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A UNIQUENESS THEOREM FOR $y' = f(x, y)$, $y(x_0) = y_0$

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ABSTRACT. Consider the initial value problem for a first-order differential equation

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

In this paper a new uniqueness criterion is proved. This criterion is related to the numeric equation

$$u = y_0 + (t - x_0)f(t, u).$$

It is also shown that some well-known uniqueness theorems are consequences of our result.

1. INTRODUCTION

Consider a first-order ordinary differential equation

$$y' = f(x, y) \tag{1}$$

with the initial condition

$$y(x_0) = y_0, \tag{2}$$

where x_0 and y_0 are given real numbers.

There are many uniqueness theorems which apply, assuming a variety of conditions on the function f. The usual assumption is some type of Lipschitz condition or generalization of one which specifically involves the difference $f(x, y_1) - f(x, y_2)$. Examples of theorems are those of Osgood, Perron, Nagumo, Tonelli, Kamke, Cafiero and many others (see Cafiero [1], Hartman [2], and Piccinini et al. [3], which also contain many references). They assumed the existence of a suitable nonnegative function $\omega(x, |y|)$, with $\omega(x, 0) \equiv 0$, such that

$$|f(x, y_1) - f(x, y_2)| \le \omega(x, |y_1 - y_2|)$$

or

$$f(x, y_1) - f(x, y_2) \le \omega(x, |y_1 - y_2|), \quad y_1 > y_2.$$

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It is often difficult to apply the above results because one has first to study the new initial value problem

$$v' = \omega(x, v), \qquad v(0) = 0.$$

While the uniqueness for the problem (1)–(2) has been investigated in many papers, results dealing with nonuniqueness have been given only by a few authors (see, for example, Kalas [4], Rudin [5], and Samimi [6]).

The aim of the present paper is to establish a necessary condition for the nonuniqueness of the initial value problem (1)–(2) and a consequent new uniqueness criterion which does not require any special condition on the difference $f(x, y_2) - f(x, y_1)$. In fact, it will be shown that one can attempt to check the uniqueness by analyzing the following auxiliary (not differential) equation

$$u = y_0 + (t - x_0)f(t, u); \tag{3}$$

here u is the unknown and t a parameter whose role will be specified in the following. In a particular case (Theorem 2) it is also possible to dispense with the continuity of f. It is remarkable that our results contain, as corollaries, some well-known uniqueness theorems (see §3).

2. Main results

We assume f maps $S \to \mathbb{R}$, where

$$S = I \times \mathbb{R}, \qquad I = [x_0, x_0 + a], \tag{4}$$

a is a given positive number, and $\mathbb{R}$ denote the real line.

By a solution to the initial value problem (1)–(2), we mean a real function $y(x)$ defined in some interval $[x_0, x_0 + d] \subseteq I$, satisfying condition (2) and such that the derivative $y'(x)$ exists and equals $f(x, y(x))$ at each point $x \in [x_0, x_0 + d]$.

Theorem 1. *Let the function f be continuous with respect to x, and such that $f(x, y_0) = 0$ for every $x \in I$. Assume that the initial value problem* (1)–(2) *admits two different solutions, defined in $[x_0, x_0 + d]$. Then for every $\epsilon > 0$, there exists $t \in]x_0, x_0 + d]$ such that* (3) *has at least two different roots u with $|u - y_0| < \epsilon$.*

Proof. It is obvious that equations (1) and (2) are satisfied by the constant function y_0; therefore, we assume that there exists a solution $z(x)$ of (1)–(2) different from the constant solution y_0.

Let $\epsilon > 0$ be given. Since $u = y_0$ is a solution of (3) for every $t \in I$, it is sufficient to show the existence of $t \in [x_0, x_0 + d]$, for which (3) is satisfied by some $u \neq y_0$, $|u - y_0| < \epsilon$.

Let a real function A be defined in $[x_0, x_0 + d]$ by putting $A(x_0) = 0$ and $A(x) = (z(x) - y_0)/(x - x_0)$ for $x \neq x_0$. Of course the function A is continuous in $[x_0, x_0 + d]$, differentiable in $]x_0, x_0 + d[$, and we have

$$A'(x) = \frac{1}{(x - x_0)^2}[(x - x_0)f(x, z(x)) - z(x) + y_0] \tag{5}$$

for every $x \in]x_0, x_0 + d[$.

Now, fix a point $x_2 \in]x_0, x_0 + d[$ with $z(x_2) \neq y_0$, such that $|z(x) - y_0| < \epsilon$ for every $x \in [x_0, x_2]$, and denote

$$x_1 = \sup\{x \in [x_0, x_2] : A(x) = 0\}.$$

We note that $z(x_1) = y_0$ and $z(x) \neq y_0$ for every $x \in]x_1, x_2[$.

At this point there are two possibilities.

If there exists $t \in]x_1, x_2]$ such that $A'(t) = 0$, then it is clear from (5) that for such a t, (3) is satisfied by $u = z(t)$. Hence the proof is accomplished just by taking these t and u.

Otherwise, if $A'(x) \neq 0$ for every $x \in]x_1, x_2]$ (hence $A'(x)$ has constant sign in $]x_1, x_2]$), then we put $u = z(x_2)$ $(\neq y_0)$, and define

$$g(x) = (x - x_0)f(x, u) - u + y_0$$

for every $x \in [x_0, x_0 + d]$. Next, we notice that if $A'(x) > 0$ for every $x \in]x_1, x_2]$, then $A(x_2) > A(x_1) = 0$, hence $u > y_0$ and consequently $g(x_0) = -u + y_0 < 0$. On the other hand, we have $g(x_2) = (x_2 - x_0)^2 A'(x_2) > 0$. Therefore, there exists $t \in]x_0, x_2[$ such that $g(t) = 0$, and the proof is accomplished. The case $A'(x) < 0$ for every $x \in]x_1, x_2]$ is similar to the previous one. This concludes the proof. □

An immediate consequence of Theorem 1 is the following uniqueness criterion:

Theorem 2. *Let f be continuous with respect to x and satisfy $f(x, y_0) = 0$ for every $x \in I$. Assume that there exist $\epsilon_* > 0$ and $x_* \in I$ $(x_* > x_0)$ such that $u = y_0$ is the only root of (3) with $|u - y_0| < \epsilon_*$ for every $t \in [x_0, x_*]$. Then the initial value problem expressed in (1)–(2) admits in the interval $[x_0, x_*]$ only the constant solution y_0.*

A crucial point in the previous theorems is the assumption $f(x, y_0) = 0$; that is, the assumption that the set of equations (1)–(2) admits a constant solution. It follows that we can use these theorems in the general case if we know a solution $\phi(x)$ of equations (1)–(2). In fact, by means of the change of variable $y = p + \phi(x) - y_0$, equations (1)–(2) become

$$p' = F(x, p), \tag{6}$$

$$p(x_0) = y_0, \tag{7}$$

where $F(x, p)$ is defined by

$$F(x, p) = f(x, p + \phi(x) - y_0) - f(x, \phi(x)), \tag{8}$$

and it is clear that any two different solutions of (1)–(2) are mapped to different solutions of (6)–(7). Moreover equations (6) and (7) admit the constant solution y_0, which corresponds to the solution $\phi(x)$ of (1)–(2). Thus, replacing (3) with

$$u = y_0 + (t - x_0)F(t, u), \tag{9}$$

with F as given by (8), we get the following theorems:

Theorem 3. *Let f be continuous in S, and let $\phi(x)$ be a solution of equations (1)–(2). Assume that the initial value problem expressed by (1) and (2) admits two different solutions, defined in $[x_0, x_0+d]$. Then, for every $\epsilon > 0$, there exists $t \in]x_0, x_0+d]$ such that (9) has at least two distinct roots u with $|u - y_0| < \epsilon$.*

Theorem 4. *Let f be continuous in S and let $\phi(x)$ be a solution of (1) and (2). Assume that there exist $\epsilon_* > 0$ and $x_* \in I$ $(x_* > x_0)$ such that $u = y_0$ is the only root of (9) with $|u - y_0| < \epsilon_*$ for every $t \in [x_0, x_*]$. Then the initial value problem expressed in (1)–(2) admits in the interval $[x_0, x_*]$ only the solution $\phi(x)$.*

3. Remarks and examples

We have already mentioned, in the Introduction, that Theorem 4 above is general enough to cover the classical case when f satisfies a Lipschitz condition with respect to y as well as the case when f is decreasing in y.

Indeed, if the function f is Lipschitzian with respect to y (with constant L), then for any two solutions u_1 and u_2 of (9) corresponding to the same t, we have:

$$\begin{aligned} |u_1 - u_2| &= (t - x_0)|F(t, u_1) - F(t, u_2)| \\ &= (t - x_0)|f(t, u_1 + \phi(t) - y_0) - f(t, u_2 + \phi(t) - y_0)| \\ &\le L(t - x_0)|u_1 - u_2|; \end{aligned}$$

hence $u_1 = u_2$, provided that $L(t - x_0) < 1$. It follows by Theorem 4 that equations (1) and (2) have a unique solution for $(x - x_0) < 1/L$. By iterating the application of Theorem 4 (if needed), we get global uniqueness. One can see, essentially by the same argument, that Theorem 4 also covers some cases when relaxed Lipschitz conditions of f are assumed.

If $f(x, y)$ is continuous in S and decreasing in y, then it is well known that equations (1) and (2) have a unique solution [1], [2], [3]. Since

$$h(t, u) = y_0 + (t - x_0)F(t, u) - u$$

is decreasing in u, it is clear that this result is a consequence of our Theorem 4.

We remark that uniqueness conditions usually require that the function f be continuous. It is worth noticing, in this connection, that Theorem 2 can also be applied to discontinue functions. This is shown by the following easy example:

$$f(x, y) = \frac{1}{y_0 - y} \quad \text{for } y \neq y_0 \text{ and } f(x, y_0) = 0.$$

We conclude this section by proving, with an example, that the converse of Theorem 1 does not hold; that is, if the Cauchy problem expressed by (1) and (2) admits only one solution (3) can admit more solutions. The same example shows also a further application of our results. Let $f = f_\alpha$ $(0 < \alpha < 2)$ given by

$$f_\alpha(x, y) = \begin{cases} |y|^\alpha \sin(1/y) & y \neq 0 \\ 0 & y = 0 \end{cases} \tag{10}$$

The initial value problem

$$\begin{cases} y' = f_\alpha(x, y) \\ y(0) = 0 \end{cases} \tag{11}$$

is satisfied only by the zero solution. This can be proved by a simple application of classical results on scalar differential inequalities [7, pp. 7–10], since

$$y = \frac{1}{k\pi}, \qquad k = \pm 1, \pm 2, \pm 3, \ldots$$

are underfunctions with respect to (11) if $k > 0$ and overfunctions if $k < 0$. In this example, the numeric equation (3) is

$$u = t|u|^\alpha \sin(1/u) \quad \text{for } u \neq 0 \tag{12}$$

and $u = 0$ otherwise.

If $\alpha = 1/2$, one can verify that

$$u = \left(\frac{\pi}{2} + 2n\pi\right)^{-1}, \qquad t = \left(\frac{\pi}{2} + 2n\pi\right)^{-1/2}, \qquad n \in N \tag{13}$$

are solutions of (12). This shows that uniqueness of initial value problem does not imply uniqueness for (3).

If $\alpha = 1$, (12) becomes

$$\operatorname{sgn}(u) = t \sin(u),$$

which cannot be satisfied if $t < 1$. This shows an application of Theorem 2 when f is neither Lipschitzian nor decreasing in y.

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