TOTALLY REAL SETS IN $C^2$

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Abstract. We establish the polynomial convexity of certain totally real disks and of annuli in the unit torus satisfying a topological condition.

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Jöricke [1] recently proved that a totally real disk contained in the unit sphere in $C^2$ is polynomially convex. More precisely, the result of [1] involves analytic extension but, by work of Stout [7] and Lupacciolu [2], the polynomial convexity follows; see also Rosay and Stout [4]. In this note we shall prove the analogous and easier result when the sphere is replaced by the set $M = \{(z, w): |z| = 1\}$.

Theorem 1. Let $K$ be a smooth totally real compact disk contained in the real hypersurface $M$. Then $K$ is polynomially convex.

One could possibly prove this by closely imitating the argument of Jöricke; however, the approach we follow, although it has some of the elements of the proof of [1], is probably shorter. Just as in [1] this is not a local result as the unit torus sits in $M$ as a totally real 2-manifold which is not polynomially convex.

Example. If we allow $K$ to fail to be totally real at a single point then it may not be polynomially convex. A simple example for such $K$ is the image of the unit disk by the map $z \rightarrow (\exp(i \cdot |z|^2), z)$. Then $K$, which is essentially the graph of the exponential, has a complex tangent only at the point $(1, 0)$ and is clearly not polynomially convex since it has circles as fibers over a subarc of the unit circle of the $z$-plane. An analogous example in the context of [1] is obtained from the map $z \rightarrow (z, \sqrt{1 - |z|^2})$. It should be noted that Wermer (see [3, p. 34]) has given an example of a totally real disk in $C^2$ which is not polynomially convex.

For a set $S$ in $C^2$ and $z \in C$ we denote the fiber $\{w \in C: (z, w) \in S\}$ by $S_z$. To prove the theorem we first claim that $K_z$ is polynomially convex.
in $C$ for all $z$. Suppose not. Then there is an $a$ in the unit circle such that $K_a$ is not polynomially convex and hence, by Runge's theorem, $C\setminus K_a$ is not connected. By Alexander duality [5, pp. 296, 334] $\hat{H}^1(K_a, \mathbb{Z})$ is nontrivial. Now $K$ is topologically a disk which we can assume sits in a copy $E$ of $\mathbb{R}^2$. Identifying $K_a$ with $\{a\} \otimes K_a \subseteq K$ and applying Alexander duality again for $K_a \subseteq E$, we conclude that $E\setminus K_a$ is not connected.

Since $K$ is totally real and $M$ has real dimension 3 there is a well-defined real tangent line bundle on $K$ given by the intersection of the complex tangent space of $M$ with the tangent space of $K$. Since $K$ is contractible, there is a unit tangent vector field $v$ which is a section of this bundle. That is, $v$ is a unit vector field on $K$ which lies pointwise in the complex tangent space to $M$ at each point of $K$; cf. [1]. Consider the integral curves of $v$ in $K$. Since $v$ is at each point a derivative in a $w$ direction, the vector field $v$ applied to the function $z$ vanishes identically. Therefore, $z$ is constant on each integral curve. If $p$ is a point of $K$ not in $K_a$, then, by the Poincaré-Bendixson theory, the integral curve through $p$ joins $p$ to the boundary of $K$. Since $z$ at $p$ is different from $a$, this integral curve is disjoint from $K_a$. This implies that $E\setminus K_a$ is connected. This is a contradiction.

Thus each $K_a$ is polynomially convex. Since $\hat{H}^1(K, C) = 0$, it follows directly from a result of Stolzenberg [6, Corollary 2.20] that $K$ is polynomially convex.

The aforementioned result of Stolzenberg requires that the set $K$ satisfy $\hat{H}^1(K, C) = 0$. However the idea of his proof holds in more general cases, for example, in the following setting. Let $T^2$ be the unit torus in $\mathbb{C}^2$. We identify the fundamental group of $T^2$ with $\mathbb{Z}^2$ as follows: $[r, s] \in \mathbb{Z}^2$ is identified with (the homotopy class of) the curve $\{(\exp(irt), \exp(ist)): 0 \leq t \leq 2\pi\}$. Let $A$ be a compact annulus contained in $T^2$ and $g$ a simple closed curve contained in $A$ that generates the fundamental group of $A$. Let $[p, q]$ be homotopic in $T^2$ to $g$; $\pm[p, q]$ is independent of the choice of $g$.

**Theorem 2.** If $pq < 0$ or $p = 0 = q$, then $A$ is polynomially convex.

**Remark.** If $pq > 0$ or exactly one of $p$ and $q$ is zero, then $A$ need not be polynomially convex. Indeed the following is easily verified.

**Lemma.** If $g$ is a simple closed curve in $T^2$ which is not null-homotopic, then $g$ is homotopic in $T^2$ to a curve $[p, q]$ with $p$ and $q$ relatively prime. In particular, if $q = 0$ then $p = 1$ or $p = -1$.

By the lemma, we can assume that $p$ and $q$ are relatively prime. This implies that $\{(z, w) \in T^2: z^q = w^p\}$ is a (connected!) simple closed curve in $T^2$ which is not polynomially convex. Then a tubular neighborhood of this curve in $T^2$ provides an example of a nonpolynomially convex annulus $A$ with $g = [p, q]$. 

Proof of Theorem 2. If \( pq < 0 \), then by symmetry we can assume that \( p < 0 \) and \( q > 0 \). Let \( a = -p \) and set \( f(z, w) = z^q w^a \). Then \( f \) is identically 1 on the curve \([p, q]\). Since \( f \) has modulus 1 on \( T^2 \) and since \( g \) is homotopic to the curve \([p, q]\) in \( T^2 \), it follows that \( f \) restricted to \( A \) lifts to a map \( F \) of \( A \) into \( C \) such that \( \exp \circ F = f \); i.e. \( F \) is a logarithm of \( f \) on \( A \). Then \( F \) extends to be a logarithm of \( f \) on a neighborhood \( N \) of \( A \) in \( C^2 \).

We claim that \( (A)_z = A_z \) for all \( z \) in the unit circle. Since \( (T^2)_z \) is a peak set of \( T^2 \), it is enough to show that \( A_z \) is a proper subset of the unit circle. Suppose not. Then \( A \) contains a circle \( k \) of the form \( (T^2)_z \) for some \( z \). But then, as the fundamental group of \( A \) is singlely generated, it follows that \( k \) is homotopic in \( A \) to some multiple of \( g \). Hence \( k \) is also homotopic in \( T^2 \) to a multiple of \([p, q]\); this is clearly false—a contradiction.

To prove that \( A \) is polynomially convex we argue by contradiction and suppose otherwise. For \( r < 1 \) we set \( Q = A \cap \{(z, w): r \leq |z| \leq 1 \} \). If \( r \) is sufficiently close to 1, by the last paragraph, \( Q \) is contained in \( N \). By the local maximum modulus principle (cf. [6]) the Shilov boundary of the algebra of functions on \( Q \) which are locally in \( P(Q) \) is contained in the union of \( Q \cap T^2 = A \) and \( Q \cap \{(z, w): |z| = r \} \). On the first set \( f \) has modulus 1 and on the second set \( f \) has modulus \( \leq r^q \). Since \( F = \log(f) \) is locally in \( P(Q) \), the boundary of \( F(Q) \) is contained in the union of two sets: the vertical line \( \{\text{Re}(z) = 0\} \) and the set \( \{\text{Re}(z) \leq q \cdot \log(r)\} \). This implies that \( F(Q) \) does not meet \( \{z: q \cdot \log r < \text{Re}(z) < 0\} \). Hence \( A \) is relatively open in \( A \). This implies that \( A \) is polynomially convex.

If \( p = 0 = q \) then \( g \) is null-homotopic in \( T^2 \); hence \( g \) bounds a disk in \( T^2 \). Thus \( A \) is contained in a compact disk \( K \) in \( T^2 \) and the polynomial convexity of \( A \) follows, say by Theorem 1.

References


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