lambda)))_{\Phi_2 \times \Phi_2}
\end{equation}

\begin{equation}
= (V'f, y(s^{-1}(\lambda)))_{\Phi_1 \times \Phi_1},
\end{equation}

\begin{equation}
f^2(\lambda) = V'f^2(s^{-1}(\lambda)),
\end{equation}

or

\begin{equation}
f^2(s(\lambda)) = V'f^2(\lambda),
\end{equation}

and so, taking the inverse $y$-Fourier transform, for any $f \in D_{\tilde{V}'}$, $V'f = \tilde{A}_s f$ holds in $H_2$. In other words

\begin{equation}
\tilde{V}' f = \tilde{A}_s f.
\end{equation}
The spectral functions are

\[ f = \int \int \overline{f^1(\lambda)} \varphi(\lambda) \, d\Gamma_1(\lambda) = \int \overline{f^2(s^{-1}(\lambda))} y(s^{-1}(\lambda)) \, d\Gamma_1(\lambda) \]

\[ = \int \overline{f^2(\lambda)} y(\lambda) \, d\Gamma_1(s(\lambda)). \]

On the other hand,

\[ f = \int \int \overline{f^2(\lambda)} y(\lambda) \, d\Gamma_2(\lambda). \]

Therefore

\[ d\Gamma_1(s(\lambda)) = d\Gamma_2(\lambda). \]

**Theorem 11.** Let \( L_2 \) be a self-adjoint operator acting in the rigged Hilbert space \( \Phi_2 \hookrightarrow H_2 \hookrightarrow \Phi_2' \). If \( L_1 = s(L_2) \), where \( s \) is an increasing function, then

\[ \Gamma_1(s(\lambda)) = \Gamma_2(\lambda) \quad \text{for any } f \in D_{\tilde{V}'}, \quad \tilde{V}' f = \tilde{A}_s f. \]

We now discuss the case where we are given two spectral functions, \( \Gamma_1(\lambda) \) and \( \Gamma_2(\lambda) \) such that \( \Gamma_1(s(\lambda)) = \Gamma_2(\lambda) \), where \( s(\lambda) \) is a one-to-one mapping \( \sigma_2 \to \sigma_1 \). The operators are not supposed to commute, and so they are not functions of each other.

\( \Gamma_1(s(\lambda)) \) is \( \text{ABS-}\text{d}\Gamma_2(\lambda) \) since \( g(\lambda) = (d\Gamma_1(s)/d\Gamma_2)(\lambda) = 1 \), and so \( \tilde{g}(L_2) = \text{Id} \) on \( D_{\tilde{g}(L_2)} \subset H_2 \to H_2 \). The shift operator is given by

\[ V \varphi(s(\lambda)) = y(\lambda) \quad \text{for } \lambda \in \sigma_2. \]

From Corollary 4 we deduce that \( \tilde{g}(L_2) = [\tilde{V}']'\tilde{V}' \) or that \( \text{Id} = [\tilde{V}']'\tilde{V}' \). Hence \( \tilde{V}' \) is a unitary operator, in fact

\[ (\tilde{V}' f, \tilde{V}' \psi)_{H_1} = (f, \psi)_{H_2}. \]

We can decompose \( V \) into two shifts:

\[ \varphi(s(\lambda)) \xrightarrow{R} \varphi(\lambda) \xrightarrow{W} y(\lambda), \]

so \( V = W \cdot R \). By definition \( R \varphi(s(\lambda)) = \varphi(\lambda) \) or \( R \varphi(\lambda) = \varphi(s^{-1}(\lambda)) \). So, by Theorem 11, if \( \tilde{R}' \) is the closure of \( R' \) in \( H_1 \), \( \tilde{f}^1(s^{-1}(\lambda)) = \tilde{R}' f(\lambda) \Rightarrow \tilde{A}_s = \tilde{R}' \). Therefore

\[ \text{Id} = [\tilde{V}']'\tilde{V}' = \tilde{W}'\tilde{R}'\tilde{W}' = [\tilde{W}']'(\tilde{A}_s - 1)'(\tilde{A}_s - 1)\tilde{W}'. \]

5. Examples

**Example.** As an example we shall show how to apply the above ideas and obtain an extension of the Gelfand Levitan theory to the generalized second order differential operator \( L_2 = -d^2/w(x) \, dx^2 \), where \( w(x) \geq 0 \). Suppose we are given two second order differential operators such that Theorem 4 holds:

\[ L_1 = L_2 + q(x) \]
where
\begin{equation}
L_2 = -\frac{d^2}{w(x) \, dx^2},
\end{equation}

\(w(x) \geq 0\), and the boundary conditions are included in the definition of the operators. Notice that the operators \(L_1\) and \(L_2\) act in the same space \(L^2_{w(x) \, dx}(0, \infty)\). In what concerns the rigged Hilbert space structure we refer to the construction done by Aleksandriyan. Or, since the Fourier transform is defined, one can simply take \(\Phi \equiv \hat{S}^{-1}\) where \(S\) is a space of rapidly decreasing functions which is nuclear and invariant by the multiplication by \(\lambda\). In this way \(\Phi\) is also an \(N\)-space. The shift operator is given by
\begin{equation}
V = 1 + H,
\end{equation}

where
\begin{equation}
1(f) = f
\end{equation}

and
\begin{equation}
H(f) = \int_0^x H(x, t)f(t)w(t) \, dt.
\end{equation}

The relation between the functions \(H(x, t)\) and \(q(x)\) is as follows: we have shown in (3.2) that
\begin{equation}
L_2 V = V L_1 \quad \text{in} \quad \Phi'_2
\end{equation}
or
\begin{equation}
L_2(1 + H) = (1 + H)(L_2 + q).
\end{equation}

So \(L_2 H - HL_2 = q + Hq\), which is a hyperbolic equation, and \(G = VV'\) or, in other words,
\begin{equation}
G = 1 + H + H' + HH',
\end{equation}

which means that
\begin{equation}
\{G - 1\}f(x) = \{H + H' + HH'\}f(x)
\end{equation}

where \(f(x)\) is a smooth function with compact support. Now notice that the term on the left-hand side of (5.2) is nothing other than
\begin{equation}
[G - 1]\hat{f}(\lambda) = \int \hat{f}^2(\lambda) y(x, \lambda) \, d[\Gamma_1(\lambda) - \Gamma_2(\lambda)].
\end{equation}

Now, using the expression of the Fourier transform, we have
\begin{equation}
\hat{f}^2(\lambda) = \int f(t)y(t, \lambda)w(t) \, dt.
\end{equation}

Assuming that
\begin{equation}
P(x, t) \equiv \int y(t, \lambda)y(x, \lambda) \, d[\Gamma_1 - \Gamma_2](\lambda)
\end{equation}
is a continuous function of $t$ and $x$ and, using the fact that $f(x)$ is of compact support, we obtain, by applying Fubini's theorem,

$$
(G - 1)f(x) = \int f(t) \left[ \int y(t, \lambda)y(x, \lambda) d[\Gamma_1 - \Gamma_2](\lambda) \right] w(t) \, dt
$$

(5.4)

$$
= \int f(t)P(t, x)w(t) \, dt.
$$

We can now express the right-hand side of (5.2):

$$
[H + H' + HH']f(x) = \int_0^x H(x, t)f(t)w(t) \, dt + \int_x^\infty H(t, x)f(t)w(t) \, dt
$$

(5.5)

$$
+ \int_0^x H(x, s)\int_s^\infty H(s, t)f(t)w(t) \, dt w(s) \, ds.
$$

The last term can be written as

$$
\int_0^x \int_0^x H(x, s)H(s, t)w(s) \, ds f(t)w(t) \, dt
$$

(5.5)

$$
+ \int_x^\infty \int_0^x H(x, s)H(s, t)w(s) \, ds f(t)w(t) \, dt,
$$

and so in the weak sense we do have the result

$$
P(t, x) = H(x, t) + \int_0^x H(x, s)H(s, t)w(t) \, dt, \quad t < x.
$$

(5.6)

Suppose that $V^{-1}$ exists and $V^{-1} = 1 + K$. Then we do have

$$
V^{-1}G = V
$$

$$
(1 + K)G = 1 + H^*
$$

$$
(1 + K)Gf(x) = (1 + H^*)f(x)
$$

(5.7)

$$
Gf(x) + \int_0^x K(x, t)Gf(t)w(t) \, dt = f(x) + \int_x^\infty H(t, x)f(t)w(t) \, dt
$$

or

$$
[G - 1]f(x) + \int_0^x K(x, t)f(t)w(t) \, dt + \int_0^x K(x, t)[G - 1]f(t)w(t) \, dt
$$

$$
= \int_x^\infty H(t, x)f(t)w(t) \, dt.
$$

But $[G - 1]f(x) = \int P(t, x)f(t)w(t) \, dt$, so we have

(5.8)

$$
\int P(t, x)f(t)w(t) \, dt + \int_0^x K(x, t)f(t)w(t) \, dt
$$

$$
+ \int_0^x K(x, s)P(s, t)w(s) \, ds f(t)w(t) \, dt = \int_x^\infty H(t, x)f(t)w(t) \, dt
$$

Hence,

(5.9)

$$
P(t, x) + K(x, t) + \int_0^x K(x, s)P(s, t)w(s) \, ds = H(x, t).
$$
Observe that $H(t, x) = 0$ if $t < x$; hence

$$P(t, x) + K(x, t) + \int_0^x K(x, s)P(s, t)w(s)\,ds = 0$$

where $0 \leq t < x$.

**Example 2.** Everitt and Zettl computed the Weyl–Titchmarsh function, associated with the operator

$$L_2f \equiv \frac{-1}{x^\alpha} f''(x), \quad x \in [0, \infty).$$

They proved that the spectral function associated with

$$L_2f \equiv \frac{-1}{x^\alpha} \frac{d^2}{dx^2} f(x), \quad x \geq 0,$$

$$f'(0) = 0$$

is of the form

$$\Gamma_2(\lambda) = \begin{cases} c \cdot \lambda^{(\alpha+1)/(\alpha+2)} & \text{for } \lambda \geq 0 \\ 0 & \text{for } \lambda < 0. \end{cases}$$

For further details see [2].

Let us prove the same result but using our method. We shall define another operator $L_1$ and then obtain the shift operator $V$. Having $V$ we shall use $(d\Gamma_1/d\Gamma_2)(L_2) = VV'$ to obtain an equation for $\Gamma_1(\lambda)$.

Denote by $y(x, \lambda)$ the eigenfunctions of $L_2y(x, \lambda) = \lambda y(x, \lambda)$

$$y''(x, \lambda) + \lambda x^\alpha y(x, \lambda) = 0$$

(5.11)

$$y(x, \lambda) = 1 \quad \text{and} \quad y'(x, \lambda) = 0.$$

Solutions of (5.11) can be written in terms of the Bessel functions ($\lambda = \mu^2$):

$$y(x, \mu^2) = Ar(x)\sqrt{t(x)}\mu J_\nu(t(x)\mu) + Br(x)\sqrt{t(x)}\mu J_{-\nu}(t(x)\mu),$$

where $r(x) = x^{-\alpha/4}$, $t(x) = 2\nu x^{1/2\nu}$ and $\nu = 1/(\alpha + 2)$. $A$ and $B$ are determined from the boundary conditions

$$y(0, \lambda) = 1 \quad \text{and} \quad y'(0, \lambda) = 0.$$

So

$$A = 0 \quad \text{and} \quad B = \mu^{-1/2} \cdot C(\nu),$$

where $C(\nu)$ is a function of $\nu$ only.

Define an operator

$$L^2 \overset{\mathcal{T}}{\rightarrow} L_2^2dx$$

$$f \rightarrow \mathcal{T}f(x) = r(x)f(t(x)).$$

We do have $TT' = \text{Id}$. Since $T'g(t) = x(t)^{\alpha/4}g(x(t))$, where $x(t)$ is the inverse function of $t(x)$. Thus

$$y(x, \mu^2) = \mu^{-1/2} \cdot C(\nu) \cdot T(\sqrt{t}J_{-\nu}(t\mu)).$$
Now it is time to find the operator $L_1$. If
\[ \phi(t, \mu^2) = \sqrt{\mu} J_{-\nu}(t \mu) \]
are the eigenfunctionals of $L_1$, then, from the inversion formula of the Bessel functions,
\[ F(\lambda) = \int_0^\infty f(t) \sqrt{t} J_{-\nu}(t \sqrt{\lambda}) \, dt \]
\[ f(t) = \int_0^\infty F(\lambda) \sqrt{t} J_{-\nu}(t \sqrt{\lambda}) \, d\sqrt{\lambda}. \]
We deduce that $d\Gamma_1(\lambda) = d\sqrt{\lambda}$.

The shift operator is given by
\[ y(x, \mu^2) = \mu^{\nu-1/2} \cdot C(\mu) \cdot T[\phi(x, \mu^2)], \]
since
\[ y(x, \lambda) = \lambda^{(\nu-1/2)/2} \cdot C(\nu) \cdot T[\phi(x, \lambda)]. \]
Using the fact that $L_2 y = \lambda y$ we have
\[ L_2^{(\nu-1/2)/2} y = \lambda^{(\nu-1/2)/2} y \]
\[ = \lambda^{(\nu-1/2)/2} \cdot C(\nu) \cdot L_2^{(\nu-1/2)/2} = T[\phi] \]
Simplifying by $\lambda^{(\nu-1/2)/2}$,
\[ y(x, \lambda) = C(\nu) \cdot L_2^{(\nu-1/2)/2} \cdot T[\phi]. \]
Recall that
\[ y \equiv V \phi, \]
hence
\[ V[\cdot] = C(\nu) \cdot L_2^{(\nu-1/2)/2} \cdot T[\cdot]. \]
From Theorem 4 we obtain that
\[ \frac{d\Gamma_1}{d\Gamma_2}(L_2) = V V' = C(\nu)^2 \cdot L_2^{(\nu-1/2)/2} \cdot T \cdot T' L_2^{(\nu-1/2)/2}. \]
Since $TT' = 1$
\[ \frac{d\Gamma_1}{d\Gamma_2}(L_2) = C(\nu)^2 \cdot L_2^{(\nu-1/2)}. \]
Hence
\[ \frac{d\Gamma_1}{d\Gamma_2}(\lambda) = C(\nu)^2 \cdot \lambda^{(\nu-1/2)}. \]
Let us solve the above differential equation:

\[ d\Gamma_2(\lambda) = \frac{1}{C(\nu)^2} \lambda^{1/2-\nu} d\Gamma_1(\lambda) \]

\[ = \frac{1}{C(\nu)^2} \lambda^{1/2-\nu} d\sqrt{\lambda} \]

\[ \Gamma_2(\lambda) = \frac{1}{C(\nu)^2} \int_0^{\sqrt{\lambda}} s^{2(1/2-\nu)} ds \]

\[ = \frac{1}{2-2\nu} \cdot \frac{1}{C(\nu)^2} \int_0^{\sqrt{\lambda}} ds^{2-2\nu} \]

\[ \Gamma_2(\lambda) = \frac{1}{2-2\nu} \cdot \frac{1}{C(\nu)^2} \lambda^{1-\nu}. \]

So

\[ \Gamma_2(\lambda) = c\lambda^{1-1/(\alpha+2)}, \]

where \( c \) is a constant. That is what Everitt and Zettl have shown.

References


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