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SIERPINSKI SETS AND STRONG FIRST CATEGORY

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ABSTRACT. We prove that if S is a Sierpinski set and $N \subseteq \mathbb{R}$ is an F_σ set of measure zero, then $(N+t) \cap S = \varnothing$ for some $t \in \mathbb{R}$. A similar result holds for generalized Sierpinski sets under Martin's Axiom.

INTRODUCTION

An uncountable set $S \subseteq \mathbb{R}$ is called a Sierpinski set if $|N \cap S| \leq \omega$ whenever $N \subseteq \mathbb{R}$ is of measure zero. We say that $X \subseteq \mathbb{R}$ has strong first category (SFC), if, for any measure-zero set $N \subseteq \mathbb{R}$, there exists $t \in \mathbb{R}$ such that $(N+t) \cap X = \varnothing$. It is well known that Lusin sets, the category analog of the Sierpinski sets, have strong measure zero (see [M] for details). This leads to the question posed by F. Galvin [M, p. 210]: Do all Sierpinski sets have SFC? It is known that under CH there are Sierpinski sets which have SFC. Simply let $\{N_\alpha\}_{\alpha<\omega_1}$ be all G_δ measure zero sets and inductively pick

$$s_\alpha \in \mathbb{R} \setminus \bigcup_{\beta<\alpha} ((F_\beta \cup \{s_\beta\}) \cup (F_\beta + t_\beta)),$$

where t_α is such that $\{s_\beta : \beta \subseteq \alpha\} \cap (F_\alpha + t_\alpha) = \varnothing$. Then $S = \{s_\alpha : \alpha < \omega_1\}$ is an SFC Sierpinski set. Recently Bartoszynski and Ihoda proved that under CH every Sierpinski set is a union of two SFC sets (although it is not known if the union of two SFC sets always has SFC) and that Con(ZF) $\Rightarrow$ Con(ZFC + there are Sierpinski sets + all Sierpinski sets have SFC) [BI].

MAIN RESULT

Let μ be Lebesgue measure on $\mathbb{R}$ and let $L_0 = \{N \subseteq \mathbb{R} : \mu(N) = 0\}$. For $X, Y \subseteq \mathbb{R}$ $X \pm Y = \{x \pm y : x \in X \text{ and } y \in Y\}$.

The following is a partial answer to Galvin's question.

Theorem 1. *If $N \in L_0$ is an F_σ set and S a Sierpinski set, then there exists $t \in \mathbb{R}$ such that $(N+t) \cap S = \varnothing$.*

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The proof is based on the following simple observation:

Lemma 1. *Let $N \in L_0$ and $Q \subseteq \mathbb{R}$ be such that*

(1) *$N + Q \in L_0$ and*
(2) *N can be shifted out of every countable set by a member of Q; i.e., for every countable set $X \subseteq \mathbb{R}$, $Q \nsubseteq X - N$.*

Then N can be shifted out of every Sierpinski set.

Proof. Let S be a Sierpinski set and let $X = S \cap (N + Q)$, so that X is countable. Then any member of Q that shifts N out of X shifts N out of S; i.e., if $t \in Q$ and $(N + t) \cap X = \varnothing$, then $(N + t) \cap S = \varnothing$.

[EKM, Lemma 9] shows that, if $N \in L_0$, then there exists a perfect set Q such that $N + Q \in L_0$. It follows that every Sierpinski set can be shifted out of any measure-zero set N such that countably many translates of N cannot cover any perfect set. Recall that a set $X \subseteq \mathbb{R}$ has universal measure zero iff, for all atomless measures ν on the Borel sets, there is a Borel set of ν-measure zero covering X. Also recall that X has property s_0 iff for any perfect set P there exists a perfect set $Q \subseteq P$ disjoint from X. It is well known that neither universal-measure-zero sets nor sets with property s_0 can cover a perfect set by countably many translates (for this and other properties of these sets see [M]). We have the following:

Corollary. *If $M \subseteq \mathbb{R}$ is universally null or $M \in L_0$ and has property s_0 and S is a Sierpinski set, then $(M + t) \cap S = \varnothing$ for some $t \in \mathbb{R}$.*

For our main result, we need $\{p_k\}_{k\in\omega}$-expansions. Let $\{p_k\}_{k\in\omega}$ be a sequence of natural numbers each greater than 1. Every real number $x \in [0, 1)$ has a unique $\{p_k\}_{k\in\omega}$-expansion of the form $x = \sum_{k\in\omega} \frac{x_k}{p_0 \cdot p_1 \cdot \dots \cdot p_k}$, where $x_k \in p_k$ and $x_k \neq p_k - 1$ for infinitely many k. (We identify p_k with $\{0, 1, 2, \dots, p_k - 1\}$.) In what follows, x_k will be denoted by $(x)_k$.

Lemma 2. *Let $N \in L_0$ be an F_σ subset of $[0, 1)$. Then there exists a sequence $\{p_k\}_{k\in\omega}$ and a sequence $\{\Delta_k\}_{k\in\omega}$ such that, for all $k \in \omega$,*

$$p_k \geq 6, \Delta_k \subseteq p_k \quad \text{and} \quad |\Delta_k| \leq \frac{p_k}{3}, \tag{1}$$

and there exists a countable set $Y \subseteq \mathbb{R}$ such that $N \subseteq D + Y$, where

$$D = \left\{ \sum_{k\in\omega} \frac{d_k}{p_0 \cdot p_1 \cdot \dots \cdot p_k} : d_k \in \Delta_k \right\}.$$

Proof. Let $N_0, N_1, \dots \subseteq [0, 1)$ be closed sets such that $N = \bigcup_{k\in\omega} N_k$. Define $I_n^i = [\frac{i}{2^n} \frac{i+1}{2^n})$, where $n \in \omega$ and $i \in 2^n$. Also, let $Q_k(n) = |\{i : I_n^i \cap N_k \neq \varnothing\}|$. It is not hard to show that, since N_k is closed (in $\mathbb{R}$), $\lim_{n\to\infty} \frac{Q_k(n)}{2^n} = 0$. Using this fact, we define $\{p_k\}_{k\in\omega}$ by induction.

Let $m \in \omega$ be such that $2^m \geq 6$ and $\frac{Q_0(m)}{2^m} \leq \frac{1}{3}$. Put $p_0 = 2^m$ and

$\Delta_0 = \{i: I_m^i \cap N_0 \neq \varnothing\}$. At stage $k+1$, consider $I_k = [0, \frac{1}{p_0 \cdot p_1 \cdot \dots \cdot p_k})$ and

$$B_k = \left(N_k \cup N_k - \frac{1}{p_0 \cdot p_1 \cdot \dots \cdot p_k} \cup \dots \cup N_k - \frac{p_0 \cdot p_1 \cdot \dots \cdot p_k - 1}{p_0 \cdot p_1 \cdot \dots \cdot p_k}\right) \cap I_k.$$

Now pick m so that $2^m \geq 6$. Then, if we partition I_k into 2^m equal subintervals, at most $\frac{2m}{3}$ of them have nonempty intersections with B_k. Let $p_{k+1} = 2^m$ and $\Delta_{k+1} \subseteq p_{k+1}$ be a set of indices of those subintervals whose intersections with B_k are nonempty. Clearly conditions (1) are satisfied.

Now suppose D is as in Lemma 2 and $a \in N_{k_0}$ for some $k_0 \in \omega$. For $k \geq k_0$, $(a)_k \in \Delta_k$. Therefore, if we let Y be the set of all real numbers from $(-1, 1)$ with finite $\{p_k\}_{k\in\omega}$-expansions, then $a \in D + Y$.

We now have to modify [EKM, Lemma 9].

Lemma 3. *Let $\{p_k\}_{k\in\omega}$ be a bounded sequence of natural numbers greater than 1. For each set $K \subseteq \omega$, let*

$$Q_K = \left\{\sum_{k\in\omega} \frac{q_k}{p_0 \cdot p_1 \cdot \dots \cdot p_k} : q_k \in p_k \text{ if } k \in K \text{ and } q_k = 0 \text{ otherwise}\right\}.$$

Given $N \in L_0$, there exists an infinite set $K \subseteq \omega$ such that $N + Q_K \in L_0$.

Proof. The proof is almost the same as the original proof of Erdös, Kunen, and Mauldin. Note that if $f: \omega \to \omega$ is an increasing function, $I \subseteq \mathbb{R}$ is an interval, and $n \in \omega$, then $\mu(I + Q_{\mathrm{ran}(f)}) \leq p_{f(0)} \cdot p_{f(1)} \cdot \dots \cdot p_{f(n)}(\mu(I) + \frac{1}{p_0 \cdot p_1 \cdot \dots \cdot p_{f(n)}})$. By making f increase rapidly enough we make $p_{f(0)} \cdot p_{f(1)} \cdot \dots \cdot p_{f(n)}/p_0 \cdot p_1 \cdot \dots \cdot p_{f(n)}$ as small as desired so that all steps in the proof given in [EKM] may be performed here without major changes. We leave the details to the reader. However, in the case relevant to the proof of Theorem 1, $p_k = 2^{r_k}$ for some $r_k \in \omega$. For such p_k, Lemma 3 follows easily. By [EKM, Lemma 9], we have an infinite set $H \subseteq \omega$ such that if

$$B_H = \left\{\sum_{k\in\omega} \frac{q_k}{2^k} q_k \in \{0, 1\} \text{ if } k \in H \text{ and } q_k = 0 \text{ otherwise}\right\},$$

then $N + B_H \in L_0$ and, if $b \in \omega$ is a bound for $\{r_k\}_{k\in\omega}$ and $H_k = \{k + i : i < b\}$, then there exists an infinite set $X \subseteq \omega$ such that, whenever $k, 1 \in X$, then $H_k \cap H_1 = \varnothing$ and $H_k \subseteq H$. Now simply let $K = \{k + b : k \in X\}$, and observe that $Q_k \subseteq B_H$.

Proof of Theorem 1. Let $N \in L_0$ be an F_σ set. Then $N' = (\bigcup_{n\in\omega}(N \pm n)) \cap [0, 1)$ is also an F_σ set, so that by Lemma 2 there exist sequences $\{p_k\}_{k\in\omega}$, $\{\Delta_k\}_{k\in\omega}$ and a set D such that $N' \subseteq D + Y$ for some countable $Y \subseteq \mathbb{R}$. Since in the proof of Lemma 2 the sets N_k can be made as small as desired, we may assume that the sequence $\{p_k\}_{k\in\omega}$ is bounded. Therefore, by Lemma 3, we have an infinite set $K = \{k_0, k_1, k_2, \dots\} \subseteq \omega$, so that $Q_K + N \in L_0$. By Lemma 1, it suffices to show that N does not cover Q_K by countable translations. Since N' as well as N may be covered by countably many translations of D, let us now

show that for any countable $X \subseteq \mathbb{R}$, $X + D \not\supseteq Q_K$. Let $X = \{x^{(n)} : n \in \omega\}$. By assuming that the addition in $X+D$ is modulo 1, we may consider $X \subseteq [0, 1)$. For each $i \in \omega$, pick $q_{k_i} \in (p_{k_i} - 1) \backslash [(\Delta_{k_i} \cup \Delta_{k_i} \oplus 1) \oplus (x^{(i)})_{k_i}]$, where $\oplus$ signifies algebraic addition modulo p_{k_i}. Set $q_k = 0$ for $k \notin K$ so that the number $q = \sum_{k \in \omega} \frac{q_k}{p_0 \cdot p_1 \cdot \cdots \cdot p_k}$ is in Q_K and is different from all numbers in $X + D$.

A similar result holds for generalized Sierpinski sets under Martin's Axiom (MA). Recall that $S \subseteq \mathbb{R}$ is a generalized Sierpinski set if $|S| = c$ and $|S \cap N| < c$ whenever $N \in L_0$. Clearly, Lemma 1 remains true for generalized Sierpinski sets if "countable X" is replaced by "$|X| < c$."

Theorem 2. *Assume MA. If $N \in L_0$ is an F_σ set, then for every generalized Sierpinski set S there exists a $t \in \mathbb{R}$ such that $(N + t) \cap S = \varnothing$.*

Proof. Let $N \in L_0$ be an F_σ set. It suffices to show that N satisfies the generalized Lemma 1 as shown above. Let N', $\{p_k\}_{k \in \omega}$, $\{\Delta_k\}_{k \in \omega}$, and K be as in the proof of Theorem 1. We would like to show that $D + X \not\supseteq Q_K$ for any $X \subseteq \mathbb{R}$, where $|X| < c$. Let $\mathbf{P} = \{(f, E) : E \subseteq X$ is a finite subset and f is a finite function, $\operatorname{dom}(f) \subseteq \omega$, $f(i) \in p_1$ and for every $e \in E$ there exists an $i \in K \cap \operatorname{dom}(f)$ such that $f(i) \in p_1 \backslash (D + e)_1$ and $f(i) = 0$ for $i \in \operatorname{dom}(f) \backslash K\}$. Define a partial order on $\mathbf{P}$ simply as $(f_2, E_2) \leq (f_1, E_1)$ if $f_2 \supseteq f_1$ and $E_2 \supseteq E_1$. Clearly $(\mathbf{P}, \leq)$ is a ccc poset and $D_F = \{(f, E) : E \supseteq F\}$ and $D_n = \{(f, E) : n \in \operatorname{dom}(f)\}$ are dense subsets of $\mathbf{P}$ for all finite $F \subseteq X$ and $n \in \omega$. By Martin's Axiom, there exists a filter $G \subseteq \mathbf{P}$ which meets all D_F and D_n. Let $q = \bigcup\{f : (f, E) \in G\}$. It is easy to see that

$$\sum_{k \in \omega} \frac{q(k)}{p_0 \cdot p_1 \cdot \cdots \cdot p_k} \in Q_k \backslash (D + X).$$

Added in proof. I. Reclaw and A. W. Miller have obtained independently a short, elegant proof of the following theorem.

Theorem. *If S is a Sierpinski set and F is a F_σ set of measure zero, then $\{x : (x + F) \cap S = \varnothing\}$ is comeager.*

T. Bartoszynski and I. Reclaw pointed out that for some dense G_δ sets N there are sets Q such that conditions (1) and (2) of Lemma 1 hold.

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