THE DE BRANGES–ROVNYAK MODEL

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Abstract. A characterization, extending results of A. Beurling, L. de Branges and J. Rovnyak, of those Hilbert spaces of formal power series, which are isometrically equal to a de Branges-Rovnyak (scalar-valued) function space \( \mathcal{H}(b) \), is obtained.

Let \( \mathcal{E}(z) \) denote the Hilbert space of square-summable power series \( f(z) = \sum a_n z^n \) with complex coefficients such that \( \|f(z)\|^2 = \sum |a_n|^2 \). For a power series \( b(z) \) which converges to a function which is bounded by one in the unit disk, the space \( \mathcal{H}(b) \) of L. de Branges and J. Rovnyak [4] is the Hilbert space of series \( f(z) \) in \( \mathcal{E}(z) \) with the property that

\[
\|f(z)\|^2 = \sup \{\|f(z) + b(z)g(z)\|^2 - \|g(z)\|^2\}
\]

is finite, where the supremum is taken over all elements \( g(z) \) of \( \mathcal{E}(z) \). If \( f(z) \) is in \( \mathcal{H}(b) \), then so is the series \( \frac{f(z) - f(0)}{z} \) and, moreover,

\[
\left\| \frac{f(z) - f(0)}{z} \right\|_b^2 \leq \|f(z)\|^2 - |f(0)|^2.
\]

Let \( \mathcal{H}(z) \) be a Hilbert space of formal power series such that the difference-quotient inequality

\[
\left\| \frac{f(z) - f(0)}{z} \right\|^2_{\mathcal{H}} \leq \|f(z)\|^2_{\mathcal{H}} - |f(0)|^2
\]

holds for every element \( f(z) \), and let \( T \) denote the contraction on \( \mathcal{H} \) which sends \( f(z) \) into \( \frac{f(z) - f(0)}{z} \). By the well-known invariant subspace theorem of A. Beurling [2], if \( \mathcal{H} \) is contained isometrically in \( \mathcal{E}(z) \), then \( \mathcal{H} \) is a space \( \mathcal{H}(b) \) for some inner function \( b \). (Conversely, if \( b \) is inner, then \( \mathcal{H}(b) \) is contained isometrically in \( \mathcal{E}(z) \)). More generally, de Branges and Rovnyak showed in [4, Theorem 15] that if equality holds in (2), for every \( f \), and the rank of \( 1 - TT^* \) is one, then \( \mathcal{H} = \mathcal{H}(b) \) isometrically, where \( b \) is an extreme

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point of the unit ball in the space of bounded analytic functions of the unit disk. (Conversely, if \( b \) is an extreme point, then equality holds in (1) for every \( f \), and the ranges of \( 1 - R(0)R(0)^* \) and \( 1 - R(0)^*R(0) \) are spanned by \( [b(z) - b(0)]/z \) and \( 1 - b(z)
olimits\overline{b(0)} \) respectively). In this note, we obtain a similar result for spaces \( \mathcal{H}(b) \) where \( b \) is not an extreme point, which answers a question raised in [4, p. 39].

Suppose that \( b \) is nonconstant and not an extreme point. Then the range of \( 1 - R(0)R(0)^* \) is spanned by \( [b(z) - b(0)]/z \), and the range of \( 1 - R(0)^*R(0) \) is spanned by 1 and \( b \). Moreover, it follows directly from [8, Lemmas 1-3] that if \( f(z) = [1 - b(z)b(0)]\|b(z)\|_b^2 - b(z)
olimits\overline{b(0)} \), then \( f(0) \neq 0 \) and equality holds in (1) for \( f \). These properties are sufficient to characterize \( \mathcal{H}(b) \).

**Theorem.** Let \( \mathcal{H} \) be a Hilbert space of formal power series which satisfies (2), and suppose that equality holds in (2) for some element with nonzero constant coefficient. If the ranks of \( 1 - TT^* \) and \( 1 - T^*T \) are 1 and 2 respectively, where \( T \) is the difference-quotient transformation on \( \mathcal{H} \), then \( \mathcal{H} \) is isometrically equal to a space \( \mathcal{H}(b) \) where \( b \) is not an extreme point.

The following example shows that the above hypothesis on equality in (2) may not be omitted.

**Example.** Let \( \phi \) be a nonconstant bounded analytic function, and, for \( r \geq 1 \), let \( \mathcal{H} = \mathcal{C}(z) \) with the inner product given by

\[
\langle f, g \rangle_{\mathcal{H}} = \langle (r + T\phi T\phi^*)f, g \rangle,
\]

where \( T\phi \) denotes multiplication by \( \phi \) on \( \mathcal{C}(z) \). It is easily checked that \( \mathcal{H} \) is a Hilbert space and, for every \( f \), we have that

\[
\|([f(z) - f(0)]/z)\|^2_{\mathcal{H}} = \|f\|^2_{\mathcal{H}} - |f(0)|^2 - (r - 1)|f(0)|^2 - |(T\phi^*f)(0)|^2
\]

\[
\leq \|f\|^2_{\mathcal{H}} - |f(0)|^2.
\]

Furthermore, the range of \( 1 - TT^* \) is spanned by \( [\phi(z) - \phi(0)]/z \), and the range of \( 1 - T^*T \) is spanned by 1 and \( \phi \). However, equality holds in (2) for \( f \) if and only if \( (r - 1)(f, 1) = 0 = \langle f, \phi \rangle \). Hence, by the theorem, \( \mathcal{H} = \mathcal{H}(b) \) isometrically if and only if \( r = 1 \).

The key to the verification of the theorem is the theory of complementation [3] of de Branges: A Hilbert space \( \mathcal{P} \) is contained contractively in a Hilbert space \( \mathcal{H} \) if \( \mathcal{P} \) is a vector subspace of \( \mathcal{H} \) and if the inclusion map \( i_{\mathcal{P}} \) of \( \mathcal{P} \) into \( \mathcal{H} \) is a contraction. If \( \mathcal{P} \) is contained contractively in \( \mathcal{H} \), then the space complementary to \( \mathcal{P} \) in \( \mathcal{H} \) is the Hilbert space \( \mathcal{C} \) of elements \( g \) of \( \mathcal{H} \) with the property that

\[
\|g\|^2_{\mathcal{C}} = \sup\{\|g + f\|^2_{\mathcal{H}} - \|f\|^2_{\mathcal{P}}\}
\]

is finite, where the supremum is taken over all \( f \) in \( \mathcal{P} \). The space \( \mathcal{C} \) is contained contractively in \( \mathcal{H} \) and is the unique Hilbert space such that the
inequality $\|h\|^2_\mathcal{H} \leq \|f\|^2_\mathcal{H} + \|g\|^2_\mathcal{H}$ holds whenever $h = f + g$ is a decomposition of $h$ in $\mathcal{H}$ into $f$ in $\mathcal{P}$ and $g$ in $\mathcal{C}$, and such that every $h$ in $\mathcal{H}$ admits the unique decomposition $h = (i_\mathcal{P}h) + (i_\mathcal{C}h)$, where $\|h\|^2_\mathcal{H} = \|i_\mathcal{P}h\|^2_\mathcal{H} + \|i_\mathcal{C}h\|^2_\mathcal{H}$.

By (2), $\mathcal{H}$ is contained contractively in $\mathcal{C}(z)$. Moreover, if $\mathcal{M}$ is the complementary space to $\mathcal{H}$ in $\mathcal{C}(z)$, then multiplication by $z$ is defined and bounded by one on $\mathcal{M}$. The essential result for us will be the following corollary of de Branges's extension [3, Theorem 15] of Beurling's theorem: if multiplication by $z$ is an isometry on $\mathcal{M}$, then $\mathcal{H}$ is isometrically equal to a space $\mathcal{H}(b)$. To begin the proof of the theorem, let us first observe that $T$ is onto: If the kernel of $T^*_T$ is nontrivial, then it must coincide with the range of $1 - TT^*$ since the rank of $1 - TT^*$ is 1. However, in this case, $T^*$ and hence $T$ are partial isometries so that ker $T = \text{range}(1 - T^*_T)$ is two-dimensional, which is impossible. Thus the range of $T$ is dense in $\mathcal{H}$. But $T$ has closed range since $T^*T$ has closed range ($T^*T$ is the orthogonal direct sum of the identity operator on the kernel of $1 - T^*T$ and an operator on the finite-dimensional range of $1 - T^*T$). Therefore, $\mathcal{H} = TH$.

Note that constants belong to $\mathcal{M}$ since $T(1 - T^*_T) = (1 - TT^*)T$ and rank$(1 - TT^*) < \text{rank}(1 - T^*T)$. Hence, since $T$ is onto, $\mathcal{H}$ has the property that $zf(z)$ is in $\mathcal{H}$ whenever $f(z)$ is in $\mathcal{H}$.

Consider the set $\mathcal{D}$ of those elements of $\mathcal{H}$ for which equality holds in (2). By (2) and the parallelogram law, $\mathcal{D}$ is a vector subspace of $\mathcal{H}$; since $T$ is continuous and $\mathcal{H}$ is contained contractively in $\mathcal{C}(z)$, $\mathcal{D}$ is closed. Also by (2), $\mathcal{D}$ contains the kernel of $1 - T^*T$; by assumption, $\mathcal{D}$ contains an element with nonvanishing constant coefficient which we may assume belongs to the range of $1 - T^*T$.

Let us suppose first that 1 does not belong to $\mathcal{D}$. Since $\mathcal{H}$ is thus the orthogonal direct sum of the kernel of $1 - T^*T$ and the span of 1 and some element of $\mathcal{D}$, we have that $\mathcal{H} = TH = T\mathcal{D}$. Therefore, for $g$ in the space $\mathcal{M}$, complementary to $\mathcal{H}$ in $\mathcal{C}(z)$,

$$\|g\|^2_\mathcal{H} = \sup \{ \|g + Tf\|^2_\mathcal{H} - \|Tf\|^2_\mathcal{H} : f \in \mathcal{D} \}$$

$$= \sup \{ (\|zg + f\|^2_\mathcal{H} - |f(0)|^2) - (\|f\|^2_\mathcal{H} - |f(0)|^2) : f \in \mathcal{D} \}$$

$$= \sup \{ \|zg + f\|^2_\mathcal{H} - \|f\|^2_\mathcal{H} : f \in \mathcal{D} \}$$

$$\leq \|zg\|^2_\mathcal{H}.$$ 

As above, since the reverse inequality holds, multiplication by $z$ is an isometry on $\mathcal{M}$, and hence the theorem follows in this case.

Next, suppose that 1 belongs to $\mathcal{D}$. There exists an integer $n \geq 1$ such that $\mathcal{D}$ contains $z^m (m = 0, \ldots, n - 1)$ but not $z^n$. (Otherwise $i_{\mathcal{P}}z^m = z^m$ for every $m \geq 0$, and consequently $\mathcal{H} = \mathcal{C}(z)$ isometrically. But then $T^*$ would be multiplication by $z$ on $\mathcal{C}(z)$ so that $1 - TT^* = 0$, contradicting its rank hypothesis.) It follows that $T^*n 1 \neq z^n$ and $(1 - TT^*)z^{n-1} \neq 0,$
but if \( n > 1 \) then \( T^m 1 = z^m \) and \((1 - T^*)z^{m-1} = 0\), for every \( m = 1, \ldots, n-1 \). \(((1 - T^*)z^n, 1)_\mathcal{H} = (z^n, 1)_\mathcal{H} = (z^n, z^n)_\mathcal{H} = (i\mathcal{H}z^n, 1)_\mathcal{H} = (z^n, 1) = 0\). Therefore \((1 - T^*)z^{n-1} = T(1 - T^*)z^n \neq 0\)

The \( \mathcal{M} \)-norm may be simplified in this case. Specifically, we will show that if \( f \) is in \( \mathcal{H} \) then there exists an element \( \hat{f} \) in the kernel of \( 1 - T^n T^*n \) such that, for every \( g \) in \( \mathcal{M} \),

\[
\|g + f\|^2 - \|f\|^2_{\mathcal{M}} = \|g + \hat{f}\|^2 - \|\hat{f}\|^2_{\mathcal{M}}.
\]

Let \( f \) be in \( \mathcal{H} \), and define constants \( c_0, \ldots, c_{n-1} \) recursively as follows:

\[
c_{n-1} = -\langle f, (1 - T^*)z^{n-1} \rangle_{\mathcal{H}} / \langle z^{n-1}, (1 - T^*)z^{n-1} \rangle_{\mathcal{H}};
\]

if \( n > 1 \), use

\[
c_{n-k} = -\left( f + \sum_{m=0}^{n-k-1} c_m z^m, T^{k-1}(1 - TT^*)T^{k-1}z^{n-k} \right)_{\mathcal{H}}
\]

\[
\div \left( z^{n-k}, T^{k-1}(1 - TT^*)T^{k-1}z^{n-k} \right)_{\mathcal{H}},
\]

for \( k = 2, \ldots, n \). By the previous paragraph, \( T^{k-1}z^{n-k} = z^{n-k} \), and furthermore \( \hat{f} = f + \sum_{m=0}^{n-k} c_m z^m \) is orthogonal to the range of \( T^{k-1}(1 - TT^*)T^{k-1} \), which is spanned by \( T^{k-1}(1 - TT^*)T^{k-1}z^{n-k} \), for every \( k = 1, \ldots, n \). Since \( 1 - T^n T^*n = \sum_{k=0}^{n} T^{k-1}(1 - TT^*)T^{k-1} \), we have that \( \hat{f} \) is orthogonal to the range of \( 1 - T^n T^*n \).

Let \( g \) be in \( \mathcal{M} \). Then \( \langle g, z^m \rangle = \langle i\mathcal{H} g, z^m \rangle = \langle g, i^* z^m \rangle = \langle g, 0 \rangle = 0 \) for every \( m = 0, \ldots, n - 1 \), so the first \( n \) coefficients of \( g \) vanish. Let us write \( f = (\sum_{m=0}^{n-1} a_m z^m) + z^n \hat{f} \) for some constants \( a_m \) and observe that

\[
\|f\|^2_{\mathcal{M}} = \sum_{m=0}^{n-1} |a_m|^2 + \|z^n \hat{f}\|^2_{\mathcal{M}}
\]

because \( i^* z^m = z^m \), \( m = 0, \ldots, n - 1 \). Therefore,

\[
\|g + f\|^2 - \|f\|^2_{\mathcal{M}} = \|g + z^n \hat{f}\|^2 - \|z^n \hat{f}\|^2_{\mathcal{M}},
\]

and the same expression holds for \( \|g + \hat{f}\|^2 - \|\hat{f}\|^2_{\mathcal{M}} \). Thus

\[
\|g\|^2_{\mathcal{M}} = \sup\{\|g + \hat{f}\|^2 - \|\hat{f}\|^2_{\mathcal{M}}\},
\]

where the supremum is taken over all elements \( \hat{f} \) of the kernel of \( 1 - T^n T^*n \). Moreover, for these elements, \( \|\hat{f}\|_{\mathcal{M}} = \|z^n \hat{f}\|_{\mathcal{M}} \) since \((1 - T^n T^*n)z^n \hat{f} \) is simultaneously in the kernel of \( T^n \) and orthogonal to the kernel of \( T^n \). Indeed, for \( m = 0, \ldots, n - 1 \), \( (1 - T^n T^*n)z^n \hat{f}, z^m \rangle_{\mathcal{H}} = (z^n \hat{f}, z^m) = (z^n \hat{f}, i^* z^m) = (z^n \hat{f}, z^m) = 0 \). Consequently,

\[
\|g\|^2_{\mathcal{M}} = \sup\{\|g + \hat{f}\|^2 - \|z^n \hat{f}\|^2_{\mathcal{M}} : (1 - T^n T^*n) \hat{f} = 0\}.
\]
Finally, since multiplication by $z$ is bounded by one on $\mathcal{M}$, and $\mathcal{H} = T^n \mathcal{H}$, it follows that
\[
\|z^n g\|_{\mathcal{H}}^2 \leq \|z g\|_{\mathcal{H}}^2 \\
\leq \|g\|_{\mathcal{H}}^2 \\
= \sup\{\|g + T^n f\|_{\mathcal{H}}^2 - \|z^n T^n f\|_{\mathcal{H}}^2 : (1 - T^n T^*) T^n f = 0\}
\]
\[
= \sup\{\|z^n g + f\|_{\mathcal{H}}^2 - \|f\|_{\mathcal{H}}^2 : (1 - T^n T^*) T^n f = 0\}
\leq \|z^n g\|_{\mathcal{H}}^2.
\]
Therefore, multiplication by $z$ is isometric on $\mathcal{M}$.

Remark. The referee has kindly pointed out that other equivalent conditions for $\mathcal{H} = \mathcal{H}(b)$ isometrically were recently obtained by R. B. Leech [5].

References

3. L. de Branges, Square summable power series (to appear).