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LOWER 2-ESTIMATES FOR SEQUENCES IN BANACH LATTICES

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ABSTRACT. We characterize Banach lattices in which every bounded sequence contains a subsequence which either converges in norm or satisfies a lower 2-estimate. As a consequence we obtain, for the class of all Banach lattices, a positive answer to a question of D.J. Aldous and D.H. Fremlin whether a Banach space of cotype 2 satisfies the above-mentioned property.

D. J. Aldous and D. H. Fremlin proved that every AL-space E has the following property (cf. [1, Theorem 6]):

(AF) Every bounded sequence (x_n) in E has a subsequence which either converges in norm or satisfies a lower 2-estimate.

In this note we characterize the Banach lattices satisfying (AF). As a consequence we obtain that every Banach lattice of cotype 2 has property (AF) which solves a problem of D. J. Aldous and D. H. Fremlin (cf. [1, §9]) for the class of all Banach lattices.

Our notation is standard and follows [3]. Let us recall the following definitions. A sequence (x_n) in a Banach space E is called *semi-normalized* if it is bounded and $\inf_n \|x_n\| > 0$. A sequence (x_n) in E satisfies a *lower 2-estimate* if there exists $c > 0$ such that for every $n \in \mathbf{N}$ and every choice of scalars $\alpha_1, \dots, \alpha_n \in \mathbf{R}$, we have $c(\sum_{k=1}^n |\alpha_k|^2)^{1/2} \leq \|\sum_{k=1}^n \alpha_k x_k\|$. A sequence (x_n) in a Banach lattice E is called *disjoint* if $\inf(|x_n|, |x_m|) = 0$ for all $n, m \in \mathbf{N}$ with $n \neq m$.

We begin with a lemma which occurs implicitly in the proof of Theorem 6 in [1].

Lemma. *Let E be a weakly sequentially complete Banach space. Suppose every semi-normalized weakly null sequence in E contains a subsequence satisfying a lower 2-estimate. Then E has property (AF).*

Proof. Let (x_n) be a bounded sequence in E such that no subsequence is norm convergent. If (x_n) does not have a weak Cauchy subsequence, then, by Rosenthal's l_1-theorem (cf. [2, 2.e.5]), there is a subsequence equivalent to the

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unit vector basis of l_1, which obviously satisfies a lower 2-estimate. Otherwise there exists a weak Cauchy subsequence (y_n) of (x_n). Since E is weakly sequentially complete, (y_n) converges weakly to some $y \in E$ but (y_n) does not converge in norm. Our assumption implies that a subsequence of $(y_n - y)$ satisfies a lower 2-estimate, hence, by Lemma 2 in [1], a subsequence of (y_n) satisfies a lower 2-estimate. □

We are already in the position to state our main result.

Theorem. *Let E be a Banach lattice. Then the following assertions are equivalent*:

(a) *E satisfies (AF).*
(b) *Every semi-normalized disjoint sequence in E contains a subsequence which satisfies a lower 2-estimate.*

Proof. (a) $\Rightarrow$ (b). Every disjoint norm convergent sequence in a Banach lattice converges to zero. Hence a semi-normalized disjoint sequence (x_n) in E cannot have a convergent subsequence. Since E satisfies (AF) there is a subsequence of (x_n) which satisfies a lower 2-estimate and assertion (b) follows.

(b) $\Rightarrow$ (a). The assumption on E implies that E does not contain a Banach sublattice isomorphic to c_0. As a consequence E is weakly sequentially complete [3, Remark on p. 35, 1.c.4]. By the lemma, it is enough to show that, for a semi-normalized weakly null sequence (x_n) in E there is a subsequence which satisfies a lower 2-estimate. We may assume that E is separable, otherwise we consider the closed separable sublattice of E generated by (x_n). Since E is weakly sequentially complete, the Banach lattice E has order continuous norm [3, 1.c.4, 1.a.8]. There exists a probability space (Ω, Σ, μ) such that E can be identified as a vector lattice with an ideal of $L_1(\Omega, \Sigma, \mu)$, and with this identification we have $\|jx\|_1 \leq \|x\|_E$ for all $x \in E$, where $j: E \to L_1(\Omega, \Sigma, \mu)$ denotes the canonical injection (cf. [3, 1.b.14]). Suppose there exists $\varepsilon > 0$ such that $\varepsilon\|x_n\|_E \leq \|jx_n\|_1$ for every $n \in \mathbf{N}$. Then (jx_n) is a semi-normalized weakly null sequence in $L_1(\Omega, \Sigma, \mu)$. Since $L_1(\Omega, \Sigma, \mu)$ has property (AF) [1, Theorem 6], a subsequence of (jx_n) and then also a subsequence of (x_n) satisfies a lower 2-estimate. If such an $\varepsilon > 0$ does not exist we can find a subsequence (x_{n_k}) of (x_n) and a disjoint sequence (y_k) in E such that $\|x_{n_k} - y_k\|_E \leq 2^{-k}$, $k \in \mathbf{N}$ (cf. [3, Proofs of 1.c.10 and 1.c.8]). Our assumption implies that there is a subsequence of (y_k) satisfying a lower 2-estimate. Since $\Sigma_k \|x_{n_k} - y_k\|_E < \infty$ there exists a subsequence of (x_{n_k}) which also satisfies a lower 2-estimate [1, Lemma 2]. This proves the theorem. □

D. J. Aldous and D. H. Fremlin [1, §9] asked whether every Banach space of cotype 2 has property (AF). From the Theorem we get that at least for Banach lattices this implication holds. We recall the following notions.

Let E be a Banach lattice. Then E has *cotype* q for some $q \geq 2$ if there exists a constant $M > 0$ such that, for every $n \in \mathbf{N}$ and arbitrary vectors

$x_1, \ldots, x_n \in E$, we have

$$M\left(\sum_{k=1}^{n} \|x_k\|^q\right)^{1/q} \leq \int_0^1 \left\|\sum_{k=1}^{n} r_k(t)x_k\right\| dt,$$

where (r_k) denotes the sequence of Rademacher functions on the interval $[0, 1]$ [3, 1.e.12]. The Banach lattice E satisfies a *lower q-estimate*, $1 \leq q < \infty$, if there exists a constant $c > 0$ such that, for every $n \in \mathbf{N}$ and every choice of pairwise disjoint elements $x_1, \ldots, x_n \in E$, we have

$$c\left(\sum_{k=1}^{n} \|x_k\|^q\right)^{1/q} \leq \left\|\sum_{k=1}^{n} x_k\right\|,$$

[3, 1.f.4]. If $x_1, \ldots, x_n$ are pairwise disjoint elements of a Banach lattice E then, for an arbitrary choice of signs $\varepsilon_k \in \{-1, 1\}$, $1 \leq k \leq n$, we have $\|\sum_{k=1}^{n} \varepsilon_k x_k\| = \|\sum_{k=1}^{n} x_k\|$. Since the Rademacher functions take values in $\{-1, 1\}$ we obtain $\int_0^1 \|\sum_{k=1}^{n} r_k(t)x_k\| dt = \|\sum_{k=1}^{n} x_k\|$. As a consequence every Banach lattice E of cotype q, $q \geq 2$, satisfies a lower q-estimate. On the other hand every Banach lattice E which satisfies a lower 2-estimate fulfills assertion (b) of the Theorem.

Hence, the Theorem implies the following result.

Corollary. *Let E be a Banach lattice of cotype 2, then E satisfies (AF).*

The cotype 2 condition is not necessary for a Banach lattice to have property (AF). For instance, $E := \oplus_{l_1} l_{p_n}^n$, the l_1-sum of the sequence of spaces $(\mathbf{R}^n, \|\cdot\|_{p_n})$, satisfies (AF) for every sequence (p_n) in $[1, \infty]$ (cf. [4, Corollary 2.5, 1.3 Example (b)]). However, if $\overline{\lim}_n p_n = \infty$ then E has cotype ∞.

References

1. D. J. Aldous and D. H. Fremlin, *Colacunary sequences in L-spaces*, Studia Math. **71** (1982), 297–304.
2. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces* I. *Sequence spaces*, Springer-Verlag, Berlin, Heidelberg, and New York, 1977.
3. ——, *Classical Banach Spaces* II. *Function spaces*, Springer-Verlag, Berlin, Heidelberg, and New York, 1979.
4. F. Räbiger, *Lower and upper 2-estimates for order bounded sequences and Dunford-Pettis operators between certain classes of Banach lattices* (Proc. Univ. of Texas at Austin 1987–1989), Lecture Notes in Math, Springer-Verlag (to appear).

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