AN INEQUALITY WITH APPLICATIONS TO THE SUBElliPTICITY
OF THE $\bar{\partial}$-NEUMANN PROBLEM

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Abstract. We prove an interesting inequality in this note. This inequality will be used to remove an unnecessary assumption in [2]. That paper dealt with the sufficient condition for the subellipticity of the $\bar{\partial}$-Neumann problem on nonpseudoconvex domains. We will then state the revised theorem and show why the original assumption can be removed.

Proposition 1. Let $\Omega \subset \mathbb{C}^n$, $x_0 \in b\Omega$, and $L \in T^{1,0}(b\Omega)$. Then given any $\varepsilon > 0$, there exists a neighbourhood $U$ of $x_0$ and $C > 0$ such that for all $u$, $v \in C_0^\infty(U \cap \bar{\Omega})$ we have

$$|(Lu, v)| \leq \varepsilon(\|Lu\|^2 + \|Lv\|^2) + C(\|u\|^2 + \|v\|^2).$$

Proof. We know that the adjoint $L^*$ of $L$ is given by

$$L^* = -\bar{L} + g,$$

where $g$ is smooth. Hence

$$L + L^* = L - \bar{L} + g.$$

By a change of coordinates we may assume that

$$L - \bar{L} = -ia(x)\frac{\partial}{\partial x_1},$$

where $a(x) > 0$ in $U$. Hence the symbol of $L - \bar{L}$ is given by

$$\sigma(L - \bar{L}) = a(x)\xi_1.$$

We define operators $P^+$, $P^-$, and $P^0$ as follows:

$P^+u(\xi, r) = \chi_1(\xi_1)\hat{u}(\xi, r)$

$P^-u(\xi, r) = \chi_2(\xi_1)\hat{u}(\xi, r)$

$P^0 = I - P^+ - P^-,$
where \( \chi_1, \chi_2 \in C^\infty(\mathbb{R}) \), \( 0 \leq \chi_i \leq 1, \ i = 1, 2 \), and
\[
\chi_1(\xi) = \begin{cases} 
1 & \xi \geq 1, \\
0 & \xi < 0, \\
0 & \xi > 0, \\
1 & \xi \leq -1.
\end{cases}
\]

We now define
\[
\{u, v\}^+ = \langle (L + L^*)P^+u, P^+v \rangle + K\langle P^+u, P^+v \rangle,
\]
where \( K \) is a large positive constant.

**Lemma 2.** \( \{u, v\}^+ \) is an inner product for large \( K \).

**Proof.** Since \( a(x)\xi_1 \) is nonnegative on the support of \( \chi_1 \), by Garding’s inequality we have
\[
\langle (L - L)P^+u, P^+u \rangle \geq -C\langle P^+u, P^+u \rangle.
\]

Hence for some large \( K \) we have \( \{u, u\}^+ \geq 0 \).

It is easy to see that \( \{u, u\}^+ = 0 \) if and only if \( u = 0 \), and that \( \{u, v\}^+ = \{v, u\}^+ \).

Since \( \{u, v\}^+ \) is an inner product, by Schwarz’s inequality
\[
|\{u, v\}^+| \leq (\{u, u\}^+)^{1/2}(\{v, v\}^+)^{1/2}.
\]

It is easily seen that
\[
\{u, u\}^+ = \langle (L + L^*)P^+u, P^+u \rangle + K\langle P^+u, P^+u \rangle
\]
\[
\leq \varepsilon \|Lu\|^2 + C\|u\|^2
\]
by (1). Similarly,
\[
\{v, v\}^+ \leq \varepsilon \|Lv\|^2 + C\|v\|^2.
\]

Hence
\[
\langle LP^+u, P^+v \rangle = \{u, v\}^+ - \langle L^*P^+u, P^+v \rangle - K\langle P^+u, P^+v \rangle
\]
\[
= \{u, v\}^+ + \langle P^+u, LP^+v \rangle - \langle gP^+u, P^+v \rangle - K\langle P^+u, P^+v \rangle,
\]
and we get from (2), (3), and (4) that
\[
|\langle LP^+u, P^+v \rangle| \leq \varepsilon(\|Lu\|^2 + \|Lv\|^2) + C(\|u\|^2 + \|v\|^2).
\]

Similarly, we define
\[
\{u, v\}^- = -\langle (L + L^*)P^-u, P^-v \rangle + K\langle P^-u, P^-v \rangle.
\]

Just as above, \( \{u, v\}^- \) is an inner product, and again
\[
|\langle LP^-u, P^-v \rangle| \leq \varepsilon(\|Lu\|^2 + \|Lv\|^2) + C(\|u\|^2 + \|v\|^2).
\]
Finally,

\( \langle Lu, v \rangle = \langle LP^+ u, P^+ v \rangle + \langle LP^- u, P^- v \rangle + \langle LP^+ u, P^0 v \rangle + \langle LP^+ u, P^- v \rangle + \langle LP^- u, P^+ v \rangle + \langle LP^0 u, P^+ v \rangle + \langle LP^0 u, P^- v \rangle + \langle LP^0 u, P^- v \rangle. \)

To deal with the third to ninth terms on the right-hand side of (7), we need the facts that

\[ ||[L, P]u||^2 < \text{const} \ ||u||^2, \]

where \( P = P^+, P^-, \) or \( P^0 \) and that

\[ ||(L - \Gamma)Pm||^2 < \text{const} \ ||w||^2 \]

if the symbol of \( P \) is a compactly supported function of \( \xi_1 \).

Thus, combining (5), (6), (7), and the above two facts, we get

\[ \langle Lu, v \rangle \leq \varepsilon \langle ||Lu||^2 + ||Lv||^2 \rangle + C \langle ||u||^2 + ||v||^2 \rangle. \]

We can now restate Theorem 2.2 in [2]. We refer the reader to [2] for the details and for definitions of \( A^{(k)} \) and \( I^u_{m} \). We assume that the reader is familiar with the \( \overline{\partial} \)-Neumann problem and subelliptic estimates. A detailed formulation of the problem can be found in [1] or [3].

**Theorem 3.** Let \( \Omega \) be a domain in \( \mathbb{C}^n \) with \( C^\infty \) boundary, \( x_0 \in b\Omega \), and \( L_1, \ldots, L_n \) a \( C^\infty \) basis for \( T^{1,0} \) so that \( L_1, \ldots, L_{n-1} \) are tangential on \( b\Omega \). Assume that there exists a neighborhood \( U \) of \( x_0 \) such that for some \( k \geq n - q \) the matrix \( A^{(k)} \) associated with the matrix of the Levi form is positive semidefinite in \( U \), then if \( 1 \in I^u_{m}(x_0) \) for some \( m \), there is a subelliptic estimate for \((p, q)\) forms at \( x_0 \).

The following extra assumption is made in Theorem 2.2 in [2]:

For all \( \varepsilon > 0 \), there exists \( C > 0 \) (\( C \) depends on \( S^0 \) but not on \( \phi \)) such that

\[ |\langle D\phi, S^0 \phi \rangle| \leq \varepsilon \langle ||\overline{\partial} \phi||^2 + ||\overline{\partial}^* \phi||^2 \rangle + C \langle ||\phi||^2 \rangle \]

for all \( \phi \in D^{p,q}_{U} \) where \( D \in \{L_{k+1}, \ldots, L_{n-1}\}, S^0 \) is a tangential pseudodifferential operator of order zero.

In [2] this assumption is used to verify the inequality

\[ \sum_{j \notin J or j \in (1, \ldots, k, n)} ||\overline{L}_j \phi ||^2 + \sum_{j \in J and k < j < n} ||L_j \phi ||^2 \leq C \langle ||\overline{\partial} \phi||^2 + ||\overline{\partial}^* \phi||^2 + ||\phi||^2 \rangle \]

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for all $\phi \in D_{U}^{p,q}$ from the inequality

$$
\sum_{j \notin J \text{ or } k \leq j < n} \|L_j \phi_j\|^2 + \sum_{j \in J \text{ and } k < j < n} \|L_j \phi_j\|^2 + \sum_{l,j} \int_{\partial \Omega} A^{(k)}_{l,j} \phi_l \phi_j dS + R(\phi)
$$

(10)

$$
\leq C(\|\partial \phi\|^2 + \|\partial^* \phi\|^2 + \|\phi\|^2),
$$

where $R(\phi) = \sum_{j=1}^{n} \langle L_j \phi_j, h_j \phi_K \rangle + O(\|\phi\|^2)$ for some smooth functions $h_j$.

We prove the following to remove assumption (8):

**Lemma 4.** The inequality (9) is true without assumption (8).

**Proof.** We want to prove (9) using (10) and Proposition 1. Clearly what we need to prove is that for all $j = 1, 2, \ldots, n$,

$$
\langle L_j \phi_j, h_j \phi_K \rangle \leq \varepsilon \left( \sum_{j \notin J \text{ or } k \leq j < n} \|L_j \phi_j\|^2 + \sum_{j \in J \text{ and } k < j < n} \|L_j \phi_j\|^2 \right) + C\|\phi\|^2.
$$

(11)

When $j \in \{1, 2, \ldots, k, n\}$, we have

$$
\langle L_j \phi_j, h_j \phi_K \rangle \leq \varepsilon \|L_j \phi_j\|^2 + C\|\phi\|^2.
$$

(12)

When $j \in \{k+1, \ldots, n-1\}$, if $j \notin J$ or $j \in K$, then $\|L_j \phi_j\|^2$ or $\|L_j \phi_K\|^2$ is in the right-hand side of (11). Hence, using the type of inequality in (12) or by integrating $L_j$ by parts on the left-hand side, we can absorb the term $\langle L_j \phi_j, h_j \phi_K \rangle$ in the right-hand side of (11). Finally, when $j \in J$ and $j \notin K$, we use Proposition 1, and we have

$$
\langle L_j \phi_j, h_j \phi_K \rangle \leq \varepsilon (\|L_j \phi_j\|^2 + \|L_j \phi_j\|^2) + C\|\phi\|^2.
$$

We see that the terms in the right-hand side of the above inequality are in the right-hand side of (11). This finishes the proof.

**References**