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THE RANGE OF A VECTOR MEASURE DETERMINES ITS TOTAL VARIATION

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ABSTRACT. We prove that if the ranges of two finitely additive measures with values in a normed space have the same closed convex hull, then the measures have the same total variation. We also study the monotonicity of this variation with respect to the range, proving that a normed space X is C-isomorphic to a subspace of an L^1 space if and only if, for every pair μ, ν of X-valued measures such that the range of μ lies inside the closed convex hull of the range of ν, the total variation of μ is less than or equal to C times the total variation of ν. This allows us to answer two questions raised by R. Anantharaman and J. Diestel.

INTRODUCTION AND NOTATION

In [AD] the authors asked whether there can exist two vector measures with the same range, exactly one of them having bounded variation. In Theorem 3 we prove that this is impossible; moreover, we prove that the range of a vector measure determines its total variation: measures with the same range have the same total variation. Actually the question had a finite-dimensional nature, as the proof of Theorem 3 reveals; it suffices to prove it for finite-dimensional spaces to extend it to all normed spaces.

Anantharaman and Diestel also proved, via a result of Grothendieck, that a subspace X of L^1 enjoys the property that if the range of an X-valued measure μ lies inside the range of another measure of bounded variation, then μ has bounded variation too. We again prove this result in Theorem 5 and show that this property characterizes the subspaces of L^1, answering another question in [AD]. Here we use the local structure of L^1 and a finite-dimensional characterization of its subspaces due to Lindenstrauss and Pelczynski [LP].

Let us introduce some notation. If X is a real, normed space, X^* will denote its dual space. If K is a subset of X, $\overline{\mathrm{co}}(K)$ will be the closed convex hull of K; we will use the fact that, for every f in X^*, the supremum over

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K and over $\overline{\text{co}}(K)$ coincide. As usual, l_p^n will stand for the Banach space $\mathbf{R}^n$ with the norm $\| \ \|_p$, $1 \le p \le \infty$. B_p^n will be the closed unit ball of l_p^n.

Let $\mathscr{A}$ be an algebra (field) of subsets of a set Ω, and $(X, \| \ \|)$ be a seminormed space; if $\mu: \mathscr{A} \to X$ is a (finitely additive) measure we will denote by $\|\mu\|$ the *total variation* of μ, that is

$$\|\mu\| = \sup\left\{ \sum_{A\in\mathscr{P}} \|\mu(A)\| : \mathscr{P} \text{ a finite partition of } \Omega \text{ in } \mathscr{A} \right\},$$

where we allow the supremum to be $+\infty$. This supremum is also a limit over the directed (by refinement) set of finite partitions of Ω in $\mathscr{A}$; this fact will simplify our proofs technically. A measure μ has *bounded variation* if $\|\mu\| < +\infty$. The range of μ will be denoted by $\operatorname{rg}\mu$, that is $\operatorname{rg}\mu = \{\mu(A): A \in \mathscr{A}\}$.

Throughout this paper every countably additive vector measure will be supposed to be defined on a σ-algebra of sets.

Measures with the same range

We devote the first part of this section to proving the announced result that two measures with the same range have the same total variation. We will see that, in fact, it is the closed convex hull of the range which determines the total variation of a measure. After that we will prove (Proposition 4) that, when dealing with $\mathbf{R}^n$-valued measures, the total variation is continuous with respect to the range when we consider the Hausdorff topology. We begin with two easy lemmas.

Lemma 1. *Let Ω be a set and $\mathscr{A}$ an algebra of subsets of Ω. Let $\mu: \mathscr{A} \to \mathbf{R}^n$ be a measure. Consider in $\mathbf{R}^n$ the seminorm $\|x\| = \sum_{i=1}^k |f_i(x)|$, where $f_1, \dots, f_k$ are in $(\mathbf{R}^n)^*$. Then*

$$\|\mu\| = \sum_{i=1}^k \|f_i \circ \mu\| = \sum_{i=1}^k (\sup\{f_i(x): x \in \operatorname{rg}\mu\} - \inf\{f_i(x): x \in \operatorname{rg}\mu\}).$$

Proof. Let $\mathscr{P} = \{A_j\}_{j=1}^r$ be a partition of Ω in $\mathscr{A}$; then

$$\sum_{j=1}^r \|\mu(A_j)\| = \sum_{j=1}^r \sum_{i=1}^k |f_i(\mu(A_j))| = \sum_{i=1}^k \left(\sum_{j=1}^r |f_i \circ \mu(A_j)| \right).$$

Taking limits over $\mathscr{P}$ we get the first equality.

The second equality is just the known fact that the total variation of a real-valued measure ν is given by $\sup(\operatorname{rg}\nu) - \inf(\operatorname{rg}\nu)$ [BB, Theorem 2.2.4].

Lemma 2. *Let $\| \cdot \|$ be a norm in $\mathbf{R}^n$ and $\varepsilon > 0$. Then there exist linear functionals $f_1, \dots, f_k, g_1, \dots, g_l$ in $(\mathbf{R}^n)^*$ such that*

$$\left| \|x\| - \left(\sum_{i=1}^k |f_i(x)| - \sum_{j=1}^l |g_j(x)| \right) \right| \le \varepsilon \|x\|, \quad \textit{for all } x \in \mathbf{R}^n.$$

Proof. Let S^{n-1} be the Euclidean sphere in $\mathbf{R}^n$, and $C_e(S^{n-1})$ the space of real continuous functions which are even (i.e., $f(x) = f(-x)$ for all $x \in S^{n-1}$). It is a known fact (see [C, p. 53] or [B, Theorem 2.8]) that the linear span of the set $\mathscr{H} = \{|f| : f \in (\mathbf{R}^n)^*\}$ is uniformly dense in $C_e(S^{n-1})$. Let $\|\cdot\|$ be a norm in $\mathbf{R}^n$ and set $\alpha = \min\{\|x\| : x \in S^{n-1}\} > 0$. As the norm is even, for every ε positive we can find ϕ in the linear span of $\mathscr{H}$ such that $|\phi(x) - \|x\|| \le \varepsilon\alpha$ for every $x \in S^{n-1}$, and so $|\phi(x) - \|x\|| \le \varepsilon\|x\|$ for every $x \in S^{n-1}$.

It is obvious that ϕ can be written as

$$\phi = \sum_{i=1}^{k} |f_i| - \sum_{j=1}^{l} |g_j| \; f_1, \dots, f_k, g_1, \dots, g_l \in (\mathbf{R}^n)^*.$$

By homogeneity, the result follows.

Lemma 2 remains true for a seminorm; to see that this is so, take the quotient of $\mathbf{R}^n$ over the subspace where the seminorm vanishes and apply the result there to the quotient norm.

We can now establish and prove our main result.

Theorem 3. *Let $\mathscr{A}$, $\mathscr{A}'$ be two algebras of subsets of Ω and Ω', respectively. Let X be a normed space and $\mu: \mathscr{A} \to X$, $\nu: \mathscr{A}' \to X$ be two vector measures such that*

$$\overline{\mathrm{co}}(\mathrm{rg}\,\mu) = \overline{\mathrm{co}}(\mathrm{rg}\,\nu).$$

Then $\|\mu\| = \|\nu\|$.

Proof. Suppose $\|\mu\| > \|\nu\|$. Then there is a partition $\{A_j\}_{j=1}^n$ of Ω in $\mathscr{A}$ such that

$$\sum_{j=1}^{n} \|\mu(A_j)\| > \|\nu\|.$$

For every j, choose x_j^* in X^* of norm one such that $x_j^*(\mu(A_j)) = \|\mu(A_j)\|$, and define the linear operator $T: X \to l_\infty^n$ by $Tx = (x_1^*(x), \dots, x_n^*(x))$. Then $\|T\| = 1$ and

$$\|T\mu\|_\infty \ge \sum_{j=1}^{n} \|T\mu(A_j)\| = \sum_{j=1}^{n} \|\mu(A_j)\| > \|\nu\| \ge \|T\nu\|_\infty.$$

It is obvious that $\overline{\mathrm{co}}(\mathrm{rg}\,T\mu) = \overline{\mathrm{co}}(\mathrm{rg}\,T\nu)$. So if the result is false for the normed space X, then it is false for a certain l_∞^n. Thus we will suppose that $X = l_\infty^n = (\mathbf{R}^n, \|\cdot\|_\infty)$.

If the range of μ is unbounded in l_∞^n, so it will be the range of ν, and then both measures would have unbounded variation. For bounded ranges in l_∞^n, both variations would be bounded also. The result will be proved by showing that, given ε positive, there exists an a_ε in $\mathbf{R}$ such that

$$(1) \qquad |\|\mu\|_\infty - a_\varepsilon| \le \varepsilon\|\mu\|_\infty,$$

$$(1') \qquad |\|\nu\|_\infty - a_\varepsilon| \le \varepsilon\|\nu\|_\infty.$$

By Lemma 2, there are linear functionals $f_1, \dots, f_k, g_1, \dots, g_l$ in $(\mathbf{R}^n)^*$ such that

$$\left| \|x\|_\infty - \left(\sum_{i=1}^{k} |f_i(x)| - \sum_{j=1}^{l} |g_j(x)| \right) \right| \le \varepsilon \|x\|_\infty \quad \text{for all } x \text{ in } \mathbf{R}^n. \tag{2}$$

Consider the following seminorms in $\mathbf{R}^n$:

$$\|x\|_1^\varepsilon = \sum_1^k |f_i(x)|,$$
$$\|x\|_2^\varepsilon = \sum_1^l |g_j(x)|.$$

The condition $\overline{\mathrm{co}}(\mathrm{rg}\,\mu) = \overline{\mathrm{co}}(\mathrm{rg}\,\nu)$ implies that every linear functional f in $(\mathbf{R}^n)^*$ has the same supremum over the range of μ and ν; the same holds for the infimum. This fact and Lemma 1 imply that both measures have the same variation with respect to the seminorms $\|\cdot\|_1^\varepsilon$ and $\|\cdot\|_2^\varepsilon$, that is

$$\|\mu\|_1^\varepsilon = \|\nu\|_1^\varepsilon \qquad \text{and} \qquad \|\mu\|_2^\varepsilon = \|\nu\|_2^\varepsilon.$$

Take $a_\varepsilon = \|\mu\|_1^\varepsilon - \|\mu\|_2^\varepsilon$. For each partition $\{A_j\}_{j=1}^r$ of Ω in $\mathscr{A}$ from (2), we get

$$\left| \sum_{j=1}^r \|\mu(A_j)\|_\infty - \sum_{j=1}^r \|\mu(A_j)\|_1^\varepsilon + \sum_{j=1}^r \|\mu(A_j)\|_2^\varepsilon \right| \le \varepsilon \sum_{j=1}^r \|\mu(A_j)\|_\infty \le \varepsilon \|\mu\|_\infty.$$

Taking a limit in the directed set of finite partitions in $\mathscr{A}$, we obtain (1) with $a_\varepsilon = \|\mu\|_1^\varepsilon - \|\mu\|_2^\varepsilon$. The proof of $(1')$ is similar, and the theorem follows.

Remark. Theorem 3 was first proved by J. Arias de Reyna (unpublished) in the case of two countably additive measures whose ranges are the Euclidean ball in $\mathbf{R}^n$.

A set in $\mathbf{R}^n$ is called a *zonoid* if it is the range of a countably additive atom-free measure or, equivalently, if it is the closed convex hull of the range of a bounded measure. We will write $\mathscr{Z}_n$ for the set of zonoids in $\mathbf{R}^n$. $\mathscr{Z}_n$ is stable under the usual addition of sets:

$$K_1 + K_2 = \{x_1 + x_2 : x_1 \in K_1, x_2 \in K_2\}.$$

$\mathscr{Z}_n$ becomes a complete metric space for the usual Hausdorff metric between compact sets in $\mathbf{R}^n$:

$$d(K_1, K_2) = \inf\{\varepsilon : K_1 \subseteq K_2 + \varepsilon B_2^n \text{ and } K_2 \subseteq K_1 + \varepsilon B_2^n\}.$$

We refer to [B] for proofs of these facts.

Theorem 3 allows us to define the *variation of a zonoid* with respect to a norm $\|\ \|$ in $\mathbf{R}^n$ in the obvious way; if $K \in \mathscr{Z}_n$, take a measure ν such that $\mathrm{rg}\,\nu = K$

and write $\operatorname{var}(\|\ \|, K) = \|\nu\|$. It is easy to prove that the map $\operatorname{var}(\|\ \|, \cdot)$ is additive:

$$\operatorname{var}(\|\ \|, K_1 + K_2) = \operatorname{var}(\|\ \|, K_1) + \operatorname{var}(\|\ \|, K_2) \quad K_1, K_2 \in \mathscr{Z}_n.$$

A consequence of the proof of Theorem 3 is the following:

Proposition 4. *The map* $\operatorname{var}(\|\ \|, \cdot): \mathscr{Z}_n \to \mathbf{R}$ *is continuous.*

Proof. If the norm $\|\ \|$ is as in Lemma 1, then $K_1 \subseteq K_2$ implies $\operatorname{var}(\|\ \|, K_1) \leq \operatorname{var}(\|\ \|, K_2)$; if $d(K_1, K_2) \leq \delta$, using the well known fact that the Euclidean ball is a zonoid, we have:

$$\operatorname{var}(\|\ \|, K_1) \leq \operatorname{var}(\|\ \|, K_2) + \delta \operatorname{var}(\|\ \|, B_2^n)$$
$$\operatorname{var}(\|\ \|, K_2) \leq \operatorname{var}(\|\ \|, K_1) + \delta \operatorname{var}(\|\ \|, B_2^n)$$

and so $|\operatorname{var}(\|\ \|, K_1) - \operatorname{var}(\|\ \|, K_2)| \leq \delta \operatorname{var}(\|\ \|, B_2^n)$, which yields the continuity in this case.

For an arbitrary norm $\|\ \|$, take $\|\ \|_1^\varepsilon$ and $\|\ \|_2^\varepsilon$ as in the proof of the theorem, and define $\psi_\varepsilon: \mathscr{Z}_n \to \mathbf{R}$ as

$$\psi_\varepsilon(K) = \operatorname{var}(\|\ \|_1^\varepsilon, K) - \operatorname{var}(\|\ \|_2^\varepsilon, K), \qquad K \in \mathscr{Z}_n;$$

ψ_ε is continuous and we know that for every $K \in \mathscr{Z}_n$

$$|\psi_\varepsilon(K) - \operatorname{var}(\|\ \|, K)| \leq \varepsilon \operatorname{var}(\|\ \|, K).$$

This says that there is uniform convergence of $\{\psi_\varepsilon\}$ as $\varepsilon \to 0$ in the subsets of $\mathscr{Z}_n$, where $\operatorname{var}(\|\ \|, \cdot)$ is bounded. Take $K_0 \in \mathscr{Z}_n$, as there is a constant C such that $\|x\| \leq C\|x\|_1$ for all $x \in \mathbf{R}^n$; if we take $K \in \mathscr{Z}_n$ with $d(K, K_0) \leq 1$ we have $\operatorname{var}(\|\ \|, K) \leq C \operatorname{var}(\|\ \|_1, K)$ and, by Lemma 1,

$$\operatorname{var}(\|\ \|_1, K) \leq 2n \sup\{\|x\|_\infty : x \in K)\} \leq 2n \sup\{\|x\|_2 + 1 : x \in K_0\}.$$

This yields the boundedness in a neighborhood of K_0 and, from the previous discussion, the continuity of $\operatorname{var}(\|\ \|, \cdot)$.

Example. *The variation of the Euclidean ball.*

It is a known fact that the Euclidean ball B_2^n is a zonoid. There are several ways to define a measure whose range is B_2^n; one can use the invariant rotation measure in S^{n-1}. We will use Gaussian variables in order to extend the result to the infinite-dimensional case.

Let (g_n) be a standard Gaussian sequence; that is, a sequence of independent, identically distributed real random variables defined on a probability space $(\Omega, \Sigma, \mathbf{P})$ such that every g_n has the distribution

$$\mathbf{P}(g_n < t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp(-x^2/2)\, dx \quad \text{for all } t \in \mathbf{R}.$$

For every n, define the measure $\sigma_n: \Sigma \to \mathbf{R}^n$ by

$$\sigma_n(A) = \left(\sqrt{2\pi} \int_A g_k\right)_{k=1, \dots, n}, \qquad A \in \Sigma.$$

By Liapunoff's Theorem, $\mathrm{rg}(\sigma_n)$ is a compact convex set in $\mathbf{R}^n$ which is invariant under rotations, thanks to the rotational invariance of the Gaussian distribution in $\mathbf{R}^n$. Thus $\mathrm{rg}(\sigma_n)$ is a multiple of B_2^n. Actually, we can see that $\mathrm{rg}(\sigma_n)$ is B_2^n, since

$$\begin{aligned}\text{radius of } \mathrm{rg}(\sigma_n) &= \sup\{x_1 : \mathbf{x} \in \mathrm{rg}(\sigma_n)\} \\ &= \sup\left\{\sqrt{2\pi}\int_A g_1 : A \in \Sigma\right\} \\ &= \sqrt{2\pi}\int_{\{g_1>0\}} g_1 \\ &= \int_0^{+\infty} x\exp(-x^2/2)\,dx = 1.\end{aligned}$$

It is easy to check that, for every $t \geq 0$

$$\sigma_n(\{g_1 > t\}) = (\exp(-t^2/2), 0, \dots, 0).$$

Using again the rotational invariance, we see that if $\mathbf{a} = (a_1, a_2, \dots, a_n)$ has $\|\mathbf{a}\|_2 = 1$, then

$$\sigma_n\left(\left\{\sum_{k=1}^n a_k g_k > t\right\}\right) = \exp(-t^2/2)\mathbf{a}.$$

This can be extended to the whole sequence (g_n). Define $\sigma: \Sigma \to l_2$ by

$$\sigma(A) = \left(\sqrt{2\pi}\int_A g_n\right)_{n\in N}.$$

The previous considerations show that σ is well defined and $\|\sigma(A)\|_2 \leq 1$ for every $A \in \Sigma$. Take $\mathbf{a} = (a_n)$ in l_2 with $\|\mathbf{a}\|_2 = 1$, then $\sum a_n g_n$ belongs to $L^2(\mathbf{P})$, thanks to the orthonormality of the sequence (g_n), and it can be proved that

$$\sigma\left(\left\{\sum a_n g_n > t\right\}\right) = \exp(-t^2/2)\mathbf{a} \quad \text{for every } t \geq 0.$$

Thus the range of σ is the whole closed unit ball of l_2. The measure σ is countably additive in l_2 (use the fact that $\|\sigma(A)\|_2 \leq \sqrt{2\pi\mathbf{P}(A)}$ for every $A \in \Sigma$).

We can also consider σ as valued in c_0, but even in this case it has unbounded variation, for, calling P_n the projection onto the first n coordinates in c_0, we have:

$$\|\sigma\|_{c_0} \geq \|P_n\sigma\|_{c_0} = \|\sigma_n\|_\infty = \sqrt{2\pi}\int \max(|g_1|, \dots, |g_n|)\,d\mathbf{P}$$

which can be estimated as bigger than or equal to $\beta\sqrt{\log n}$ for an absolute constant β. Thus the ball of l_2 is the range of a countably additive measure which has unbounded c_0-variation.

SUBSPACES OF L^1

We now know that two measures with the same range have the same variation, so we can wonder about the monotonicity of the variation with respect to the range. That is, if the condition $\operatorname{rg}\mu \subseteq \operatorname{rg}\nu$ implies $\|\mu\| \le \|\nu\|$ for two vector measures ν, μ. Anantharaman and Diestel [AD] pointed out that this is not true, exhibiting two c_0-valued measures ν, μ such that $\operatorname{rg}\mu \subseteq \operatorname{rg}\nu$, and ν but not μ is of bounded variation. They also observed that, thanks to a result of Grothendieck, this is not possible for measures with values in (a subspace of) L^1. The next theorem proves that the monotonicity of the variation characterizes the subspaces of L^1. The corollary that follows it shows that the construction made in c_0 can be made in every Banach space not isomorphic to a subspace of an L^1.

Theorem 5. *Let X be a normed space and $C \ge 1$. Then the following statements are equivalent*:

(a) *X is C-isomorphic to a subspace of an L^1 space.*
(b) *For every pair μ, ν of X-valued measures, the condition* $\operatorname{rg}\mu \subseteq \overline{\operatorname{co}}(\operatorname{rg}\nu)$ *implies $\|\mu\| \le C\|\nu\|$.*
(c) *For every pair μ, ν of X-valued countably additive measures, the condition* $\operatorname{rg}\mu \subseteq \operatorname{rg}\nu$ *implies $\|\mu\| \le C\|\nu\|$.*
(d) *For every pair E, F of finite subsets of X, the condition*

$$\sum_{x\in E} |f(x)| \le \sum_{y\in F} |f(y)| \quad \textit{for all } f \in X^*$$

implies $\sum_{x\in E} \|x\| \le C \sum_{y\in F} \|y\|$.

Proof. (d) $\Rightarrow$ (a) is Theorem 7.3 in [LP].

(b) $\Rightarrow$ (c) is obvious.

(a) $\Rightarrow$ (b). Let μ, ν be two X-valued measures such that $\operatorname{rg}\mu \subseteq \overline{\operatorname{co}}(\operatorname{rg}\nu)$. Suppose μ is defined on an algebra $\mathscr{A}$. Take a partition $\{A_j\}_{j=1}^r$ in $\mathscr{A}$. By (a), for every $\varepsilon > 0$ there is an operator $T\colon X \to L^1$ such that

$$\|x\| \le \|Tx\|_1 \le (C+\varepsilon)\|x\|. \tag{3}$$

Using a conditional expectation over an appropriate finite σ-algebra, we can find a norm-one linear projection $P\colon L^1 \to L^1$ which verifies that

(4) $\qquad Y = P(L^1)$ is a finite-dimensional space isometric to l_1^n

$$\|PT\mu(A_j) - T\mu(A_j)\|_1 \le \varepsilon/r, \qquad j = 1, \dots, r. \tag{5}$$

Condition (4) says that there exist $f_1, \dots, f_n \in (L^1)^*$ such that for every x in Y the norm is given by $\|x\|_1 = \sum_{i=1}^n |f_i(x)|$, so by Lemma 1 and the condition $\operatorname{rg} PT\mu \subseteq \overline{\operatorname{co}}(\operatorname{rg} PT\nu)$, we have $\|PT\mu\|_1 \le \|PT\nu\|_1$.

Then using (3), (5), and this last inequality, we get

$$\begin{aligned}\sum_{j=1}^{r}\|\mu(A_j)\| &\le \sum_{j=1}^{r}\|T\mu(A_j)\|_1\\ &\le \sum_{j=1}^{r}\|PT\mu(A_j)\|_1+\varepsilon\\ &\le \|PT\mu\|_1+\varepsilon\\ &\le \|PT\nu\|_1+\varepsilon\\ &\le \|T\nu\|_1+\varepsilon\\ &\le (C+\varepsilon)\|\nu\|+\varepsilon \text{ for all } \varepsilon>0.\end{aligned}$$

The last chain of inequalities yields $\|\mu\|\le C\|\nu\|$, as claimed.

(c) $\Rightarrow$ (d). Suppose that X satisfies (c), and let $E=\{x_1,\dots,x_n\}$, $F=\{y_1,\dots,y_m\}$ be finite subsets of X satisfying

$$\sum_{i=1}^{n}|f(x_i)|\le\sum_{j=1}^{m}|f(y_j)| \quad \text{for all } f\in X^*. \tag{6}$$

Define $\psi,\phi\colon \mathbf{R}\to X$ by

$$\phi=2\sum_{i=1}^{n}x_i\mathscr{X}_{[i-1,\,i-1/2)}-x_i\mathscr{X}_{[i-1/2,\,i)}$$

$$\psi=2\sum_{j=1}^{m}y_j\mathscr{X}_{[j-1,\,j-1/2)}-y_j\mathscr{X}_{[j-1/2,\,j)}.$$

Let μ be the measure with density ϕ with respect to the Lebesgue measure in $\mathbf{R}$, and ν the measure with density ψ. It is easy to see that

$$\operatorname{rg}\mu=\left\{\sum_{i=1}^{n}t_ix_i : t_i\in[-1,1],\ i=1,\dots,n\right\}$$

$$\operatorname{rg}\nu=\left\{\sum_{j=1}^{m}t_jy_j : t_j\in[-1,1],\ j=1,\dots,m\right\}.$$

By (6), for every continuous linear functional $f\in X^*$,

$$\sup\{f(x): x\in\operatorname{rg}\mu\}=\sum_{i=1}^{n}|f(x_i)|\le\sum_{j=1}^{m}|f(y_j)|=\sup\{f(x): x\in\operatorname{rg}\nu\}.$$

As both ranges are compact convex sets in X, this implies that $\operatorname{rg}\mu\subseteq\operatorname{rg}\nu$ and, by (c), $\|\mu\|\le C\|\nu\|$. But

$$\|\mu\|=\int_{\mathbf{R}}\|\phi(t)\|\,dt=2\sum_{i=1}^{n}\|x_i\|$$

$$\|\nu\|=\int_{\mathbf{R}}\|\psi(t)\|\,dt=2\sum_{j=1}^{m}\|y_j\|$$

so (d) follows, and the theorem is proved.

The next corollary can be viewed as the case $C = \infty$ in the previous theorem; the proof uses the same ideas.

Corollary 6. *If X is a Banach space not isomorphic to a subspace of an L^1 space, then there exist two X-valued countably additive measures ν, μ defined on the Lebesgue-measurable sets in $\mathbf{R}$ such that:*

(a) *ν has a Bochner-integrable derivative with respect to the Lebesgue measure.*

(b) *μ has unbounded variation.*

(c) $\operatorname{rg}\mu \subseteq \operatorname{rg}\nu$.

Proof. Deny (d) in Theorem 5 for an appropriate sequence (C_n) tending to infinity to produce two sequences (x_n) and (y_n) in X such that

$$\sum \|y_n\| < \infty \tag{7}$$

$$\sum \|x_n\| = \infty \tag{8}$$

$$\sum |f(x_n)| \le \sum |f(y_n)| \quad \text{for all } f \in X^*. \tag{9}$$

Define two functions ϕ, $\psi\colon \mathbf{R} \to X$ by

$$\phi = 2\sum_{n=1}^{\infty} x_n \mathscr{X}_{[n-1\,,\,n-1/2)} - x_n \mathscr{X}_{[n-1/2\,,\,n)}$$

$$\psi = 2\sum_{n=1}^{\infty} y_n \mathscr{X}_{[n-1\,,\,n-1/2)} - y_n \mathscr{X}_{[n-1/2\,,\,n)}.$$

Then (7) implies that ψ is Bochner-integrable in $\mathbf{R}$. Let ν be the X-valued measure that has ψ as derivative. It is easy to see that the range of ν is $K = \{\sum_{n=1}^{\infty} t_n y_n \colon t_n \in [-1\,,\,1]\,,\, n \in \mathbf{N}\}$, which by (7) is a compact convex subset of X.

The function ϕ is locally integrable, and as in the proof of Theorem 5, we can see that for every bounded measurable set A in $\mathbf{R}$ and for every $f \in X^*$

$$f\left(\int_A \phi\right) \le \sup\{f(x)\colon x \in K\}.$$

This implies that $\int_A \phi \in K$, since K is compact and convex. By (7) and (9), ϕ is weakly integrable. Let A be a measurable set in $\mathbf{R}$, and set $z_n = \int_{[-n\,,\,n]\cap A} \phi$. The sequence (z_n) is weakly Cauchy and lies in the compact set K, so it is norm-convergent to a z_A in K, which satisfies

$$f(z_A) = \int_A f \circ \phi \quad \text{for all } f \in X^*.$$

We have proved that ϕ is Pettis-integrable. The vector measure μ defined by

$$\mu(A) = (P) - \int_A \phi$$

is countably additive [DU, p. 53] and has its range in K. By (8), μ has unbounded variation, so the corollary is proved.

Remark. The previous corollary says that if a Banach space X is not isomorphic to a subspace of an L^1 space, there are two X-valued countably additive measures ν, μ defined on σ-algebras such that $\operatorname{rg}\mu \subseteq \operatorname{rg}\nu$, ν has bounded variation and μ does not. For normed spaces not isomorphic to subspaces of L^1 it is possible to do the same with measures defined on algebras, but not always on σ-algebras: let E be the linear span of the canonical basis of c_0, it is easy to prove, by a "sliding hump" argument or using Baire's categories, that every countably additive measure with values in E has finite dimensional range, and so it has bounded variation.

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