SIERPINSKI SETS AND STRONG FIRST CATEGORY

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ABSTRACT. We prove that if $S$ is a Sierpinski set and $N \subseteq \mathbb{R}$ is an $F_\sigma$ set of measure zero, then $(N + t) \cap S = \emptyset$ for some $t \in \mathbb{R}$. A similar result holds for generalized Sierpinski sets under Martin's Axiom.

INTRODUCTION

An uncountable set $S \subseteq \mathbb{R}$ is called a Sierpinski set if $|N \cap S| \leq \omega$ whenever $N \subseteq \mathbb{R}$ is of measure zero. We say that $A \subseteq \mathbb{R}$ has strong first category (SFC), if, for any measure-zero set $N \subseteq \mathbb{R}$, there exists $t \in \mathbb{R}$ such that $(N + t) \cap X = \emptyset$. It is well known that Lusin sets, the category analog of the Sierpinski sets, have strong measure zero (see [M] for details). This leads to the question posed by F. Galvin [M, p. 210]: Do all Sierpinski sets have SFC? It is known that under CH there are Sierpinski sets which have SFC. Simply let $\{N_\alpha\}_{\alpha < \omega_1}$ be all $G_\delta$ measure zero sets and inductively pick

$$s_\alpha \in \mathbb{R} \setminus \bigcup_{\beta < \alpha} \left( (F_\beta \cup \{s_\beta\}) \cup (F_\beta + t_\beta) \right),$$

where $t_\alpha$ is such that $\{s_\beta : \beta \leq \alpha\} \cap (F_\alpha + t_\alpha) = \emptyset$. Then $S = \{s_\alpha : \alpha < \omega_1\}$ is an SFC Sierpinski set. Recently Bartoszynski and Ihoda proved that under CH every Sierpinski set is a union of two SFC sets (although it is not known if the union of two SFC sets always has SFC) and that Con(ZF) $\Rightarrow$ Con(ZFC + there are Sierpinski sets + all Sierpinski sets have SFC) [BI].

MAIN RESULT

Let $\mu$ be Lebesgue measure on $\mathbb{R}$ and let $L_0 = \{N \subseteq \mathbb{R} : \mu(N) = 0\}$. For $X, Y \subseteq \mathbb{R}$ $X \pm Y = \{x \pm y : x \in X \text{ and } y \in Y\}$.

The following is a partial answer to Galvin’s question.

Theorem 1. If $N \in L_0$ is an $F_\sigma$ set and $S$ a Sierpinski set, then there exists $t \in \mathbb{R}$ such that $(N + t) \cap S = \emptyset$. 
The proof is based on the following simple observation:

**Lemma 1.** Let $N \in L_0$ and $Q \subseteq \mathbb{R}$ be such that

1. $N + Q \in L_0$ and
2. $N$ can be shifted out of every countable set by a member of $Q$; i.e., for every countable set $X \subseteq \mathbb{R}$, $Q \not\subseteq X - N$.

Then $N$ can be shifted out of every Sierpinski set.

**Proof.** Let $S$ be a Sierpinski set and let $X = S \cap (N + Q)$, so that $X$ is countable. Then any member of $Q$ that shifts $N$ out of $X$ shifts $N$ out of $S$; i.e., if $t \in Q$ and $(N + t) \cap X = \emptyset$, then $(N + t) \cap S = \emptyset$.

[EKM, Lemma 9] shows that, if $N \in L_0$, then there exists a perfect set $Q$ such that $N + Q \in L_0$. It follows that every Sierpinski set can be shifted out of any measure-zero set $N$ such that countably many translates of $N$ cannot cover any perfect set. Recall that a set $X \subseteq \mathbb{R}$ has universal measure zero iff, for all atomless measures $\nu$ on the Borel sets, there is a Borel set of $\nu$-measure zero covering $X$. Also recall that $X$ has property $s_0$ iff for any perfect set $P$ there exists a perfect set $Q \subseteq P$ disjoint from $X$. It is well known that neither universal-measure-zero sets nor sets with property $s_0$ can cover a perfect set by countably many translates (for this and other properties of these sets see [M]).

We have the following:

**Corollary.** If $M \subseteq \mathbb{R}$ is universally null or $M \in L_0$ and has property $s_0$ and $S$ is a Sierpinski set, then $(M + t) \cap S = \emptyset$ for some $t \in \mathbb{R}$.

For our main result, we need $\{p_k\}_{k \in \omega}$-expansions. Let $\{p_k\}_{k \in \omega}$ be a sequence of natural numbers each greater than 1. Every real number $x \in [0, 1)$ has a unique $\{p_k\}_{k \in \omega}$-expansion of the form $x = \sum_{k \in \omega} \frac{x_k}{p_0 \cdot p_1 \cdots p_k}$, where $x_k \in p_k$ and $x_k \neq p_k - 1$ for infinitely many $k$. (We identify $p_k$ with $\{0, 1, 2, \ldots, p_k - 1\}$.) In what follows, $x_k$ will be denoted by $(x)_k$.

**Lemma 2.** Let $N \in L_0$ be an $F_\sigma$ subset of $[0, 1)$. Then there exists a sequence $\{p_k\}_{k \in \omega}$ and a sequence $\{\Delta_k\}_{k \in \omega}$ such that, for all $k \in \omega$,

1. $p_k \geq 6$, $\Delta_k \subseteq p_k$ and $|\Delta_k| \leq \frac{p_k}{3}$,

and there exists a countable set $Y \subseteq \mathbb{R}$ such that $N \subseteq D + Y$, where

$$D = \left\{ \sum_{k \in \omega} \frac{d_k}{p_0 \cdot p_1 \cdots p_k} : d_k \in \Delta_k \right\}.$$

**Proof.** Let $N_0$, $N_1$, $\ldots \subseteq [0, 1)$ be closed sets such that $N = \bigcup_{k \in \omega} N_k$. Define $I_n^i = \left[ \frac{i - \frac{1}{2}}{2^n}, \frac{i + \frac{1}{2}}{2^n} \right)$, where $n \in \omega$ and $i \in 2^n$. Also, let $Q_k(n) = |\{i : I_n^i \cap N_k \neq \emptyset\}|$. It is not hard to show that, since $N_k$ is closed (in $\mathbb{R}$), $\lim_{n \to \infty} \frac{Q_k(n)}{2^n} = 0$. Using this fact, we define $\{p_k\}_{k \in \omega}$ by induction.

Let $m \in \omega$ be such that $2^m \geq 6$ and $\frac{Q_{m+1}(m)}{2^{2m}} \leq \frac{1}{3}$. Put $p_0 = 2^m$ and
$\Delta_0 = \{ i : I_m' \cap N_0 \neq \emptyset \}$. At stage $k+1$, consider $I_k = [0, \frac{1}{p_0 \cdot p_1 \cdots p_k})$ and

$$B_k = \left( N_k \cup N_k - \frac{1}{p_0 \cdot p_1 \cdots p_k} \cup \ldots \cup N_k - \frac{1}{p_0 \cdot p_1 \cdots p_k} \right) \cap I_k.$$

Now pick $m$ so that $2^m \geq 6$. Then, if we partition $I_k$ into $2^m$ equal subintervals, at most $\frac{2^m}{3}$ of them have nonempty intersections with $B_k$. Let $p_{k+1} = 2^m$ and $\Delta_{k+1} \subseteq p_{k+1}$ be a set of indices of those subintervals whose intersections with $B_k$ are nonempty. Clearly conditions (1) are satisfied.

Now suppose $D$ is as in Lemma 2 and $a \in N_{k_0}$ for some $k_0 \in \omega$. For $k \geq k_0$, $(a)_k \in \Delta_k$. Therefore, if we let $Y$ be the set of all real numbers from $(-1, 1)$ with finite \{ $p_k$ \}_{k \in \omega}$-expansions, then $a \in D + Y$.

We now have to modify [EKM, Lemma 9].

**Lemma 3.** Let $\{p_k\}_{k \in \omega}$ be a bounded sequence of natural numbers greater than 1. For each set $K \subseteq \omega$, let

$$Q_K = \left\{ \sum_{k \in \omega} \frac{q_k}{p_0 \cdot p_1 \cdots p_k} : q_k \in p_k \text{ if } k \in K \text{ and } q_k = 0 \text{ otherwise} \right\}.$$

Given $N \in L_0$, there exists an infinite set $K \subseteq \omega$ such that $N + Q_K \in L_0$.

**Proof.** The proof is almost the same as the original proof of Erdös, Kunen, and Mauldin. Note that if $f : \omega \to \omega$ is an increasing function, $I \subseteq \mathbb{R}$ is an interval, and $n \in \omega$, then $\mu(I + Q_{\text{ran}(f)}) \leq p_{f(0)} \cdot p_{f(1)} \cdots p_{f(n)}(\mu(I) + \frac{1}{p_0 \cdots p_n})$. By making $f$ increase rapidly enough we make $p_{f(0)} \cdot p_{f(1)} \cdots p_{f(n)}/p_0 \cdots p_n$ as small as desired so that all steps in the proof given in [EKM] may be performed here without major changes. We leave the details to the reader. However, in the case relevant to the proof of Theorem 1, $p_k = 2^{r_k}$ for some $r_k \in \omega$. For such $p_k$, Lemma 3 follows easily. By [EKM, Lemma 9], we have an infinite set $H \subseteq \omega$ such that if

$$B_H = \left\{ \sum_{k \in \omega} \frac{q_k}{2^k} q_k \in \{0, 1\} \text{ if } k \in H \text{ and } q_k = 0 \text{ otherwise} \right\},$$

then $N + B_H \in L_0$ and, if $b \in \omega$ is a bound for $\{r_k\}_{k \in \omega}$ and $H_k = \{ k + i : i < b \}$, then there exists an infinite set $X \subseteq \omega$ such that, whenever $k, 1 \in X$, then $H_k \cap H_1 = \emptyset$ and $H_k \subseteq H$. Now simply let $K = \{ k + b : k \in X \}$, and observe that $Q_K \subseteq B_H$.

**Proof of Theorem 1.** Let $N \in L_0$ be an $F_\sigma$ set. Then $N' = (\bigcup_{n \in \omega} (N \pm n)) \cap [0, 1)$ is also an $F_\sigma$ set, so that by Lemma 2 there exist sequences $\{p_k\}_{k \in \omega}$, $\{\Delta_k\}_{k \in \omega}$, and a set $D$ such that $N' \subseteq D + Y$ for some countable $Y \subseteq \mathbb{R}$. Since in the proof of Lemma 2 the sets $N_k$ can be made as small as desired, we may assume that the sequence $\{p_k\}_{k \in \omega}$ is bounded. Therefore, by Lemma 3, we have an infinite set $K = \{ k_0, k_1, k_2, \ldots \} \subseteq \omega$, so that $Q_K + N \in L_0$. By Lemma 1, it suffices to show that $N$ does not cover $Q_K$ by countable translations. Since $N'$ as well as $N$ may be covered by countably many translations of $D$, let us now...
show that for any countable \( X \subseteq \mathbb{R} \), \( X + D \not\subseteq Q_K \). Let \( X = \{ x^{(n)} : n \in \omega \} \). By assuming that the addition in \( X + D \) is modulo 1, we may consider \( X \subseteq [0, 1) \).

For each \( i \in \omega \), pick \( q_k \in (p_k - 1) \setminus [(\Delta_k \cup \Delta_k \oplus 1) \oplus (x^{(i)}_k)] \), where \( \oplus \) signifies algebraic addition modulo \( p_k \). Set \( q_k = 0 \) for \( k \not\in K \) so that the number \( q = \sum_{k \in \omega} \frac{q_k}{p_0 \cdot p_1 \cdots p_k} \) is in \( Q_K \) and is different from all numbers in \( X + D \).

A similar result holds for generalized Sierpinski sets under Martin’s Axiom (MA). Recall that \( S \subseteq \mathbb{R} \) is a generalized Sierpinski set if \( |S| = c \) and \( |S \cap N| < c \) whenever \( N \in L_0 \). Clearly, Lemma 1 remains true for generalized Sierpinski sets if “countable \( X \)” is replaced by “\( |X| < c \).”

**Theorem 2.** Assume MA. If \( N \in L_0 \) is an \( F_\sigma \) set, then for every generalized Sierpinski set \( S \) there exists a \( t \in \mathbb{R} \) such that \( (N + t) \cap S = \emptyset \).

**Proof.** Let \( N \in L_0 \) be an \( F_\sigma \) set. It suffices to show that \( N \) satisfies the generalized Lemma 1 as shown above. Let \( N' \), \( \{ p_k \}_{k \in \omega} \), \( \{ \Delta_k \}_{k \in \omega} \), and \( K \) be as in the proof of Theorem 1. We would like to show that \( D + X \not\subseteq Q_K \) for any \( X \subseteq \mathbb{R} \), where \( |X| < c \). Let \( P = \{ (f, E) : E \subseteq X \) is a finite subset and \( f \) is a finite function, \( \text{dom}(f) \subseteq \omega \), \( f(i) \in p_i \) and for every \( e \in E \) there exists an \( i \in K \cap \text{dom}(f) \) such that \( f(i) \in p_i \setminus (D + e)_1 \) and \( f(i) = 0 \) for \( i \in \text{dom}(f) \setminus K \). Define a partial order on \( P \) simply as \( (f_2, E_2) \leq (f_1, E_1) \) if \( f_2 \supseteq f_1 \) and \( E_2 \supseteq E_1 \). Clearly \( (P, \leq) \) is a ccc poset and \( D_F = \{ (f, E) : E \subseteq F \} \) and \( D_n = \{ (f, E) : n \in \text{dom}(f) \} \) are dense subsets of \( P \) for all finite \( F \subseteq X \) and \( n \in \omega \). By Martin’s Axiom, there exists a filter \( G \subseteq P \) which meets all \( D_F \) and \( D_n \). Let \( q = \bigcup \{ f : (f, E) \in G \} \). It is easy to see that

\[
\sum_{k \in \omega} \frac{q(k)}{p_0 \cdot p_1 \cdots p_k} \in Q_k \setminus (D + X).
\]

**Added in proof.** I. Reclaw and A. W. Miller have obtained independently a short, elegant proof of the following theorem.

**Theorem.** If \( S \) is a Sierpinski set and \( F \) is a \( F_\sigma \) set of measure zero, then \( \{ x : (x + F) \cap S = \emptyset \} \) is comeager.

T. Bartoszynski and I. Reclaw pointed out that for some dense \( G_\delta \) sets \( N \) there are sets \( Q \) such that conditions (1) and (2) of Lemma 1 hold.

**References**


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