SIERPINSKI SETS AND STRONG FIRST CATEGORY

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Abstract. We prove that if $S$ is a Sierpinski set and $N \subseteq \mathbb{R}$ is an $F_\sigma$ set of measure zero, then $(N + t) \cap S = \emptyset$ for some $t \in \mathbb{R}$. A similar result holds for generalized Sierpinski sets under Martin's Axiom.

Introduction

An uncountable set $S \subseteq \mathbb{R}$ is called a Sierpinski set if $|N \cap S| \leq \omega$ whenever $N \subseteq \mathbb{R}$ is of measure zero. We say that $A \subseteq \mathbb{R}$ has strong first category (SFC), if, for any measure-zero set $N \subseteq \mathbb{R}$, there exists $t \in \mathbb{R}$ such that $(N + t) \cap X = \emptyset$. It is well known that Lusin sets, the category analog of the Sierpinski sets, have strong measure zero (see [M] for details). This leads to the question posed by F. Galvin [M, p. 210]: Do all Sierpinski sets have SFC? It is known that under CH there are Sierpinski sets which have SFC. Simply let $\{N_\beta\}_{\beta \in \omega}$ be all $G_\delta$ measure zero sets and inductively pick

$$s_\alpha \in \mathbb{R} \setminus \bigcup_{\beta < \alpha} ((F_\beta \cup \{s_\beta\}) \cup (F_\beta + t_\beta)),$$

where $t_\alpha$ is such that $\{s_\beta: \beta \leq \alpha\} \cap (F_\alpha + t_\alpha) = \emptyset$. Then $S = \{s_\alpha: \alpha < \omega_1\}$ is an SFC Sierpinski set. Recently Bartoszynski and Ihoda proved that under CH every Sierpinski set is a union of two SFC sets (although it is not known if the union of two SFC sets always has SFC) and that $\text{Con}(ZF) \Rightarrow \text{Con}(ZFC + \text{there are Sierpinski sets + all Sierpinski sets have SFC})$ [BI].

Main result

Let $\mu$ be Lebesgue measure on $\mathbb{R}$ and let $L_0 = \{N \subseteq \mathbb{R}: \mu(N) = 0\}$. For $X, Y \subseteq \mathbb{R}$, $X \pm Y = \{x \pm y: x \in X \text{ and } y \in Y\}$.

The following is a partial answer to Galvin’s question.

Theorem 1. If $N \in L_0$ is an $F_\sigma$ set and $S$ a Sierpinski set, then there exists $t \in \mathbb{R}$ such that $(N + t) \cap S = \emptyset$.

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The proof is based on the following simple observation:

**Lemma 1.** Let \( N \in L_0 \) and \( Q \subseteq \mathbb{R} \) be such that

1. \( N + Q \in L_0 \)
2. \( N \) can be shifted out of every countable set by a member of \( Q \); i.e., for every countable set \( X \subseteq \mathbb{R} \), \( Q \not\subseteq X - N \).

Then \( N \) can be shifted out of every Sierpinski set.

**Proof.** Let \( S \) be a Sierpinski set and let \( X = S \cap (N + Q) \), so that \( X \) is countable. Then any member of \( Q \) that shifts \( N \) out of \( X \) shifts \( N \) out of \( S \); i.e., if \( t \in Q \) and \( (N + t) \cap X = \emptyset \), then \( (N + t) \cap S = \emptyset \).

[EKM, Lemma 9] shows that, if \( N \in L_0 \), then there exists a perfect set \( Q \) such that \( N + Q \in L_0 \). It follows that every Sierpinski set can be shifted out of any measure-zero set \( N \) such that countably many translates of \( N \) cannot cover any perfect set. Recall that a set \( X \subseteq \mathbb{R} \) has universal measure zero iff, for all atomless measures \( \nu \) on the Borel sets, there is a Borel set of \( \nu \)-measure zero covering \( X \). Also recall that \( X \) has property \( s_0 \) iff for any perfect set \( P \) there exists a perfect set \( Q \subseteq P \) disjoint from \( X \). It is well known that neither universal-measure-zero sets nor sets with property \( s_0 \) can cover a perfect set by countably many translates (for this and other properties of these sets see [M]).

We have the following:

**Corollary.** If \( M \subseteq \mathbb{R} \) is universally null or \( M \in L_0 \) and has property \( s_0 \) and \( S \) is a Sierpinski set, then \( (M + t) \cap S = \emptyset \) for some \( t \in \mathbb{R} \).

For our main result, we need \( \{p_k\}_{k \in \omega} \)-expansions. Let \( \{p_k\}_{k \in \omega} \) be a sequence of natural numbers each greater than 1. Every real number \( x \in [0, 1) \) has a unique \( \{p_k\}_{k \in \omega} \)-expansion of the form \( x = \sum_{k \in \omega} x_k p_k \), where \( x_k \in p_k \) and \( x_k \neq p_k - 1 \) for infinitely many \( k \). (We identify \( p_k \) with \( \{0, 1, 2, \ldots, p_k - 1\} \).) In what follows, \( x_k \) will be denoted by \( (x)_k \).

**Lemma 2.** Let \( N \in L_0 \) be an \( F_\sigma \) subset of \([0, 1)\). Then there exists a sequence \( \{p_k\}_{k \in \omega} \) and a sequence \( \{\Delta_k\}_{k \in \omega} \) such that, for all \( k \in \omega \),

1. \( p_k \geq 6 \), \( \Delta_k \subseteq p_k \) and \( |\Delta_k| \leq \frac{p_k}{3} \),

and there exists a countable set \( Y \subseteq \mathbb{R} \) such that \( N \subseteq D + Y \), where

\[
D = \left\{ \sum_{k \in \omega} d_k \cdot p_0 \cdot p_1 \cdot \ldots \cdot p_k : d_k \in \Delta_k \right\}.
\]

**Proof.** Let \( N_0, N_1, \ldots \subseteq [0, 1) \) be closed sets such that \( N = \bigcup_{k \in \omega} N_k \). Define \( I_n^i = [\frac{i}{2^n}, \frac{i+1}{2^n}] \), where \( n \in \omega \) and \( i \in 2^n \). Also, let \( Q_n^i (n) = \{i : I_n^i \cap N_k \neq \emptyset\} \). It is not hard to show that, since \( N_k \) is closed (in \( \mathbb{R} \)), \( \lim_{n \to \infty} \frac{Q_n^i (n)}{2^n} = 0 \). Using this fact, we define \( \{p_k\}_{k \in \omega} \) by induction.

Let \( m \in \omega \) be such that \( 2^m \geq 6 \) and \( \frac{Q_{\infty}(m)}{2^m} \leq \frac{1}{3} \). Put \( p_0 = 2^m \) and
\[ \Delta_0 = \{ i : I'_m \cap N_0 \neq \emptyset \} \]. At stage \( k + 1 \), consider \( I_k = [0, \frac{1}{p_0 \cdot p_1 \cdots p_k}) \) and
\[ B_k = \left( N_k \cup N_{k-1} - \frac{1}{p_0 \cdot p_1 \cdots p_k} \cup \cdots \cup N_k - \frac{1}{p_0 \cdot p_1 \cdots p_k} \right) \cap I_k. \]

Now pick \( m \) so that \( 2^m \geq 6 \). Then, if we partition \( I_k \) into \( 2^m \) equal subintervals, at most \( \frac{2^m}{3} \) of them have nonempty intersections with \( B_k \). Let \( p_{k+1} = 2^m \) and \( \Delta_{k+1} \subseteq p_{k+1} \) be a set of indices of those subintervals whose intersections with \( B_k \) are nonempty. Clearly conditions (1) are satisfied.

Now suppose \( D \) is as in Lemma 2 and \( a \in N_{k_0} \) for some \( k_0 \in \omega \). For \( k \geq k_0 \), \( (a)_k \in \Delta_k \). Therefore, if we let \( Y \) be the set of all real numbers from \((-1, 1)\) with finite \( \{p_k\}_{k \in \omega} \)-expansions, then \( a \in D + Y \).

We now have to modify \([EKM, Lemma 9]\).

**Lemma 3.** Let \( \{p_k\}_{k \in \omega} \) be a bounded sequence of natural numbers greater than 1. For each set \( K \subseteq \omega \), let
\[ Q_K = \left\{ \sum_{k \in \omega} \frac{q_k}{p_0 \cdot p_1 \cdots p_k} : q_k \in p_k \text{ if } k \in K \text{ and } q_k = 0 \text{ otherwise} \right\}. \]

Given \( N \in L_0 \), there exists an infinite set \( K \subseteq \omega \) such that \( N + Q_K \in L_0 \).

**Proof.** The proof is almost the same as the original proof of Erdős, Kunen, and Mauldin. Note that if \( f : \omega \to \omega \) is an increasing function, \( I \subseteq \mathbb{R} \) is an interval, and \( n \in \omega \), then \( \mu(I + Q_{\text{ran}(f)}) \leq p_{f(0)} \cdot p_{f(1)} \cdots p_{f(n)} \cdot (\mu(I) + \frac{1}{p_0 \cdot p_1 \cdots p_n}) \). By making \( f \) increase rapidly enough we make \( p_{f(0)} \cdot p_{f(1)} \cdots p_{f(n)} / p_0 \cdot p_1 \cdots p_n \) as small as desired so that all steps in the proof given in \([EKM]\) may be performed here without major changes. We leave the details to the reader. However, in the case relevant to the proof of Theorem 1, \( p_k = 2^{r_k} \) for some \( r_k \in \omega \). For such \( p_k \), Lemma 3 follows easily. By \([EKM, Lemma 9]\), we have an infinite set \( H \subseteq \omega \) such that if
\[ B_H = \left\{ \sum_{k \in \omega} \frac{q_k}{2^k} : q_k \in \{0, 1\} \text{ if } k \in H \text{ and } q_k = 0 \text{ otherwise} \right\}, \]
then \( N + B_H \in L_0 \) and, if \( b \in \omega \) is a bound for \( \{r_k\}_{k \in \omega} \) and \( H_k = \{k + i : i < b\} \), then there exists an infinite set \( X \subseteq \omega \) such that, whenever \( k, 1 \in X \), then \( H_k \cap H'_1 = \emptyset \) and \( H_k \subseteq H \). Now simply let \( K = \{k + b : k \in X\} \), and observe that \( Q_K \in B_H \).

**Proof of Theorem 1.** Let \( N \in L_0 \) be an \( F_\sigma \) set. Then \( N' = (\bigcup_{n \in \omega}(N \pm n)) \cap \{0, 1\} \) is also an \( F_\sigma \) set, so that by Lemma 2 there exist sequences \( \{p_k\}_{k \in \omega}, \{\Delta_k\}_{k \in \omega} \) and a set \( D \) such that \( N' \subseteq D + Y \) for some countable \( Y \subseteq \mathbb{R} \). Since in the proof of Lemma 2 the sets \( N_k \) can be made as small as desired, we may assume that the sequence \( \{p_k\}_{k \in \omega} \) is bounded. Therefore, by Lemma 3, we have an infinite set \( K = \{k_0, k_1, k_2, \ldots\} \subseteq \omega \), so that \( Q_K + N \in L_0 \). By Lemma 1, it suffices to show that \( N \) does not cover \( Q_K \) by countable translations. Since \( N' \) as well as \( N \) may be covered by countably many translations of \( D \), let us now
show that for any countable \( X \subseteq \mathbb{R} \), \( X + D \not\subseteq Q_K \). Let \( X = \{x^{(n)} : n \in \omega\} \). By assuming that the addition in \( X + D \) is modulo 1, we may consider \( X \subseteq [0, 1) \).

For each \( i \in \omega \), pick \( q_k \in (p_k - 1) \backslash \{(\Delta_k \cup \Delta_k \oplus 1) \oplus (x^{(i)}_{k})_{k} \} \), where \( \oplus \) signifies algebraic addition modulo \( p_k \). Set \( q_k = 0 \) for \( k \notin K \) so that the number \( q = \sum_{k \in \omega} \frac{q_k}{p_0 \cdot p_1 \cdots p_k} \) is in \( Q_K \) and is different from all numbers in \( X + D \).

A similar result holds for generalized Sierpinski sets under Martin’s Axiom (MA). Recall that \( S \subseteq \mathbb{R} \) is a generalized Sierpinski set if \( |S| = c \) and \( |S \cap \mathbb{N}| < c \) whenever \( N \in L_0 \). Clearly, Lemma 1 remains true for generalized Sierpinski sets if “countable \( X \)” is replaced by “\( |X| < c \).”

**Theorem 2.** Assume MA. If \( N \in L_0 \) is an \( F_\sigma \) set, then for every generalized Sierpinski set \( S \) there exists a \( t \in \mathbb{R} \) such that \( (N + t) \cap S = \emptyset \).

**Proof.** Let \( N \in L_0 \) be an \( F_\sigma \) set. It suffices to show that \( N \) satisfies the generalized Lemma 1 as shown above. Let \( N', \{p_k\}_{k \in \omega}, \{\Delta_k\}_{k \in \omega}, \) and \( K \) be as in the proof of Theorem 1. We would like to show that \( D + X \not\subseteq Q_K \) for any \( X \subseteq \mathbb{R} \), where \( |X| < c \). Let \( P = \{(f, E) : E \subseteq X \text{ is a finite subset and } f \text{ is a finite function, } \text{dom}(f) \subseteq \omega, (f \in p_1) \}\) and for every \( e \in E \) there exists an \( i \in K \cap \text{dom}(f) \) such that \( f(i) \in p_1 \backslash (D + e) \) and \( f(i) = 0 \) for \( i \in \text{dom}(f) \backslash K \). Define a partial order on \( P \) simply as \( (f_2, E_2) \leq (f_1, E_1) \) if \( f_2 \supseteq f_1 \) and \( E_2 \supseteq E_1 \). Clearly \( (P, \leq) \) is a ccc poset and \( D_F = \{(f, E) : E \supseteq F \} \) and \( D_n = \{(f, E) : n \in \text{dom}(f) \} \) are dense subsets of \( P \) for all finite \( F \subseteq X \) and \( n \in \omega \). By Martin’s Axiom, there exists a filter \( G \subseteq P \) which meets all \( D_F \) and \( D_n \). Let \( q = \bigcup\{f : (f, E) \in G\} \). It is easy to see that

\[
\sum_{k \in \omega} \frac{q(k)}{p_0 \cdot p_1 \cdots p_k} \in Q_K \backslash (D + X).
\]

**Added in proof.** I. Reclaw and A. W. Miller have obtained independently a short, elegant proof of the following theorem.

**Theorem.** If \( S \) is a Sierpinski set and \( F \) is a \( F_\sigma \) set of measure zero, then \( \{x : (x + F) \cap S = \emptyset\} \) is comeager.

T. Bartoszynski and I. Reclaw pointed out that for some dense \( G_\delta \) sets \( N \) there are sets \( Q \) such that conditions (1) and (2) of Lemma 1 hold.

**References**


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