

## SURJECTIVE MAPPINGS WHOSE DIFFERENTIAL IS NOWHERE SURJECTIVE

Y. YOMDIN

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**ABSTRACT.** Examples of  $C^k$ -mappings  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $n \geq m > 2$ , are given, with  $\text{rank } df(x) \leq s$  at any  $x \in \mathbb{R}^n$ ,  $2 \leq s < m$ , but  $f(\mathbb{R}^n) = \mathbb{R}^m$ , for any  $k < (n - s + 2)/(m - s + 2)$ . Thus a weak form of Sard's theorem (if all the points in the source are critical, the image has measure zero) does not hold for mappings of low smoothness.

1. The structure of critical points and values of differentiable mappings has been studied intensively since the classical work of H. Whitney [5], [6] and M. Morse [3]. Sard's (or, better, M. Morse's–A. P. Morse's–A. Sard's) Theorem [4] claims that critical values of a sufficiently smooth mapping form a set of measure zero. Whitney's paper [5] provides examples of functions which are nonconstant on a connected set of critical points, thus showing the sharpness of the differentiability assumptions in Sard's theorem.

However, the following closely related question seems to be open: Is it possible for a differentiable mapping  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ , with  $\text{rank } df < m$  at every point, to be onto (or to have an image with a nonempty interior)?

Clearly, for  $m = 1$ , any  $C^1$ -function with all the points critical is a constant, so the question makes sense only for mappings into  $\mathbb{R}^m$ ,  $m \geq 2$ . If Sard's theorem is valid, the answer is, of course, negative. But what happens for mappings of a lower smoothness than that required in Sard's theorem?

Besides its theoretical interest, this question is important in various applications. Recently it has attracted the attention of specialists in control theory [2, p. 59].

In this note, we answer this question positively, constructing highly differentiable surjective mappings  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  with  $\text{rank } df < m$  everywhere.

2. For  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ , let us denote as  $\sum_s(f)$  the set  $\{x \in \mathbb{R}^n, \text{rank } df(x) \leq s\}$ , and let  $\Delta_s(f) = f(\sum_s(f))$  be the corresponding set of critical values.

Define the set  $\Omega_s(f)$  of *strongly critical values* as follows:

$\Omega_s(f) = \{y \in \mathbb{R}^m / \text{rank } df(x) \leq s \text{ for any } x \in f^{-1}(y)\}$ . While the usual

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Sard's theorem describes the sets  $\Delta(f)$ , the question above concerns  $\Omega(f)$ . We have a diagram of inclusions:

$$\begin{array}{ccccccc} \Delta_0(f) & \subseteq & \Delta_1(f) & \subseteq & \dots & \subseteq & \Delta_m(f) = \text{Im } f \\ \cup & & \cup & & & & \parallel \\ \Omega_0(f) & \subseteq & \Omega_1(f) & \subseteq & \dots & \subseteq & \Omega_m(f) \end{array}$$

Let  $I^p \subseteq \mathbb{R}^p$  denote a  $p$ -dimensional unit cube  $[0, 1]^p$ . Clearly, it is enough to construct corresponding examples of mappings  $f : I^n \rightarrow I^m$ .

**Theorem.** For any  $m \geq 3$ ,  $n \geq m$ ,  $2 \leq s \leq m - 1$  and  $k < (n - s + 2) / (m - s + 2)$  there is a  $C^k$ -mapping  $f : I^n \rightarrow I^m$  with  $\Omega_s(f) = I^m$  (i.e.,  $\text{rank } df \leq s$  everywhere and  $f(I^n) = I^m$ ).

*Proof.* Consider first the case  $s = 2$ . We obtain  $f$  as a composition of two mappings  $f_1 : I^n \rightarrow I^2$  and  $f_2 : I^2 \rightarrow I^m$ .

Fix three integers  $N_1, N_2, N_3$  such that  $N_1^n = N_2^2 = N_3^m = N$ ; e.g.,  $N_1 = 4^m, N_2 = 2^{mn}, N_3 = 4^n$ . Fix also some  $\alpha, 0 < \alpha < 1$ . Subdivide  $I^n$  (resp.  $I^2, I^m$ ) into  $N$  subcubes  $I_j^n$  (resp.  $I_j^2, I_j^m$ ),  $j = 1, \dots, N$ , of the size  $1/N_1$  (resp.  $1/N_2, 1/N_3$ ), fixing in each case one of the possible orderings of the corresponding grid.

Inscribe in each  $I_j^n$  (resp.  $I_j^2$ ) a concentric subcube  $J_j^n$  (resp.  $J_j^2$ ) of the size  $\alpha/N_1$  (resp.  $\alpha/N_2$ ). Denote by  $x_j$  (resp.  $y_j, z_j$ ) the lower, left-hand vertices of  $J_j^n$  (resp.  $J_j^2, I_j^m$ ),  $j = 1, \dots, N$ , and let  $x_0 = 0 \in \mathbb{R}^n$  ( $y_0 = 0 \in \mathbb{R}^2, z_0 = 0 \in \mathbb{R}^m$ , respectively; see Figure 1).

Now let  $\varphi : I^n \rightarrow I^2$  be a  $C^\infty$  mapping, such that

1.  $\varphi \equiv y_0$  on a neighborhood of the boundary  $\partial I^n$ .
2.  $\varphi \equiv y_j$  on a neighborhood of a subcube  $J_j^n, j = 1, \dots, N$ .
3.  $\varphi(I^n)$  does not intersect the interiors of  $J_j^2$ .

To construct  $\varphi$ , join the points  $y_0, y_1, \dots, y_N$  by a  $C^\infty$ -smooth curve  $L$  in  $I^2$ , which does not intersect the interiors of  $J_j^2$  (Figure 1). Define  $\varphi$  to be  $y_0 \in L$  on a neighborhood of  $\partial I^n$  and to be  $y_j \in L$  on a neighborhood of each  $J_j^n, j = 1, \dots, N$ . Consider  $\varphi$  as a partial mapping of  $I^n$  into  $L$  (diffeomorphic to  $[0, 1]$ ) and extend it to  $I^n$  in a  $C^\infty$  manner by Whitney's theorem [6].

Let  $\psi : I^2 \rightarrow I^m$  be a  $C^\infty$ -mapping with the following properties:

1.  $\psi \equiv z_0$  on a neighborhood of  $\partial I^2$ .
2.  $\psi \equiv z_j$  on a neighborhood of each  $J_j^2, j = 1, \dots, N$ .

Construction of  $\psi$  is immediate.

Now let  $f_1^0 = \varphi, f_2^0 = \psi$ . To construct  $f_1^1$  (resp.  $f_2^1$ ) replace  $\varphi$  (resp.  $\psi$ ) on each  $J_j^n$  (resp.  $J_j^2$ ) by an appropriately rescaled copy of  $\varphi$  which maps  $J_j^n \rightarrow J_j^2$  (copy of  $\psi$  which maps  $J_j^2 \rightarrow I_j^m$ ). Properties 1 and 2 of  $\varphi$  and  $\psi$  provide a  $C^\infty$ -smooth contact of the "old" and "new" parts. Applying

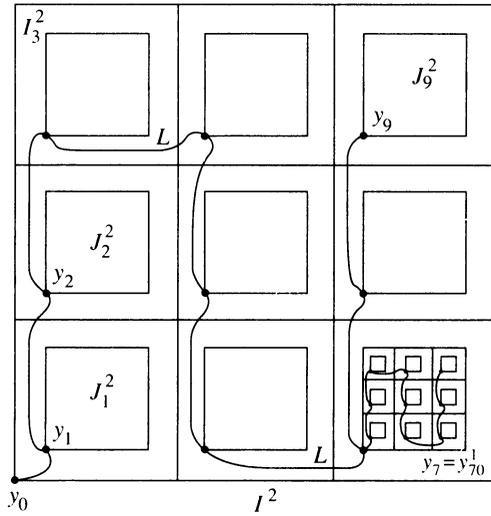


FIGURE 1

this procedure  $r$  times, we obtain  $C^\infty$ -smooth mappings  $f_1^r : I^n \rightarrow I^2$  and  $f_2^r : I^2 \rightarrow I^m$ , and these mappings differ for  $r - 1$  and  $r$  only on the union  $\mathcal{D}_r^n \subset I^n$  ( $\mathcal{D}_r^2 \subset I^2$ ) of the subcubes  $J_i^n$  ( $J_i^2$ ) of the  $r$ th order.

Let  $f_1 = \lim_{r \rightarrow \infty} f_1^r$ ,  $f_2 = \lim_{r \rightarrow \infty} f_2^r$ .

Denote by  $G^n \subset I^n$  ( $G^2 \subset I^2$ ) the Cantor sets  $\bigcap_{r=1}^\infty \mathcal{D}_r^n$  ( $\bigcap_{r=1}^\infty \mathcal{D}_r^2$ , respectively). Then  $G^2 \subseteq f_1(G^n)$  and  $f_2(G^2) = I^m$ . This follows by considering the grids  $x_j^r, y_j^r, z_j^r$  of all the orders  $r$ , which are mapped onto one another by  $f_1$  and  $f_2$ . In particular,  $f_2 \circ f_1(I^n) = I^m$ . By property 3 of  $\varphi$   $f_1(I^n \setminus G^n) \subseteq I^2 \setminus G^2$  and hence  $f_2 \circ f_1$  is  $C^\infty$  on  $I^n \setminus G^n$  (by construction,  $f_1$  is  $C^\infty$  on  $I^n \setminus G^n$  and  $f_2$  is  $C^\infty$  on  $I^2 \setminus G^2$ ).

It remains to estimate the derivatives of  $f_2 \circ f_1(x)$  as  $x \in I^n \setminus G^n$  approaches  $G^n$ .

Once more by property 3 of  $\varphi$ , for  $x \in \mathcal{D}_r^n \setminus \mathcal{D}_{r+1}^n$   $f_1(x) \in \mathcal{D}_r^2 \setminus \mathcal{D}_{r+1}^2$ , for any  $r$ . Hence computing rescaling factors shows that

$$\begin{aligned} f(x) &= f_2 \circ f_1(x) = [(1/N_3)^r \psi(N_2/\alpha)^r] \circ [(\alpha/N_2)^r \varphi(N_1/\alpha)^r] \\ &= (1/4)^{nr} \psi \circ \varphi(4^m/\alpha)^r. \end{aligned}$$

The  $k$ th derivative of  $f$  satisfies

$$d^k f(x) = (1/4)^{nr} d^k(\psi \circ \varphi)(4^m/\alpha)^{rk} = (4^{mk-n}/\alpha^k)^r d^k(\psi \circ \varphi).$$

For any  $k < n/m$ , this expression tends to zero as  $r \rightarrow \infty$  (or  $x$  approaches  $G^n$ ), assuming  $\alpha$  to be sufficiently close to 1. Hence the mapping  $f$  constructed is at least  $C^k$ -smooth, with all the derivatives of order  $\leq k$  vanishing on  $G^n$ . Since  $f$  goes through  $\mathbb{R}^2$ ,  $\text{rank } df \leq 2$  on  $I^n \setminus G^n$ , and it is zero on  $G^n$ .

This proves the theorem for  $s = 2$ . For  $s > 2$ , consider  $I^n$  and  $I^m$  as  $I^{n-s+2} \times I^{s-2}$  and  $I^{m-s+2} \times I^{s-2}$ , respectively, and define  $f$  to be as above on the first factor and to be identical on the second.

### Open questions.

1. What is the theorem corresponding to Sard's theorem for the sets  $\Omega_s$ . The remaining differentiability gap is rather small: in the examples above,  $f \in C^k$ , with any  $k < (n - s + 2)/(m - s + 2)$ , and for  $f \in C^l$ ,  $l \geq (n - s)/(m - s)$ , the set  $\Delta_s$ , containing  $\Omega_s$ , has measure zero, by the usual Sard's theorem. What is a corresponding result for near-critical values (compare [7])?
2. Is it possible to construct  $C^\infty$ -examples as above on infinite-dimensional spaces? In particular, is it possible for input-to-state mappings of nonlinear control systems? (See [1] for  $C^\infty$ -counterexamples to the usual Sard's theorem in this framework).
3. Our construction excludes  $s = 1$  (the mean value theorem prevents constructing  $\varphi : I^n \rightarrow I$  with property 3 above). Are there examples of surjective mappings  $f : I^n \rightarrow I^m$  with rank  $df \leq 1$  everywhere?

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