THE RELATIONSHIPS OF SPANS OF CONVEX CONTINUA IN $\mathbb{R}^n$

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Abstract. It has been conjectured that $\sigma^*(X) \geq \frac{1}{2} \sigma_0^*(X)$ for each nonempty connected metric space $X$. In this paper we show that $\sigma^*(X) \geq \frac{\sqrt{15}}{4} \sigma_0^*(X)$ for each convex continuum $X$ in $\mathbb{R}^n$. We also show that under certain conditions a lower bound for the ratio $\sigma^*(X)/\sigma_0^*(X)$ is larger than $\frac{\sqrt{15}}{4}$. It has also been conjectured that $\sigma^*(X) \geq \sigma(X)/2$ and that $\sigma_0^*(X) \geq \sigma_0(X)/2$ for each nonempty connected metric space $X$. We show that these two inequalities hold when $X$ is a convex continuum in $\mathbb{R}^n$.

Generally speaking, the spans of an object are connectedness-type analogues of its diameter. We follow the definitions from [1]: Let $X$ be a nonempty connected metric space. The standard projections of the product $X \times X$ onto $X$ are denoted by $p_1$ and $p_2$; that is, $p_1(x, x') = x$ and $p_2(x, x') = x'$ for $(x, x') \in X \times X$. The surjective span $\sigma^*(X)$ of $X$ is the least upper bound of real numbers $\alpha$ such that there exist nonempty connected sets $C_\alpha \subset X \times X$ with $\text{dist}(x, x') \geq \alpha$ for $(x, x') \in C_\alpha$ and $p_1(C_\alpha) = p_2(C_\alpha) = X$. Relaxing the last condition to $p_1(C_\alpha) = p_2(C_\alpha)$, or $p_2(C_\alpha) = X$, or $p_1(C_\alpha) \subset p_2(C_\alpha)$, one obtains the definitions of the span $\sigma(X)$, the surjective semispan $\sigma^*(X)$, and the semispan $\sigma_0(X)$ of $X$, respectively. Hence,

\begin{align*}
(1) \quad 0 \leq \sigma^*(X) &\leq \sigma(X) \leq \sigma_0(X) \leq \text{diam } X \\
(2) \quad 0 \leq \sigma^*(X) &\leq \sigma_0^*(X) \leq \sigma_0(X) \leq \text{diam } X.
\end{align*}

From the definitions, it is clear that if $T$ is an isometry, then $\sigma(X) = \sigma[T(X)]$, $\sigma^*(X) = \sigma^*[T(X)]$, $\sigma_0(X) = \sigma_0[T(X)]$, and $\sigma_0^*(X) = \sigma_0^*[T(X)]$. When $X$ is a continuum, that is, a nonempty, compact, connected metric space, we need to consider only those sets $C_\alpha$ which are closed in $X \times X$. It has been shown that when $X$ is a continuum, the requirement that $\text{dist}(x, x') \geq \alpha$ for $(x, x') \in C_\alpha$ can be replaced in the above definitions by $\text{dist}(x, x') = \alpha$ for $(x, x') \in C_\alpha$ [2, p. 169].

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Theorem 1. Let \( Y = \{(x_1, x_2, \ldots, x_n) \mid x_2 = x_3 = \cdots = x_n = 0, \ 0 \leq x_1 \leq d\} \cup \{(x_1, x_2, \ldots, x_n) \mid 0 \leq x_1 \leq d/2, \ (x_1 - d)^2 + x_2^2 + \cdots + x_n^2 = d^2\} \cup \{(x_1, x_2, \ldots, x_n) \mid d/2 \leq x_1 \leq d, \ x_1^2 + x_2^2 + \cdots + x_n^2 = d^2\}. \) Then for each \( p \in Y, \ \rho(p, (d/2, 0, \ldots, 0)) \leq \frac{\sqrt{3}}{2} d. \)

Proof. We need just to observe that

\[
f : [d/2, d] \rightarrow \mathbb{R}, \quad \text{where} \quad f(t) = \sqrt{(t - d/2)^2 + d^2 - t^2},
\]

is a decreasing function. Hence the maximum value of \( f \) occurs when \( t = d/2. \)

Also, \( g : [0, d/2] \rightarrow \mathbb{R} \) where \( g(t) = \sqrt{(t - d/2)^2 + d^2 - (t - d)^2} \) is an increasing function. Hence the maximum value of \( g \) occurs when \( t = d/2. \) Also, \( g(d/2) = f(d/2) = \frac{\sqrt{3}}{2} d. \) Consequently, if \( p \in Y, \) then \( \rho(p, (d/2, 0, \ldots, 0)) \leq \frac{\sqrt{3}}{2} d. \)

Let \( a, b \in \mathbb{R}^n; \) we denote the straight line segment in \( \mathbb{R}^n \) with endpoints \( a \) and \( b \) by \( \overline{ab}. \)

Lemma 2. If \( X \subset \mathbb{R}^n \) and \( T \) is an isometry from \( X \) into \( \mathbb{R}^n, \) then for \( x, y \in X, \ T[xy] = \overline{T(x)T(y)}. \)

Lemma 3. If \( X \) is a convex continuum in \( \mathbb{R}^n, \) then \( \sigma_0^*(X) \leq \frac{\sqrt{3}}{2} \text{diam} \ X. \)

Proof. Let \( a, b \in X \) such that \( \rho(a, b) = \text{diam} \ X = d. \) Let \( q \) be the midpoint of \( \overline{ab}. \)

Let \( C \subset X \times X \) such that \( p_2(C) = X \) and, for each \( (x, y) \in C, \ \rho(x, y) = \sigma_0^*(X). \) Hence, for some \( p \in X, \) \( \rho(p, q) \in C, \) since \( q \in \overline{ab} \subset X \) and \( p_1(C) = X. \) Clearly, there is an isometry \( T \) from \( X \) into \( \mathbb{R}^n \) which is the composition of a translation and one or two rotations such that \( T(a) = \theta, \ T(b) = (d, 0, \ldots, 0), \) and \( T(p) \) is in the plane \( \mathbb{R} \times \mathbb{R} \times \{0\} \times \cdots \times \{0\}. \) Note also that \( T(q) = (d/2, 0, \ldots, 0). \) The set

\[
T[X] \subset \{(x_1, x_2, \ldots, x_n) \mid x_1^2 + x_2^2 + \cdots + x_n^2 \leq d^2\}
\]

\[
\cap \{(x_1, x_2, \ldots, x_n) \mid (x_1 - d)^2 + x_2^2 + \cdots + x_n^2 \leq d^2\} = Z,
\]

since \( \rho(a, b) = \rho(T(a), T(b)) = \text{diam} \ X = d. \) Let \( z \) be the point where the ray \( \overline{T(q)T(p)} \) through the point \( T(p), \) inter-
sects the boundary of $Z$. Hence $\rho(q, p) = \rho(T(q), T(p)) \leq \rho(T(q), z) = \rho((d/2, 0, \ldots, 0), z)$ and $\rho((d/2, 0, \ldots, 0), z) \leq \sqrt{d^2}$ by Lemma 1. Consequently, $\sigma^*_0(X) \leq \sqrt{\frac{3}{2}} \text{diam } X$.

Lemma 4. Let $p_t \in \{(x, y) \in \mathbb{R}^2 \mid (x - d)^2 + y^2 = t^2d^2\}$, where $0 \leq t \leq 1$ \cap \{(x, y) \mid x^2 + y^2 \leq d^2\}$. Then $\rho((d, 0), \theta p_t) \geq \frac{dt\sqrt{4 - t^2}}{2}$.

Proof. The lemma is trivial when $d \leq 0$. We just need to consider the case when $d > 0$. Clearly, the lemma is true when $t = 0$. We will just consider the case where $0 < t \leq 1$. Also, we will only consider the case when $\pi_2(p_t) \geq 0$, where $\pi_2$ is the standard projection from $\mathbb{R}^2$ into the second factor space. The case when $\pi_2(p_t) < 0$ is similar.

Let $r_t = (d(\frac{2-t^2}{2}), \frac{dt\sqrt{4-t^2}}{2})$. Let $q$ be the point on $\theta r_t$, the line through $\theta$ and $r_t$, such that $q(d, 0)$, the line through $q$ and $(d, 0)$, are perpendicular to each other. The triangles $\Delta \theta((\frac{2-t^2}{2}), 0)r_t$ and $\Delta \theta(d(\frac{2-t^2}{2}), 0)$ are congruent, since angles $\theta((\frac{2-t^2}{2}), 0)r_t$ and $\theta(d(\frac{2-t^2}{2}), 0)$ are right angles, $<r, \theta((\frac{2-t^2}{2}), 0) =< q\theta(d, 0)$, and $\rho(\theta, r_t) = \rho(\theta, (d, 0)) = d$. Consequently, $\rho((d, 0), q) = \rho((\frac{2-t^2}{2}), 0) = \frac{dt\sqrt{4-t^2}}{2}$.

Let $p_t \in \{(x, y) \mid (x - d)^2 + y^2 = t^2d^2\}$, where $0 \leq t \leq 1$ \cap \{(x, y) \mid x^2 + y^2 \leq d^2\}$. There are two cases to consider.

Case 1. $m(\theta p_t) \geq m(\theta r_t)$.

Here $m(\theta p_t)$ and $m(\theta r_t)$ represent the slopes of the corresponding line segments. Clearly, $\rho((d, 0), \theta p_t) \geq \rho((d, 0), \theta r_t) = \frac{dt\sqrt{4-t^2}}{2}$.

Case 2. $m(\theta p_t) < m(\theta r_t)$.

Let $(\frac{2-t^2}{2}, s)$ be a point on $\theta p_t$. Then $0 \leq s < \frac{dt\sqrt{4-t^2}}{2}$, since $m(\theta p_t) = m(\theta p_t) < m(\theta r_t)$ and $\pi_2(p_t) \geq 0$. Consequently, $\theta p_t$ intersects the interior of the circle $(x - d)^2 + y^2 = t^2d^2$. Hence $\theta p_t$ intersects this circle in two points, $u$ and $v$, and either $u = p_t$ or $v = p_t$. Either $\pi_1(u) < \frac{2-t^2}{2} < \pi_1(v)$, where $\pi_1$ is the standard projection from $\mathbb{R}^2$ into the first factor space, or $\pi_1(v) < \frac{2-t^2}{2} < \pi_1(u)$. We will assume that $\frac{2-t^2}{2} < \pi_1(v)$. Either $d < \pi_1(v)$ or $\frac{2-t^2}{2} < \pi_1(v) \leq d$. If $d < \pi_1(v)$, then $v \not\in \{(x, y) \mid x^2 + y^2 \leq d^2\}$. Hence $v \neq p_t$. If $\frac{2-t^2}{2} < \pi_1(v) \leq d$, then $\frac{dt\sqrt{4-t^2}}{2} < \pi_2(v) \leq td$, where $\pi_2$ is the standard projection map from $\mathbb{R}^2$ into the second factor space. However, if $(x, y) \in \{(x, y) \mid x^2 + y^2 = d^2\}$ and $\frac{2-t^2}{2} < x \leq d$, then $0 \leq y < \frac{dt\sqrt{4-t^2}}{2}$. Hence, $v \neq p_t$. So $u = p_t$. Observe that $\theta p_t$ does not intersect the
interior of the circle \((x - d)^2 + y^2 = r^2d^2\). Hence, \(\rho(\bar{p}_t, (d, 0)) = td > \frac{d\sqrt{4-r^2}}{2}\).

In both cases we see that \(\rho(\bar{p}_t, (d, 0)) \geq \frac{d\sqrt{4-r^2}}{2}\), as desired.

**Theorem 1.** Let \(X\) be a nondegenerate convex continuum in \(R^n\). Then \(\sigma^*(X) \geq \sqrt{4 - \left[\frac{\sigma_0^*(X)/\text{diam } X}{2}\right]^2}\).

**Proof.** Let \(C \subset X \times X\) such that \(C\) is closed and connected, \(p_2(C) = X\), and \(\rho(x, y) = \sigma_0^*(X)\) for each \((x, y) \in C\). Let \(a, b \in X\) such that \(\rho(a, b) = \text{diam } X = d\). There must be points \(a_1, b_1 \in X\) such that \((a_1, b)\) and \((b_1, a)\) are elements of \(C\), since \(p_2(C) = X\). The line segment \(\overline{a_1a} \subset X\), since \(X\) is convex. Also, \(b^1 \subset X\). Hence, \(\overline{a_1a} \times \{b\} \subset X \times X\) and \(\{a\} \times b^1 \subset X \times X\).

Let \(C' = C \cup (\overline{a_1a} \times \{b\}) \cup (\{a\} \times b^1) \cup C^{-1}\). So, \(p_1(C') = p_2(C') = X\), since \(X = p_2(C) \subseteq p_2(C')\) and \(X = p_2(C) = p_1(C^{-1}) \subseteq p_1(C')\). Each of the sets \(\overline{a_1a} \times \{b\}\), \(\{a\} \times b^1\), and \(C^{-1}\) is connected. Also, \((a_1, b) \in C \cap (\overline{a_1a} \times \{b\})\), \((a, b) \in (\overline{a_1a} \times \{b\}) \cap (\{a\} \times b^1)\), and \((a, b_1) \in (\{a\} \times b^1) \cap C^{-1}\). Hence, \(C'\) is connected.

Let \(T\) be an isometry from \(X\) into \(R^n\) such that \(T(a) = \theta, T(b) = (d, 0, \ldots, 0)\), and \(T|_{\{a\}} = 0\) where \(p\) is a point in the plane \(R \times R \times \{0\} \times \cdots \times \{0\}\). Hence \(T[\overline{a_1a}] = \overline{d\theta}p\) and \(\rho(a_1, b) = \rho((d, 0, \ldots, 0), p) = \sigma_0^*(X)\). Also, \(\sigma_0^*(X) = dt\) where \(0 \leq t \leq \sqrt{3}/2\), by Lemma 3 and the fact that \(\sigma_0^*(X) \geq 0\). Note also that \(\rho(\bar{p}_t, (d, 0, \ldots, 0)) \geq \frac{d\sqrt{4-r^2}}{2}\), by Lemma 4. Similarly, \(\rho(b^1, a) \geq \frac{d\sqrt{4-r^2}}{2}\). So \(\sigma^*(X) \geq \frac{d\sqrt{4-r^2}}{2}\) and \(\frac{d\sqrt{4-r^2}}{2} = \left[\sqrt{4 - \left[\frac{\sigma_0^*(X)/\text{diam } X}{2}\right]^2}\right]\sigma_0^*(X)\).

**Corollary 1.1.** If \(X\) is a convex continuum in \(R^n\), then \(\sigma^*(X) \geq \frac{\sqrt{13}}{4}\sigma_0^*(X)\).

**Proof.** If \(\text{diam } X \neq 0\), then \(0 \leq \sigma_0^*(X)/\text{diam } X \leq \sqrt{3}/2\), by Lemma 3 and the fact that \(\sigma_0^*(X)\) and \(\text{diam } X\) are both nonnegative. Consequently, the minimal value of \(\sqrt{4 - \sigma_0^*(X)/\text{diam } X}\) is \(\sqrt{13}/4\). If \(\text{diam } X = 0\), then \(\sigma^*(X) = \sigma_0^*(X) = 0\), and the inequality holds.

**Theorem 2.** Let \(X\) be a nondegenerate convex continuum in \(R^n\). Then \(\sigma^*(X) \geq \sigma_0(X)/2\).

**Proof.** Let \(C \subset X \times X\) such that \(C\) is closed and connected, \(p_1(C) \subset p_2(C)\), and \(\rho(x, y) = \sigma_0(X)\) for each \((x, y) \in C\). Let \(a, b \in p_2(C)\) such that \(\rho(a, b) = \text{diam } p_2(C) = d\). Clearly, \(d \geq \sigma_0(X)\). There must be points \(a_1, b_1 \in p_1(C)\) such that \((a_1, b) \in C\) and \((b_1, a) \in C\), since \(a, b \in p_2(C)\). Let \(T\) be an isometry from \(X\) into \(R^n\) such that \(T(a) = \theta, T(b) = (d, 0, \ldots, 0)\), and \(T(a_1) = p\) where \(p\) is a point in the plane \(R \times R \times \{0\} \times \cdots \times \{0\}\). Let \(q\) be the
midpoint of the straight line segment $\overline{ab}$. Clearly, $\pi_1[T(q)] = (d/2, 0, \ldots, 0)$.

Let $X_L = \{ x \in X \mid \pi_1[T(x)] \leq d/2 \}$ and $X_R = \{ x \in X \mid \pi_1[T(x)] \geq d/2 \}$. Let $SX_L = \bigcup \{ xq \mid x \in X_L \}$ and $SX_R = \bigcup \{ qx \mid x \in X_R \}$. The set $SX_L$ is connected, since it is the union of arcs and each of these arcs contains the point $q$. Similarly, $SX_R$ is connected. Also, for each line segment $\overline{xq}$, where $x \in X_L$, if $y \in \overline{xq}$, then $\pi_1[T(y)] \leq d/2$ since $\pi_1[T(x)] \leq d/2$, $\pi_1[T(q)] = d/2$, and $T(\overline{xq}) = T(x)T(q)$. Similarly, for each line segment $\overline{qx}$, where $x \in X_R$, if $y \in \overline{qx}$, then $\pi_1[T(y)] > d/2$.

Let $C' = C \cup (\overline{aa_1} \times \{ b \}) \cup (SX_L \times \{ b \}) \cup (\{ a \} \times SX_R) \cup (b \times \{ a \}) \cup (SX_R \times \{ a \}) \cup (\{ b \} \times SX_L)$. The set $p_2(C') = X$, since $p_2(\{ a \} \times SX_R) = SX_R$, $p_2(\{ b \} \times SX_L) = SX_L$, and $X = SX_L \cup SX_R$. Similarly, $p_1(C') = X$.

Each of the sets $C$, $\overline{aa_1} \times \{ b \}$, $SX_L \times \{ b \}$, $\{ a \} \times SX_R$, $\overline{b_1b} \times \{ a \}$, $SX_R \times \{ a \}$, and $\{ b \} \times SX_L$ is connected. The set $C'$ is connected since $(a, b) \in C \cap (\overline{aa_1} \times \{ b \}) \cap (SX_L \times \{ b \}) \cap (\{ a \} \times SX_R) \cap (b_1b \times \{ a \}) \cap (\{ b \} \times SX_L)$. And $X = SX_L \cup SX_R$. Similarly, $p_1(C') = X$.

Next, we will see that if $(x, y) \in C'$, then $\rho(x, y) \geq \sigma_0(X)/2$. Let $(x, y) \in C'$. If $(x, y) \in C$, then $\rho(x, y) = \sigma_0(X)$. If $(x, y) = (x, b) \in \overline{aa_1} \times \{ b \}$, then $\rho(x, b) = \rho(T(x), T(b)) = \rho(T(x), (d, 0, \ldots, 0))$. In addition, $\rho(T(x), (d, 0, \ldots, 0)) \geq \rho(\overline{dd}, (d, 0, \ldots, 0))$, since $x \in \overline{aa_1}$ and $T(x) \in T[\overline{aa_1}] = \overline{dd}$. Clearly, $\sigma_0(X) = dt$ where $0 \leq t \leq 1$. Hence, $\sigma_0(X) = \rho(a_1, b) = \rho(p, (d, 0, \ldots, 0)) = dt$. So $p \in \{(x_1, x_2, 0, \ldots, 0) \mid (x_1 - d)^2 + x_2^2 = t^2d^2\} \cap \{(x_1, x_2, 0, \ldots, 0) \mid x_1^2 + x_2^2 \leq d^2\}$. So, $\rho(\overline{dd}, (d, 0, \ldots, 0)) \geq \sqrt{4 - t^2(\sigma_0(X)/2)} \geq \sqrt{3}(\sigma_0(X)/2)$, by Lemma 4. So, $\rho(x, b) \geq \sigma_0(X)/2$. Similarly, if $(x, y) = (x, a) \in b_1b \times \{ a \}$, then $\rho(x, a) \geq \sigma_0(X)/2$. If $(x, y) \in SX_L \times \{ b \}$, then $\rho(x, y) = \rho(T(x), T(b)) \geq d/2 \geq \sigma_0(X)/2$, since $\pi_1[T(x)] \leq d/2$ and $\pi_1[T(b)] = d$. Similarly, if $(x, y) \in (\{ a \} \times SX_R) \cup (SX_R \times \{ a \}) \cup (\{ b \} \times SX_L)$, then $\rho(x, y) \geq \sigma_0(X)/2$. Consequently, in each case $\rho(x, y) \geq \sigma_0(X)/2$ and $\sigma^*(X) \geq \sigma_0(X)/2$.

**Corollary 2.1.** Let $X$ be a convex continuum in $\mathbb{R}^n$. Then $\sigma^*(X) \geq \sigma(X)/2$.

**Proof.** This follows from Theorem 2 and (1).

**Corollary 2.2.** Let $X$ be a convex continuum in $\mathbb{R}^n$. Then $\sigma_0^*(X) \geq \sigma_0(X)/2$.

**Proof.** This follows from Theorem 2 and (2).

**References**