PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 111, Number 1, January 1991

MULTIPLICATIVE DECOMPOSITION OF PROBABILITY MEASURES

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(Communicated by R. Daniel Mauldin)

ABSTRACT. We investigate decompositions of probability measures having the feature that the component of interest in the decomposition of a product probability measure is the product of the corresponding components in the decomposition of the factors.

1. INTRODUCTION

It is well known that the Lebesgue decomposition is multiplicative in the following sense. Let P_i and Q_i be probability measures (p-measures) on some σ-algebra $\mathfrak{U}_i$, and let μ_i be the Q_i-continuous component of P_i in the Lebesgue decomposition of P_i w.r.t. Q_i, $i = 1, \dots, n$. Then the product measure $\bigotimes_1^n \mu_i$ is the $\bigotimes_1^n Q_i$-continuous component of $\bigotimes_1^n P_i$ in the Lebesgue decomposition of $\bigotimes_1^n P_i$ w.r.t. $\bigotimes_1^n Q_i$ (cf. [8, p. 54]). This is the origin of the question of which other decompositions share the feature of being multiplicative with the Lebesgue decomposition. In this paper we consider decompositions w.r.t. "band-defining" properties of p-measures, such as being monogenic, admitting extensions, being uniquely extensible, and so forth. Observing that, in the above example, $\bigotimes_1^n \mu_i$ is the band projection of $\bigotimes_1^n P_i$ onto the band of $\bigotimes_1^n Q_i$-continuous, bounded, signed measures, we introduce the notion of a multiplicative band projection and derive a characterization of the band-defining properties which admit a multiplicative band projection. It turns out that exactly those properties are multiplicative that are stable under (direct) products and under forming marginals of measures majorized by a (direct) product (Theorem 2). Several examples are discussed, and a generalization to the setting of invariant measures is given (Theorem 3).

A basic fact. Let us recall a useful characterization of bands of measures. Let $\mathfrak{U}$ be a σ-algebra of subsets of some set Ω and let $\mathrm{ca}(\mathfrak{U})$ denote the order-

Received by the editors November 14, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 28A35.
Key words and phrases. Multiplicative band projections, decompositions of product measures, invariant measures.
Research supported by a Heisenberg grant of the Deutsche Forschungsgemeinschaft.

complete vector lattice of all bounded, signed measures on $\mathfrak{U}$. For a subset M of $\mathrm{ca}(\mathfrak{U})$, let $B(M)$ denote the band generated by M. For details on bands and related notions see [9].

The following characterization of bands in $\mathrm{ca}(\mathfrak{U})$ is contained in [6] (see [1] for a different proof).

Proposition 1. *Let* $M \subset \mathrm{ca}(\mathfrak{U})_+$. *Then* $B = \{\mu \in \mathrm{ca}(\mathfrak{U}) \colon |\mu| \in M\}$ *is a band in* $\mathrm{ca}(\mathfrak{U})$ *if and only if* M *has the following two properties*:

(i) $\mu \in M$, $\nu \in \mathrm{ca}(\mathfrak{U})_+$, $\nu \ll \mu$ *imply* $\nu \in M$.

(ii) $\mu \in \mathrm{ca}(\mathfrak{U})_+$, $\mu(A_n \cap \cdot) \in M$ *for every* $n \in \mathbb{N}$, *where* $(A_n)_{n\in\mathbb{N}}$ *is a countable partition of* Ω *in* $\mathfrak{U}$, *imply* $\mu \in M$.

Proof. Since $B(\mu) = \{v \in \mathrm{ca}(\mathfrak{U}) \colon |v| \ll \mu\}$ (cf. [9, p. 114]), the "only if" part is clear. The "if" part: By [6, Lemma 2 and Remark], it follows from (i) and (ii) that $B(M)_+ = M$. This gives $B(M) = B$ and hence, B is a band. □

2. Multiplicative band projections

Let $\mathfrak{U}_i$ be a σ-algebra of subsets of some set Ω_i and let B_i denote a band in $\mathrm{ca}(\mathfrak{U}_i)$, $i = 1, \dots, n$. Let B be a band in $\mathrm{ca}(\bigotimes_1^n \mathfrak{U}_i)$. The band projection φ_B of $\mathrm{ca}(\bigotimes_1^n \mathfrak{U}_i)$ onto B is said to be multiplicative w.r.t. B_i if

$$\varphi_B\Big(\bigotimes_1^n P_i\Big) = \bigotimes_1^n \varphi_{B_i}(P_i)$$

for every p-measure P_i on $\mathfrak{U}_i$, $i = 1, \dots, n$. Then by the Riesz decomposition theorem (cf. [9, p. 62]), $\bigotimes_1^n P_i$ has a unique decomposition

$$\bigotimes_1^n P_i = \bigotimes_1^n \varphi_{B_i}(P_i) + \mu, \quad \text{where } \mu \in B^{\perp}.$$

Note that multiplicative band projections satisfy $\varphi_B(\bigotimes_1^n \nu_i) = \bigotimes_1^n \varphi_{B_i}(\nu_i)$ for every $\nu_i \in \mathrm{ca}(\mathfrak{U}_i)_+$. Concerning applications, the bands B and $B_1, \dots, B_n$ will always be related by the same defining property. In the sequel, we deal only with the case $n = 2$. Extensions to arbitrary n are obvious.

Define product bands by

$$B_1 \check{\otimes} B_2 = B(\{Q_1 \otimes Q_2 \colon Q_i \in B_i \ p\text{-measure},\ i = 1, 2\})$$

and

$$B_1 \hat{\otimes} B_2 = B(\{Q \in \mathrm{ca}(\mathfrak{U}_1 \otimes \mathfrak{U}_2) \colon Q \ p\text{-measure},\ \pi_i(Q) \in B_i,\ i = 1, 2\}),$$

where $\pi_i(Q)$ denotes the ith marginal of Q. Using Proposition 1, it is not difficult to check that

$$B_1 \hat{\otimes} B_2 = \{\mu \in \mathrm{ca}(\mathfrak{U}_1 \otimes \mathfrak{U}_2) \colon \pi_i(|\mu|) \in B_i,\ i = 1, 2\}.$$

Obviously, we have $B_1 \check{\otimes} B_2 \subset B_1 \hat{\otimes} B_2$. To see that, in general, $B_1 \check{\otimes} B_2 \neq B_1 \hat{\otimes} B_2$, let $\Omega_1 = \Omega_2 = [0, 1]$, $\mathfrak{U}_1 = \mathfrak{U}_2 = \mathfrak{U}$ the Borel σ-algebra, and let $B_1 = B_2 = B(\lambda)$, where λ is the Lebesgue measure. Then $B_1 \check{\otimes} B_2 = B(\lambda^2)$ and, for the p-

measure Q on $\mathfrak{A}^2$ defined by $Q(A) = \int 1_A(\omega, \omega)\, d\lambda(\omega)$, we have $\pi_i(Q) = \lambda$, and hence $Q \in B(\lambda)\hat{\otimes}B(\lambda)$, but $Q \notin B(\lambda^2)$.

The bands B having a multiplicative band projection can be characterized in terms of the product bands of B_i as follows:

Theorem 2. *The following statements are equivalent*:

(i) φ_B *is multiplicative w.r.t.* B_1 *and* B_2.

(ii) $B_1\tilde{\otimes}B_2 \subset B$ *and* $B \cap [0, P_1 \otimes P_2] \subset B_1\hat{\otimes}B_2$ *for every p-measure* P_i *on* $\mathfrak{A}_i$, $i = 1, 2$.

(iii) $B \cap [0, P_1 \otimes P_2] = B_1\tilde{\otimes}B_2 \cap [0, P_1 \otimes P_2]$ *for every p-measure* P_i *on* $\mathfrak{A}_i$, $i = 1, 2$.

As a consequence, we obtain that the multiplicative property of band projections is stable under intersections of bands.

Corollary. *Let* D_i *and* D *be bands in* $\mathrm{ca}(\mathfrak{A}_i)$ *and* $\mathrm{ca}(\mathfrak{A})$, *respectively,* $i = 1, 2$. *If* φ_D *is multiplicative w.r.t.* D_1, D_2, *and* φ_B *is multiplicative w.r.t.* B_1, B_2, *then* $\varphi_{B\cap D}$ *is multiplicative w.r.t.* $B_1 \cap D_1$ *and* $B_2 \cap D_2$.

Proof. Immediately from Theorem 2, in view of the relations

$$B_1 \cap D_1\tilde{\otimes}B_2 \cap D_2 \subset B_1\tilde{\otimes}B_2 \cap D_1\tilde{\otimes}D_2$$

and

$$B_1 \cap D_1\hat{\otimes}B_2 \cap D_2 = B_1\hat{\otimes}B_2 \cap D_1\hat{\otimes}D_2. \quad \square$$

The proof of the theorem is based on two lemmas.

Lemma 1. *If* $B_1\tilde{\otimes}B_2 \subset B \subset B_1\hat{\otimes}B_2$, *then* φ_B *is multiplicative w.r.t.* B_1 *and* B_2.

Proof. Let P_i be a p-measure on $\mathfrak{A}_i$, $i = 1, 2$. Since $B_1\tilde{\otimes}B_2 \subset B$, we have $\varphi_{B_1}(P_1)\otimes\varphi_{B_2}(P_2) \in B$. Let $Q \in B$ be a p-measure. Then $\pi_i(Q) \in B_i$, $i = 1, 2$, because $B \subset B_1\hat{\otimes}B_2$. Thus $P_i - \varphi_{B_i}(P_i)\perp\pi_i(Q)$, and so there exists $A_i \in \mathfrak{A}_i$ such that $P_i(A_i) - \varphi_{B_i}(P_i)(A_i) = 0$ and $\pi_i(Q)(A_i) = 1$. For $A = A_1 \times A_2$, we obtain

$$P_1 \otimes P_2(A) - \varphi_{B_1}(P_1) \otimes \varphi_{B_2}(P_2)(A) = 0$$

and

$$Q(A) = Q(A_1 \times \Omega_2 \cap \Omega_1 \times A_2) = 1.$$

This yields $P_1 \otimes P_2 - \varphi_{B_1}(P_1) \otimes \varphi_{B_2}(P_2) \perp Q$. Hence, Q being arbitrary, $P_1 \otimes P_2 - \varphi_{B_1}(P_1) \otimes \varphi_{B_2}(P_2) \in B^{\perp}$. This establishes $\varphi_B(P_1 \otimes P_2) = \varphi_{B_1}(P_1) \otimes \varphi_{B_2}(P_2)$. $\square$

From the preceding lemma one can deduce that the band projections of the product bands coincide when applied to product measures.

Lemma 2. $B_1\tilde{\otimes}B_2 \cap [0, P_1 \otimes P_2] = B_1\hat{\otimes}B_2 \cap [0, P_1 \otimes P_2]$ *for every p-measure* P_i *on* $\mathfrak{A}_i$, $i = 1, 2$.

Proof. For $\mu \in B_1\hat{\otimes}B_2$, $0 \le \mu \le P_1 \otimes P_2$, we have by Lemma 1

$$\mu \le \varphi_{B_1 \widehat{\bigotimes} B_2}(P_1 \otimes P_2) = \varphi_{B_1}(P_1) \otimes \varphi_{B_2}(P_2).$$

Since the latter product measure belongs to $B_1\tilde{\otimes}B_2$, we get $\mu\in B_1\tilde{\otimes}B_2$. □

Proof of Theorem 2. (i) $\Rightarrow$ (ii). For p-measures $Q_i\in B_i$, we have $\varphi_B(Q_1\otimes Q_2)=\varphi_{B_1}(Q_1)\otimes\varphi_{B_2}(Q_2)=Q_1\otimes Q_2$ and hence, $Q_1\otimes Q_2\in B$. This yields $B_1\tilde{\otimes}B_2\subset B$. Moreover, for $\mu\in B$ satisfying $0\le\mu\le P_1\otimes P_2$ for some p-measures P_i on $\mathfrak{U}_i$, we obtain $\mu\le\varphi_B(P_1\otimes P_2)=\varphi_{B_1}(P_1)\otimes\varphi_{B_2}(P_2)$. Since the latter product measure belongs to $B_1\tilde{\otimes}B_2\subset B_1\hat{\otimes}B_2$, it follows that $\mu\in B_1\hat{\otimes}B_2$.

(ii) $\Rightarrow$ (iii) is an immediate consequence of Lemma 2.

(iii) $\Rightarrow$ (i). Let P_i be a p-measure on $\mathfrak{U}_i$, $i=1,2$. Then

$$\begin{aligned}\varphi_B(P_1\otimes P_2)&=\sup B\cap[0,P_1\otimes P_2]=\sup B_1\tilde{\otimes}B_2\cap[0,P_1\otimes P_2]\\&=\varphi_{B_1\widetilde{\bigotimes}B_2}(P_1\otimes P_2)=\varphi_{B_1}(P_1)\otimes\varphi_{B_2}(P_2).\end{aligned}$$

where the last equation follows from Lemma 1. □

3. Examples

3.1. **Generalized Lebesgue decomposition.** Let $\mathfrak{Q}_i$ be a set of p-measures on $\mathfrak{U}_i$, $i=1,2$, and let $\mathfrak{Q}_1\otimes\mathfrak{Q}_2=\{Q_1\otimes Q_2\colon Q_i\in\mathfrak{Q}_i,\ i=1,2\}$. We claim that

$$B(\mathfrak{Q}_1\otimes\mathfrak{Q}_2)=B(\mathfrak{Q}_1)\tilde{\otimes}B(\mathfrak{Q}_2).$$

The inclusion $\subset$ is obvious. To prove the converse inclusion, note that μ belongs to $B(\mathfrak{Q})$ if and only if $|\mu|\ll\mathfrak{Q}_0(Q(A)=0$ for every $Q\in\mathfrak{Q}_0$ implies $|\mu|(A)=0)$ for some countable subset $\mathfrak{Q}_0$ of $\mathfrak{Q}$. This follows from Proposition 1. Now let $P_i\in B(\mathfrak{Q}_i)$ be a p-measure, and choose a countable subset $\mathfrak{Q}_{i0}$ of $\mathfrak{Q}_i$ such that $P_i\ll\mathfrak{Q}_{i0}$, $i=1,2$. Then $P_1\otimes P_2\ll\mathfrak{Q}_{10}\otimes\mathfrak{Q}_{20}$, hence $P_1\otimes P_2\in B(\mathfrak{Q}_1\otimes\mathfrak{Q}_2)$, and our claim is proved. It follows from Theorem 2 (or Lemma 1) that $\varphi_{B(\mathfrak{Q}_1\otimes\mathfrak{Q}_2)}$ is multiplicative w.r.t. $B(\mathfrak{Q}_1)$ and $B(\mathfrak{Q}_2)$. In particular, $\varphi_{B(Q_1\otimes Q_2)}$ is multiplicative w.r.t. $B(Q_1)$ and $B(Q_2)$ for every p-measure Q_i on $\mathfrak{U}_i$. This re-establishes the known fact that the Lebesgue decomposition is multiplicative.

3.2. A variant of 3.1 is as follows. Let $B_i=\{\nu\in\mathrm{ca}(\mathfrak{U}_i)\colon|\nu|\ll\mathfrak{Q}_i\}$, and $B=\{\mu\in\mathrm{ca}(\mathfrak{U}_1\otimes\mathfrak{U}_2)\colon|\mu|\ll\mathfrak{Q}_1\otimes\mathfrak{Q}_2\}$. By Proposition 1, B and B_i are bands. Here we obviously have

$$B_1\tilde{\otimes}B_2\subset B\subset B_1\hat{\otimes}B_2.$$

Thus φ_B is multiplicative w.r.t. B_1 and B_2.

3.3. **Monogenic measures.** Let $\mathfrak{B}_i$ be a sub-σ-algebra of $\mathfrak{U}_i$, $i=1,2$. A measure $\nu\in\mathrm{ca}(\mathfrak{U}_i)_+$ is said to be $\mathfrak{B}_i$-monogenic if $\lambda\in\mathrm{ca}(\mathfrak{U}_i)_+$, $\lambda|\mathfrak{B}_i=\nu|\mathfrak{B}_i$ imply $\lambda=\nu$. In other words, the set $E(\nu|\mathfrak{B}_i,\mathfrak{U}_i)=\{\lambda\in\mathrm{ca}(\mathfrak{U}_i)_+\colon\lambda|\mathfrak{B}_i=\nu|\mathfrak{B}_i\}$ of all measure extensions of $\nu|\mathfrak{B}_i$ to $\mathfrak{U}_i$ coincides with $\{\nu\}$. Let $B_i=\{\nu\in\mathrm{ca}(\mathfrak{U}_i)\colon|\nu|$ is $\mathfrak{B}_i$-monogenic$\}$ and $B=\{\mu\in\mathrm{ca}(\mathfrak{U}_1\otimes\mathfrak{U}_2)\colon|\mu|$ is $\mathfrak{B}_1\otimes\mathfrak{B}_2$-monogenic$\}$. By [3], B and B_i are bands, $B_1\hat{\otimes}B_2\subset B$ and, in general, $B_1\hat{\otimes}B_2\ne B$. However, we have

$$B\cap[0,P_1\otimes P_2]\subset B_1\hat{\otimes}B_2$$

for every p-measure P_i on $\mathfrak{U}_i$, $i = 1, 2$. This is a consequence of the following lemma. Thus, by Theorem 2, being monogenic is a multiplicative property.

Lemma 3. *Suppose* $\mu \in \mathrm{ca}(\mathfrak{U}_1 \otimes \mathfrak{U}_2)_+$ *satisfies* $\mu \ll P_1 \otimes P_2$ *for some p-measures* P_i *on* $\mathfrak{U}_i$. *Then for* $i = 1, 2$, $\{\pi_i(\rho)\colon \rho \in E(\mu|\mathfrak{B}_1 \otimes \mathfrak{B}_2, \mathfrak{U}_1 \otimes \mathfrak{U}_2)\} = E(\pi_i(\mu)|\mathfrak{B}_i, \mathfrak{U}_i)$.

Proof. The inclusion $\subset$ is clear. To prove the converse inclusion, let f denote the Radon-Nikodym derivative of $\mu|\mathfrak{B}_1 \otimes \mathfrak{B}_2$ w.r.t. $P_1 \otimes P_2|\mathfrak{B}_1 \otimes \mathfrak{B}_2$. Then $g = \int f(\cdot, \omega_2)\, dP_2(\omega_2)$ is the Radon-Nikodym derivative of $\pi_1(\mu)|\mathfrak{B}_1$ w.r.t. $P_1|\mathfrak{B}_1$. Let $N = \{g = 0\}$, and define a Markov kernel K from $(\Omega_1, \mathfrak{U}_1)$ to $(\Omega_2, \mathfrak{U}_2)$ by

$$K(\omega_1, A_2) = \frac{\int_{A_2} f(\omega_1, \omega_2)\, dP_2(\omega_2)}{g(\omega_1)}$$

if $\omega_1 \notin N$ and

$$K(\omega_1, A_2) = Q(A_2)$$

otherwise, where Q is an arbitrary p-measure on $\mathfrak{U}_2$. Then $N \in \mathfrak{B}_1$, $\pi_1(\mu)(N) = 0$ and for the generalized product measure $\pi_1(\mu) \otimes K \in \mathrm{ca}(\mathfrak{U}_1 \otimes \mathfrak{U}_2)_+$ given by

$$\pi_1(\mu) \otimes K(A_1 \times A_2) = \int_{A_1} K(\omega_1, A_2)\, d\pi_1(\mu)(\omega_1),$$

we have $\pi_1(\mu) \otimes K|\mathfrak{B}_1 \otimes \mathfrak{B}_2 = \mu|\mathfrak{B}_1 \otimes \mathfrak{B}_2$. Now let $\lambda \in E(\pi_1(\mu)|\mathfrak{B}_1, \mathfrak{U}_1)$, and let $\rho = \lambda \otimes K$. Then $\pi_1(\rho) = \lambda$ and since $K(\cdot, B_2)$ is $\mathfrak{B}_1$-measurable for every $B_2 \in \mathfrak{B}_2$, we obtain $\rho|\mathfrak{B}_1 \otimes \mathfrak{B}_2 = \pi_1(\mu) \otimes K|\mathfrak{B}_1 \otimes \mathfrak{B}_2 = \mu|\mathfrak{B}_1 \otimes \mathfrak{B}_2$. This completes the proof (for $i = 1$). □

3.4. **Extensible measures.** Let $\mathfrak{C}_i$ be a σ-algebra on Ω_i with $\mathfrak{U}_i \subset \mathfrak{C}_i$, $i = 1, 2$. Let $B_i = \{\nu \in \mathrm{ca}(\mathfrak{U}_i)\colon\ E(|\nu|, \mathfrak{C}_i) \neq \emptyset\}$ and let $B = \{\mu \in \mathrm{ca}(\mathfrak{U}_1 \otimes \mathfrak{U}_2)\colon E(|\mu|, \mathfrak{C}_1 \otimes \mathfrak{C}_2) \neq \emptyset\}$. By [4], [5], B and B_i are bands (which is also an easy consequence of Proposition 1; cf. [1]), and clearly

$$B_1 \tilde{\otimes} B_2 \subset B \subset B_1 \hat{\otimes} B_2 .$$

Thus the property of admitting a measure extension is a multiplicative property.

3.5. **Unique measure extensions.** In the situation of 3.4, let $B_i = \{\nu \in \mathrm{ca}(\mathfrak{U}_i)\colon \#E(|\nu|, \mathfrak{C}_i) = 1\}$ and $B = \{\mu \in \mathrm{ca}(\mathfrak{U}_1 \otimes \mathfrak{U}_2)\colon \#E(|\mu|, \mathfrak{C}_1 \otimes \mathfrak{C}_2) = 1\}$. By [4], B and B_i are bands (cf. also [1]). From 3.3 it follows that $B_1 \tilde{\otimes} B_2 \subset B$. Note that, in general, $B \not\subset B_1 \hat{\otimes} B_2$ (cf. [3]). The following lemma shows that

$$B \cap [0, P_1 \otimes P_2] \subset B_1 \hat{\otimes} B_2$$

for every p-measure P_i on $\mathfrak{U}_i$ and therefore, unique extensibility of measures is a multiplicative property.

Lemma 4. *Suppose* $\mu \in \mathrm{ca}(\mathfrak{U}_1 \otimes \mathfrak{U}_2)_+$ *satisfies* $E(\mu, \mathfrak{C}_1 \otimes \mathfrak{C}_2) \neq \emptyset$ *and* $\mu \leq P_1 \otimes P_2$ *for some p-measures* P_i *on* $\mathfrak{U}_i$. *Then for* $i = 1, 2$, $\{\pi_i(\rho)\colon \rho \in E(\mu, \mathfrak{C}_1 \otimes \mathfrak{C}_2)\} = E(\pi_i(\mu), \mathfrak{C}_i)$.

Proof. We claim that there exists an extension of μ to $\mathfrak{C}_1 \otimes \mathfrak{C}_2$ which is majorized by a (direct) product measure. By 3.4, extensibility is a multiplicative property. Therefore, the band component of $P_1 \otimes P_2$ w.r.t. this property is a product measure $\nu_1 \otimes \nu_2$, $\nu_i \in \mathrm{ca}(\mathfrak{U}_i)_+$. From the assumptions follow $\mu \leq \nu_1 \otimes \nu_2$. Let f denote the Radon-Nikodym derivative of μ w.r.t. $\nu_1 \otimes \nu_2$, and let $\lambda_i \in E(\nu_i, \mathfrak{C}_i)$. Define $\rho \in \mathrm{ca}(\mathfrak{C}_1 \otimes \mathfrak{C}_2)_+$ by $\rho(C) = \int_C f\, d\lambda_1 \otimes \lambda_2$. Then $\rho \in E(\mu, \mathfrak{C}_1 \otimes \mathfrak{C}_2)$, and $\rho \leq \lambda_1 \otimes \lambda_2$. Our claim is proved. Now the assertion follows from Lemma 3. □

3.6. **Extreme extensions.** In the situation of 3.4, let

$$B_i = \{\nu \in \mathrm{ca}(\mathfrak{U})_i \colon \mathrm{ex}\, E(|\nu|, \mathfrak{C}_i) \neq \varnothing\}, \qquad \text{where } \mathrm{ex}\, E(|\nu|, \mathfrak{C}_i)$$

denotes the set of all extreme points of the convex set $E(|\nu|, \mathfrak{C}_i)$. This set is characterized by the "Douglas criterion" (cf. [7]): $\lambda \in E(|\nu|, \mathfrak{C}_i)$ is an extreme point if and only if, for each $C \in \mathfrak{C}_i$ there exists $A \in \mathfrak{U}_i$ such that $\lambda(C \Delta A) = 0$. Let $B = \{\mu \in \mathrm{ca}(\mathfrak{U}_1 \otimes \mathfrak{U}_2) \colon \mathrm{ex}\, E(|\mu|, \mathfrak{C}_1 \otimes \mathfrak{C}_2) \neq \varnothing\}$. By [6], B and B_i are bands. We have

$$B_1 \tilde{\otimes} B_2 \subset B.$$

In fact, let $Q_i \in B_i$ be p-measures, $P_i \in \mathrm{ex}\, E(Q_i, \mathfrak{C}_i)$, and $C_i \in \mathfrak{C}_i$, $i = 1, 2$. By the Douglas criterion, one can find sets $A_i \in \mathfrak{U}_i$ such that $P_i(C_i \Delta A_i) = 0$. Then

$$C_1 \times C_2 \Delta A_1 \times A_2 \subset C_1 \Delta A_1 \times \Omega_2 \cup \Omega_1 \times C_2 \Delta A_2$$

and hence, $P_1 \otimes P_2(C_1 \times C_2 \Delta A_1 \times A_2) = 0$. Again by the Douglas criterion, this gives $P_1 \otimes P_2 \in \mathrm{ex}\, E(Q_1 \otimes Q_2, \mathfrak{C}_1 \otimes \mathfrak{C}_2)$ because $\{C \in \mathfrak{C}_1 \otimes \mathfrak{C}_2 \colon$ there exists $A \in \mathfrak{U}_1 \otimes \mathfrak{U}_2$ with $P(C \Delta A) = 0\}$ is a σ-algebra for every p-measure P on $\mathfrak{C}_1 \otimes \mathfrak{C}_2$. Thus $Q_1 \otimes Q_2 \in B$.

However, $Q_1 \otimes Q_2 \in B$ does not imply $Q_i \in B_i$, $i = 1, 2$. Thus, by Theorem 2, existence of an extreme extension is not a multiplicative property. To see this, let $\Omega_1 = \Omega_2 = [0, 1]$, $\mathfrak{U}_1 = \mathfrak{C}_1 = \mathfrak{C}_2 = \mathfrak{C}$ (the Borel σ-algebra), and $\mathfrak{U}_2 = \mathfrak{U}$ (the σ-algebra consisting of all countable sets and their complements). Let $Q = \lambda \otimes \lambda|\mathfrak{U}$, where λ denotes the Lebesgue measure. Clearly $\mathrm{ex}\, E(\lambda|\mathfrak{U}, \mathfrak{C}) = \varnothing$. Define a p-measure P on $\mathfrak{C}^2$ by $P(C) = \int 1_C(\omega, \omega)\, d\lambda(\omega)$. Then $P|\mathfrak{C} \otimes \mathfrak{U} = Q$. Let $C_i \in \mathfrak{C}$, $i = 1, 2$, and let $A = C_1 \cap C_2 \times [0, 1]$. Since $C_1 \times C_2 \Delta A$ is contained in the complement of the diagonal and P is concentrated on the diagonal, $P(C_1 \times C_2 \Delta A) = 0$. From the Douglas criterion it follows that $P \in \mathrm{ex}\, E(Q, \mathfrak{C}^2)$.

4. Invariant measures

In this section, we establish a generalization of Theorem 2 to invariant measures. Let G_i be a semigroup acting on Ω_i (from the left) and assume that $\mathfrak{U}_i$ is a G_i-invariant σ-algebra; i.e., $g^{-1}A = \{\omega \in \Omega_i \colon g\omega \in A\} \in \mathfrak{U}_i$ for every $g \in G_i$, $A \in \mathfrak{U}_i$, $i = 1, 2$. Then G_i acts on $\mathrm{ca}(\mathfrak{U}_i)$ by $g\nu(A) = \nu(g^{-1}A)$. Let $\mathrm{ca}(\mathfrak{U}_i)_{G_i} = \{\nu \in \mathrm{ca}(\mathfrak{U}_i) \colon g\nu = \nu$ for every $g \in G_i\}$ be the order-complete vector sublattice of $\mathrm{ca}(\mathfrak{U}_i)$ consisting of G_i-invariant elements (cf. [4]). Furthermore, let G be a semigroup acting on $\Omega_1 \times \Omega_2$ such that $\mathfrak{U}_1 \otimes \mathfrak{U}_2$ is G-invariant. To relate the semigroup actions, make the following assumptions:

(C.1) If $\nu_i \in \mathrm{ca}(\mathfrak{U}_i)_+$ is G_i-invariant, $i = 1, 2$, then $\nu_1 \otimes \nu_2$ is G-invariant.
(C.2) If $\mu \in \mathrm{ca}(\mathfrak{U}_1 \otimes \mathfrak{U}_2)_+$ is G-invariant, then $\pi_i(\mu)$ is G_i-invariant, $i = 1, 2$.

For instance, if $G = G_1 \times G_2$, the direct product of G_1 and G_2, acting on $\Omega_1 \times \Omega_2$ by $(g_1, g_2)(\omega_1, \omega_2) = (g_1\omega_1, g_2\omega_2)$, then $\mathfrak{U}_1 \otimes \mathfrak{U}_2$ is G-invariant and (C.1), (C.2) are satisfied. Also, if $G_1 = G_2 = G$, where G acts componentwise on $\Omega_1 \times \Omega_2$, then the above conditions are satisfied.

Let B_i be a band in $\mathrm{ca}(\mathfrak{U}_i)_{G_i}$, and let B be a band in $\mathrm{ca}(\mathfrak{U}_1 \otimes \mathfrak{U}_2)_G$. The band projection $\varphi_{B,G}$ of $\mathrm{ca}(\mathfrak{U}_1 \otimes \mathfrak{U}_2)_G$ onto B is said to be multiplicative w.r.t. B_1 and B_2 if

$$\varphi_{B,G}(P_1 \otimes P_2) = \varphi_{B_1,G_1}(P_1) \otimes \varphi_{B_2,G_2}(P_2)$$

for every G_i-invariant p-measure P_i on $\mathfrak{U}_i$, $i = 1, 2$. Here we use (C.1). The product bands of B_1 and B_2 in $\mathrm{ca}(\mathfrak{U}_1 \otimes \mathfrak{U}_2)_G$ are defined as in §2 with band generation in $\mathrm{ca}(\mathfrak{U}_1 \otimes \mathfrak{U}_2)_G$ and are denoted by $B_1 \widetilde{\otimes}_G B_2$ and $B_1 \widehat{\otimes}_G B_2$. Using (C.2) and, e.g., [6, Lemma 2], one can easily see that

$$B_1 \widehat{\bigotimes}_G B_2 = \{\mu \in \mathrm{ca}(\mathfrak{U}_1 \otimes \mathfrak{U}_2)_G : \pi_i(|\mu|) \in B_i,\ i = 1, 2\}.$$

Analogous to Theorem 2 we have the following:

Theorem 3. *Suppose* (C.1) *and* (C.2) *are satisfied. Then the following statements are equivalent*:

(i) $\varphi_{B,G}$ *is multiplicative w.r.t.* B_1 *and* B_2.
(ii) $B_1 \widetilde{\otimes}_G B_2 \subset B$ *and* $B \cap [0, P_1 \otimes P_2]_G \subset B_1 \widehat{\otimes}_G B_2$ *for every* G_i*-invariant* p*-measure* P_i *on* $\mathfrak{U}_i$, $i = 1, 2$ *(*$[0, P_1 \otimes P_2]_G$ *is the order interval in* $\mathrm{ca}(\mathfrak{U}_1 \otimes \mathfrak{U}_2)_G$*)*.
(iii) $B \cap [0, P_1 \otimes P_2]_G = B_1 \widetilde{\otimes}_G B_2 \cap [0, P_1 \otimes P_2]_G$. □

Combining this theorem with results in [2], [4], and [5], one may check that under suitable assumptions on the semigroups, invariant versions of the examples given in §3 are valid.

Acknowledgment

I would like to thank to D. Plachky, who brought this topic to my attention.

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