LOWER 2-ESTIMATES FOR SEQUENCES IN BANACH LATTICES

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Abstract. We characterize Banach lattices in which every bounded sequence contains a subsequence which either converges in norm or satisfies a lower 2-estimate. As a consequence we obtain, for the class of all Banach lattices, a positive answer to a question of D.J. Aldous and D.H. Fremlin whether a Banach space of cotype 2 satisfies the above-mentioned property.

D. J. Aldous and D. H. Fremlin proved that every AL-space $E$ has the following property (cf. [1, Theorem 6]):

(AF) Every bounded sequence $(x_n)$ in $E$ has a subsequence which either converges in norm or satisfies a lower 2-estimate.

In this note we characterize the Banach lattices satisfying (AF). As a consequence we obtain that every Banach lattice of cotype 2 has property (AF) which solves a problem of D. J. Aldous and D. H. Fremlin (cf. [1, §9]) for the class of all Banach lattices.

Our notation is standard and follows [3]. Let us recall the following definitions. A sequence $(x_n)$ in a Banach space $E$ is called semi-normalized if it is bounded and $\inf_n \|x_n\| > 0$. A sequence $(x_n)$ in $E$ satisfies a lower 2-estimate if there exists $c > 0$ such that for every $n \in \mathbb{N}$ and every choice of scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, we have $c(\sum_{k=1}^n |\alpha_k|^2)^{1/2} \leq \|\sum_{k=1}^n \alpha_k x_k\|$. A sequence $(x_n)$ in a Banach lattice $E$ is called disjoint if $\inf(|x_n|, |x_m|) = 0$ for all $n, m \in \mathbb{N}$ with $n \neq m$.

We begin with a lemma which occurs implicitly in the proof of Theorem 6 in [1].

Lemma. Let $E$ be a weakly sequentially complete Banach space. Suppose every semi-normalized weakly null sequence in $E$ contains a subsequence satisfying a lower 2-estimate. Then $E$ has property (AF).

Proof. Let $(x_n)$ be a bounded sequence in $E$ such that no subsequence is norm convergent. If $(x_n)$ does not have a weak Cauchy subsequence, then, by Rosenthal's $l_1$-theorem (cf. [2, 2.e.5]), there is a subsequence equivalent to the...
unit vector basis of \( l_1 \), which obviously satisfies a lower 2-estimate. Otherwise there exists a weak Cauchy subsequence \((y_n)\) of \((x_n)\). Since \( E \) is weakly sequentially complete, \((y_n)\) converges weakly to some \( y \in E \) but \((y_n)\) does not converge in norm. Our assumption implies that a subsequence of \((y_n - y)\) satisfies a lower 2-estimate, hence, by Lemma 2 in [1], a subsequence of \((y_n)\) satisfies a lower 2-estimate. □

We are already in the position to state our main result.

**Theorem.** Let \( E \) be a Banach lattice. Then the following assertions are equivalent:

(a) \( E \) satisfies (AF).

(b) Every semi-normalized disjoint sequence in \( E \) contains a subsequence which satisfies a lower 2-estimate.

**Proof.** (a) \(\Rightarrow\) (b). Every disjoint norm convergent sequence in a Banach lattice converges to zero. Hence a semi-normalized disjoint sequence \((x_n)\) in \( E \) cannot have a convergent subsequence. Since \( E \) satisfies (AF) there is a subsequence of \((x_n)\) which satisfies a lower 2-estimate and assertion (b) follows.

(b) \(\Rightarrow\) (a). The assumption on \( E \) implies that \( E \) does not contain a Banach sublattice isomorphic to \( c_0 \). As a consequence \( E \) is weakly sequentially complete [3, Remark on p. 35, 1.c.4]. By the lemma, it is enough to show that, for a semi-normalized weakly null sequence \((x_n)\) in \( E \) there is a subsequence which satisfies a lower 2-estimate. We may assume that \( E \) is separable, otherwise we consider the closed separable sublattice of \( E \) generated by \((x_n)\). Since \( E \) is weakly sequentially complete, the Banach lattice \( E \) has order continuous norm [3, 1.c.4, 1.a.8]. There exists a probability space \((\Omega, \Sigma, \mu)\) such that \( E \) can be identified as a vector lattice with an ideal of \( L_1(\Omega, \Sigma, \mu) \), and with this identification we have \( \|jx\|_1 \leq \|x\|_E \) for all \( x \in E \), where \( j: E \rightarrow L_1(\Omega, \Sigma, \mu) \) denotes the canonical injection (cf. [3, 1.b.14]). Suppose there exists \( \varepsilon > 0 \) such that \( \varepsilon \|x_n\|_E \leq \|jx_n\|_1 \) for every \( n \in \mathbb{N} \). Then \((jx_n)\) is a semi-normalized weakly null sequence in \( L_1(\Omega, \Sigma, \mu) \). Since \( L_1(\Omega, \Sigma, \mu) \) has property (AF) [1, Theorem 6], a subsequence of \((jx_n)\) and then also a subsequence of \((x_n)\) satisfies a lower 2-estimate. If such an \( \varepsilon > 0 \) does not exist we can find a subsequence \((x_{n_k})\) of \((x_n)\) and a disjoint sequence \((y_k)\) in \( E \) such that \( \|x_{n_k} - y_k\|_E \leq 2^{-k} \), \( k \in \mathbb{N} \) (cf. [3, Proofs of 1.c.10 and 1.c.8]). Our assumption implies that there is a subsequence of \((y_k)\) satisfying a lower 2-estimate. Since \( \Sigma_k \|x_{n_k} - y_k\|_E < \infty \) there exists a subsequence of \((x_{n_k})\) which also satisfies a lower 2-estimate [1, Lemma 2]. This proves the theorem. □

D. J. Aldous and D. H. Fremlin [1, §9] asked whether every Banach space of cotype 2 has property (AF). From the Theorem we get that at least for Banach lattices this implication holds. We recall the following notions.

Let \( E \) be a Banach lattice. Then \( E \) has cotype \( q \) for some \( q \geq 2 \) if there exists a constant \( M > 0 \) such that, for every \( n \in \mathbb{N} \) and arbitrary vectors
\(x_1, \ldots, x_n \in E\), we have

\[
M \left( \sum_{k=1}^{n} \|x_k\|^q \right)^{1/q} \leq \int_0^1 \left\| \sum_{k=1}^{n} r_k(t)x_k \right\| \, dt,
\]

where \((r_k)\) denotes the sequence of Rademacher functions on the interval \([0, 1]\) [3, 1.e.12]. The Banach lattice \(E\) satisfies a lower \(q\)-estimate, \(1 \leq q < \infty\), if there exists a constant \(c > 0\) such that, for every \(n \in \mathbb{N}\) and every choice of pairwise disjoint elements \(x_1, \ldots, x_n \in E\), we have

\[
c \left( \sum_{k=1}^{n} \|x_k\|^q \right)^{1/q} \leq \left\| \sum_{k=1}^{n} x_k \right\|.
\]

[3, 1.f.4] If \(x_1, \ldots, x_n\) are pairwise disjoint elements of a Banach lattice \(E\) then, for an arbitrary choice of signs \(e_k \in \{-1, 1\}\), \(1 \leq k \leq n\), we have

\[
\| \sum_{k=1}^{n} e_k x_k \| = \| \sum_{k=1}^{n} x_k \|. 
\]

Since the Rademacher functions take values in \(\{-1, 1\}\) we obtain

\[
\int_0^1 \| \sum_{k=1}^{n} r_k(t)x_k \| \, dt = \| \sum_{k=1}^{n} x_k \|. 
\]

As a consequence every Banach lattice \(E\) of cotype \(q\), \(q \geq 2\), satisfies a lower \(q\)-estimate. On the other hand every Banach lattice \(E\) which satisfies a lower 2-estimate fulfills assertion (b) of the Theorem.

Hence, the Theorem implies the following result.

**Corollary.** Let \(E\) be a Banach lattice of cotype 2, then \(E\) satisfies \((AF)\).

The cotype 2 condition is not necessary for a Banach lattice to have property \((AF)\). For instance, \(E := \oplus l^n_{p_n}\), the \(l^n\)-sum of the sequence of spaces \((\mathbf{R}^n, \| \cdot \|_{p_n})\), satisfies \((AF)\) for every sequence \((p_n)\) in \([1, \infty]\) (cf. [4, Corollary 2.5, 1.3 Example (b)]). However, if \(\lim_n p_n = \infty\) then \(E\) has cotype \(\infty\).

**REFERENCES**


