PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 111, Number 1, January 1991

CONSTRUCTION OF CHARACTERISTIC FUNCTIONS FOR CLASSES OF INFINITELY DIFFERENTIABLE FUNCTIONS

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(Communicated by R. Daniel Mauldin)

ABSTRACT. Characteristic functions are constructed for classes of infinitely differentiable functions defined on a half-line and for Paley–Wiener classes. A corresponding result is given for normal operators defined in Hilbert space.

1. INTRODUCTION

Let M_n be a sequence of positive numbers. We denote by $C\{M_n\}$ the class of infinitely differentiable functions $f(x)$, real valued and satisfying the inequality

$$|f^{(n)}(x)| \leq AB^n M_n, \qquad n = 0, 1, \ldots, \tag{1.1}$$

for all $x \in \mathbf{R}$ and some constants A and B (which may depend on f). According to Gorny's inequalities [6], we can replace the sequence M_n by one that is logarithmically convex, and this, without altering the class $C\{M_n\}$. So now we assume the sequence M_n to be logarithmically convex. Following Bang's terminology, we shall term a function $h(x)$ characteristic of the class $C\{M_n\}$ if it belongs to this class and if, for some positive constants C and D, $CD^n M_n \leq \sup_{-\infty<x<\infty} |h^{(n)}(x)|$, $n \geq 0$. Gorny [6] was first to construct such functions. Using a method of Cartan [4], who dealt with this problem in the case of functions defined on a bounded interval, Bang [3] has given another construction of a characteristic function of $C\{M_n\}$. It can be written as

$$h(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\cos(-\frac{1}{4} + r_k x)}{T(r_k)}, \qquad -\infty < x < \infty, \tag{1.2}$$

where $T(r) = \sup_{n\geq 0} r^n/M_n$, $r > 0$, and r_k is chosen so that $M_k = r_k^k/T(r_k)$ (Bang's choice of r_k is M_{k+1}/M_k).

Our aim is to extend Cartan's method to other types of classes.

Received by the editors August 16, 1989.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 26E10; Secondary 47B15.

Key words and phrases. Classes of infinitely differentiable functions, characteristic functions, normal operators.

2. The classes $C_d\{M_n\}$

The class $C_d\{M_n\}$ consists of those functions $f(x)$ defined and infinitely differentiable on the half-line $x \geq 0$, which satisfy (1.1). As a consequence of Gorny's inequalities, we have $C_d\{M_n\} = C_d\{M_n^d\}$. Here M_n^d is the upper envelope of all sequences bounded by M_n and of the form ar^n/n^n, $n \geq 0$, where a and r are nonnegative constants. We will assume that $M_n = M_n^d$. We construct first a characteristic function for $C_d\{M_n\}$ in the case $M_n = ar^n/n^n$ (in the case $C\{M_n\}$ where M_n is logarithmically convex, this sequence M_n is the upper envelope of sequences of the form ar^n; a characteristic function for $C\{ar^n\}$ is then $a\cos(-\pi/4+rx)$; compare with (1.2); in all these subcases one must obtain the same constants A, B, C, and D).

Lemma 1. *Let a and r be two positive constants. If $h_{a,r}(x) = a\cos(\sqrt{rx})$, $x \geq 0$, then*

$$|h_{a,r}^{(n)}(x)| \leq e^n(ar^n/n^n), \qquad n \geq 0, \ x \geq 0, \tag{2.1}$$

$$h_{a,r}^{(n)}(0) = (-1)^n ar^n n!/(2n)!, \qquad n \geq 0. \tag{2.2}$$

Proof. We can assume $a = r = 1$. Then $h_{1,1}(x) = h(x)$, where $h(z) = \sum_{n=0}^{\infty}(-z)^n/(2n)!$, $z \in \mathbf{C}$, and (2.2) follows.

For $\rho > 0$, let $\Omega_\rho = \{z = x+iy \in \mathbf{C}\colon$ either $|z| < \rho$, or $x > 0$ and $|y| < \rho\}$. Now, the image of $\Omega_{\sqrt{\rho}}$ by the mapping $z \mapsto z^2$ contains Ω_ρ so, since $h(z^2) = \cos z$ and $|\cos z| \leq e^{|y|}$, we obtain

$$\sup_{z\in\Omega_\rho} |h(z)| \leq e^{\sqrt{\rho}}, \qquad \rho > 0. \tag{2.3}$$

If we apply Cauchy's inequalities to $h(z)$ with a disk $D_{n^2}(x_0) \subset \Omega_{n^2}$ centered at $x_0 \geq 0$ and with radius n^2 we obtain, using (2.3),

$$|h^{(n)}(x_0)| \leq e^n n!/n^{2n} \leq e^n/n^n, \qquad n \geq 0,$$

and (2.1) follows. □

Theorem 1. *Let M_n be a sequence of positive numbers with $M_n = M_n^d$. Then, there exists a function $h(x) \in C_d\{M_n\}$ such that, for some positive constants C and D,*

$$CD^n M_n \leq |h^{(n)}(0)|, \qquad n \geq 0.$$

Proof. If one defines $H(r) = \sup_{n\geq 0} r^n/(n^n M_n)$, $r > 0$ (see [1]) then, since $M_n = M_n^d$, we have

$$M_n = \sup_{r>0} \frac{r^n}{H(r)n^n}, \qquad n \geq 0. \tag{2.4}$$

For each n, let $r_n > 0$ be a value of r giving the supremum. We put (compare with (1.2))

$$h(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\cos(\sqrt{r_k x})}{H(r_k)}, \qquad x \geq 0. \tag{2.5}$$

Using (2.1), we obtain for $n \geq 0$ and $x \geq 0$,

$$|h^{(n)}(x)| \leq \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{e^n r_k^n}{H(r_k) n^n} \leq 2e^n M_n,$$

so that $h(x) \in C_d\{M_n\}$.

Also from (2.2) we have that the nth derivative at 0 of all terms in (2.5) are of the same sign, and so, for $n \geq 0$,

$$|h^{(n)}(0)| \geq \frac{1}{2n} \frac{r_n^n n!}{H(r_n)(2n)!} \geq \frac{1}{4n} \frac{r_n^n}{H(r_n) n^n} = \frac{1}{4^n} M_n. \quad \square$$

Remarks. The classes $C_d\{M_n\}$ are introduced by Mandelbrojt [9] in the context of a theory of generalized quasi-analyticity. Agmon [1] studied the problem of the equivalence (or, more generally, the inclusion) of such classes: $C_d\{M_n\} \subset C_d\{L_n\}$ if and only if $M_n^d \leq AB^n L_n$, $n \geq 0$, for some constants A and B. This follows from the equality $C_d\{M_n\} = C_d\{M_n^d\}$ and Theorem 1. However, Agmon's statement that $\underline{\lim}\, M_n^{1/n} < \infty$ implies $C_d\{M_n\} = C_d\{1\}$ is not correct. For instance we have $C_d\{n^{-n}\} \neq C_d\{1\}$. Then he used Laguerre polynomials in conjunction with the condition $\underline{\lim}\, M_n^{1/n} = \infty$. Mandelbrojt (see [10, p. 230]) also considered the question under the assumption $\underline{\lim}\, M_n^{1/n} > 0$.

The function $\cos(\sqrt{x})$ is a special case of the Mittag–Leffler functions $E_\alpha(z)$ (see, i.e. [11]). Korenbljum [7] had used these functions to show, in particular, that $C_d\{M_n\} \neq \{0\}$ if $\underline{\lim}\, n M_n^{1/n} > 0$.

3. The Paley-Wiener classes

For a sequence of positive numbers M_n and for $1 \leq p < \infty$, let $L^p\{M_n\}$ denote the class of complex-valued, infinitely differentiable functions $f(x)$, $x \in \mathbf{R}$, which satisfy, for some constants A and B,

$$\left\{ \int_{-\infty}^{\infty} |f^{(n)}(x)|^p \, dx \right\}^{1/p} \leq AB^n M_n, \qquad n \geq 0.$$

As in the $C\{M_n\}$ case, we can assume that the sequence M_n is logarithmically convex (see, i.e., [13]).

Let $\varphi(\xi)$, $\xi \in \mathbf{R}$, be a function of class C^2, nonnegative, not identically zero, and with compact support contained in $(-\infty, 0]$. We put

$$\varphi_r(\xi) = \varphi(\xi - r), \qquad h_r(x) = \hat{\varphi}_r(x) = \int_{-\infty}^{\infty} \varphi_r(\xi) e^{ix\xi} \, d\xi$$

and

$$h(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{h_{r_k}(x)}{T(r_k)}, \qquad x \in \mathbf{R}, \tag{3.1}$$

where $T(r)$ and r_k are as in §1.

Lemma 2. *Let $K > 0$ be such that* $\operatorname{supp} \varphi \subset [-K, 0]$. *Then there exist constants A and B such that for all $r > 0$ and $1 \le p < \infty$,*

$$\|h_r^{(n)}\|_p \le A(B \max(r, K))^n, \qquad n \ge 0. \tag{3.2}$$

We also have, if $r > K$,

$$\|h_r^{(n)}\|_2 \ge \sqrt{2\pi}\|\varphi\|_2 (r - K)^n, \qquad n \ge 0. \tag{3.3}$$

Proof. Since $h_r^{(n)}(x)$ is the Fourier transform of $(i\xi)^n \varphi_r(\xi)$, we have

$$\|h_r^{(n)}\|_\infty \le \|\xi^n \varphi_r(\xi)\|_1 \le r_0^n \|\varphi\|_1, \qquad n \ge 0 \tag{3.4}$$

where $r_0 = \max(r, K)$. Also $-x^2 h_r^{(n)}(x)$ is the Fourier transform of

$$d^2((i\xi)^n \varphi_r(\xi))/d\xi^2$$

and

$$\begin{aligned} \|x^2 h_r^{(n)}(x)\|_\infty &\le \left\| \frac{d^2}{d\xi^2}((i\xi)^n \varphi_r(\xi)) \right\|_1 \\ &\le n(n-1) r_0^{n-2} \|\varphi\|_1 + 2n r_0^{n-1} \|\varphi'\|_1 + r_0^n \|\varphi''\|_1 . \end{aligned} \tag{3.5}$$

It follows from (3.4) and (3.5) that for some constants A_1 and B_1, $|h_r^{(n)}(x)| \le A_1 B_1^n r_0^n \min(1, x^{-2})$, $x \in \mathbf{R}$. From this inequality, we obtain (3.2).

Now we assume $r > K$. By Plancherel's formula, we have

$$\|h_r^{(n)}\|_2^2 = 2\pi \|\xi^n \varphi_r(\xi)\|_2^2 \ge 2\pi (r - K)^{2n} \|\varphi\|_2^2$$

and obtain (3.3). □

Lemma 3. *Let $f(x)$, $x \in \mathbf{R}$, be measurable. If $\|f\|_1 \le A$ and $\|f\|_2 \ge B$ then $\|f\|_p > (A/B)^{2/p}(B^2/A)$ for $p \ge 2$. If $\|f\|_p \le A$ for all $p \ge 2$ and $\|f\|_2 \ge B$, then $\|f\|_p \ge B$ for $1 \le p \le 2$.*

Proof. By Hölder's inequality, $\log \|f\|_p^p$ is a convex function of p. The second statement of the lemma follows immediately. Also, if $\alpha = \log(B^2/A)$ and $\beta = \log(A/B)^2$, $\alpha p + \beta$ is the linear function of p, equal to $\log A$ at $p = 1$ and to $\log B^2$ at $p = 2$, and we obtain for $p \ge 2$, $\log((B^2/A)^p (A/B)^2) = \alpha p + \beta \le \log \|f\|_p^p$. This gives the first statement. □

Now we will show that $h(x)$, as given by (3.1), is characteristic of all the classes $L^p\{M_n\}$, $1 \le p < \infty$.

Theorem 2. *Let* M_n *be a sequence of positive numbers, logarithmically convex. Then the function* $h(x)$, *as given by* (3.1), *belongs to* $L^p\{M_n\}$, $1 \le p < \infty$. *Also, for each such* p, *there exist positive constants* C *and* D *such that*

$$CD^n \le \|h^{(n)}\|_p, \qquad n \ge 0. \tag{3.6}$$

Proof. Let K be as in Lemma 2. Then, using (3.2),

$$\|h^{(n)}\|_p \le A(BK)^n \sum_{r_k<K} \frac{1}{2^k}\frac{1}{T(r_k)} + AB^n \sum_{r_k \ge K} \frac{1}{2^k}\frac{r_k^n}{T(r_k)}.$$

The first sum is bounded by a constant D (dependent only on K and the sequence M_n). Also, for some $\delta > 0$, $\delta^n \le M_n$, $n \ge 0$, since M_n is logarithmically convex. We thus obtain, using the definition of $T(r)$ (see §1), $\|h^{(n)}\|_p \le AD(BK/\delta)^n M_n + AB^n M_n$ and it follows that $h \in L^p\{M_n\}$.

By Lemma 3 it is sufficient to show (3.6) for $p = 2$. First we assume that the sequence $M_n^{1/n}$ is unbounded. Then $\underline{\lim}\, r_k = \infty$. Since it is also sufficient to prove (3.6) for large values of n, we may assume that n is such that $r_n > 2K$. By Plancherel's formula and the positivity of φ, we obtain

$$\begin{aligned} \|h^{(n)}\|_2^2 &\ge 2\pi \int_0^\infty \left| \xi^n \sum_{k=0}^\infty \frac{1}{2^k} \frac{\varphi_{r_k}(\xi)}{T(r_k)} \right|^2 d\xi \\ &\ge \frac{\pi}{2^{n-1}} \int_0^\infty \xi^{2n} \left(\frac{\varphi_{r_n}(\xi)}{T(r_n)} \right)^2 d\xi \\ &= \frac{\pi}{2^{n-1} T(r_n)^2} \|h_{r_n}^{(n)}\|_2^2. \end{aligned} \tag{3.7}$$

Now using (3.3), the inequality $r_n > 2K$ and the defining property of r_n (see §1) we deduce that $\|h^{(n)}\|_2^2 \ge \pi^2 \|_2^2 M_n^2$ and obtain (3.6).

Finally, we assume that the sequence $M_n^{1/n}$ is bounded. It is then sufficient to verify (3.6) with $M_n \equiv 1$. By Plancherel's theorem and the positivity of φ, we have

$$\|h^{(n)}\|_2^2 \ge 2\pi \int_{-\infty}^\infty \xi^{2n} f(\xi)\, d\xi,$$

where $f(\xi) \ge 0$ is continuous and not identically zero. On some closed interval I not containing zero, $f(\xi) \ge \delta > 0$. So, if $D = \inf_{\xi \in I} |\xi|^2$, we obtain $\|f^{(n)}\|_2^2 \ge 2\pi\delta |I| D^n$ and (3.6) follows. □

Remarks. The classes $L^2\{M_n\}$ were introduced by Paley and Wiener [12] in their study of the problem of quasi-analyticity by means of the Fourier transformation.

Mandelbrojt [8] proved the following statement. Let M_n and L_n be two sequences of positive numbers, and assume M_n is logarithmically convex; then $L^2\{M_n\} \subset L^2\{L_n\}$ if and only if $M_n \le AB^n L_n$, $n \ge 0$, provided $\log M_n = O(n^2)$. The statement follows immediately, without this last condition, from

Theorem 2. Using a topological method of Hormander, Couture [5] solved the problem of the equivalence of the classes $L^p\{M_n\}$, $1 \le p < \infty$ (same statement as for $p = 2$). This also follows from Theorem 2.

4. The classes $H^T\{M_n\}$

Let H be a complex Hilbert space and T, densely defined (possibly unbounded) normal operator in H. Let M_n be a sequence of positive numbers. We define $H^T\{M_n\}$ as the "class" of vectors $v \in H$ belonging to the domain of definition of all iterates T^n of T and satisfying

$$\|T^n v\| \le AB^n M_n, \qquad n \ge 0,$$

for some constants A and B. Since $\|T^n v\|$ is a logarithmically convex sequence (see, for instance the appendix in [2]), we may assume the same of M_n. If we take, for example, $H = L^2(\mathbf{R})$ and $T = d/dx$ we have $H^T\{M_n\} = L^2\{M_n\}$.

Let dP denote the spectral decomposition of T, and $P_{r,s}$, $r < s$, the subspace of H corresponding by dP to the ring $\{z \in \mathbf{C}\colon r < |z| \le s\}$.

Lemma 4. *If $v \in P_{r,s}$ has norm $\|v\| = 1$, then $r^n \le \|T^n v\| \le s^n$, $n \ge 0$.*

Proof. We have for all $n \ge 0$,

$$\|T^n v\| = \int_{\mathbf{C}} |\lambda|^{2n} \|dP_\lambda v\|^2 = \int_{r<|\lambda|\le s} |\lambda|^{2n} \|dP_\lambda v\|^2 .$$

The lemma follows. □

Theorem 3. *We assume that there exists a constant δ, $0 < \delta < 1$, such that $P_{\delta r, r} \ne \{0\}$ if r is sufficiently large. Then there exists a characteristic vector v for the class $H^T\{M_n\}$; that is, $v \in H^T\{M_n\}$ and, for some positive constants C and D, $CD^n \le \|T^n v\|$, $n \ge 0$.*

Proof. Take r_0 so that $r \ge r_0$ implies $P_r = P_{\delta r, r} \ne \{0\}$. First let us assume that the sequence M_{n+1}/M_n is unbounded. We choose two sequences of real numbers, k_i and r_i, $i \ge 1$, such that

(i) $r_i > r_0$ and k_i are positive integers;

(ii) $M_{k_i} = r_i^{k_i}/T(r_i)$ (see §1);

(iii) k_{i+1} is the smallest integer k such that $\delta^{-1} r_i \le M_k/M_{k-1}$, $i \ge 1$.

Condition (ii) implies $M_{k_i}/M_{k_i-1} \le r_i$ so, $k_{i+1} > k_i$ and $r_{i+1} \ge \delta^{-1} r_i$ by (iii). The sequence k_i is thus strictly increasing and the rings $\{\lambda \in \mathbf{C}\colon \delta r_i < |\lambda| < r_i\}$, $i \ge 1$, are disjoint. The corresponding subspaces P_{r_i} are then mutually orthogonal. By (i), $P_{r_i} \ne \{0\}$ and we choose a vector $v_i \in P_{r_i}$ with

$\|v_i\| = 1$. We then put

$$v = \sum_{i=1}^{\infty} \frac{1}{2^{k_i}} \frac{v_i}{T(r_i)}.$$

As $T^n v_i \in P_{r_i}$, the vectors $T^n v_i$, $i \geq 1$, are mutually orthogonal and we have

$$\|T^n v\|^2 = \sum_{i=1}^{\infty} \frac{1}{2^{2k_i}} \frac{\|T^n v_i\|^2}{T(r_i)^2}, \qquad n \geq 0. \tag{4.1}$$

By Lemma 4, $\|T^n v\|^2 \leq \sum_{i=1}^{\infty} (1/2^{2k_i})(r_i^{2n}/T(r_i)^2)$ and so, using the definition of $T(r)$ (see §1), $\|T^n v\| \leq \{\sum_{i=1}^{\infty} 1/2^{2k_i}\}^{1/2} M_n$, $n \geq 0$. We have shown that $v \in H^T\{M_n\}$.

On the other hand, given an integer $n \geq k_1$, we take i as the greatest integer such that $k_i \leq n$ and using (4.1) we obtain Lemma 4 and (ii),

$$\|T^n v\| \geq \frac{1}{2^{2k_i}} \frac{(\delta r_i)^n}{T(r_i)} \geq \frac{r_i^{n-k_i}}{(4\delta^{-1})^n} M_{k_i}. \tag{4.2}$$

Since $n < k_{i+1}$, by (iii) we have $M_n/M_{n-1} \leq \delta^{-1} r_i$ and so, using logarithmic convexity, $M_n \leq M_{k_i}(\delta^{-1} r_i)^{n-k_i}$. Combining this inequality with (4.2) we obtain $\|T^n v\| \geq (\delta^2/4)^n M_n$, and v satisfies the requirements of the theorem.

Finally, if the sequence M_{n+1}/M_n is bounded, one may simply choose r so that $P_r \neq \{0\}$ and take v equal to a vector of norm 1 in P_r. Using Lemma 4, one easily obtains that this vector v is characteristic of the class $H^T\{M_n\}$. □

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