PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 111, Number 1, January 1991

A FURTHER GENERALIZATION OF THE KNASTER-KURATOWSKI-MAZURKIEWICZ THEOREM

NAOKI SHIOJI

(Communicated by R. Daniel Mauldin)

ABSTRACT. Granas and Dugundji obtained the following generalization of the Knaster–Kuratowski–Mazurkiewicz theorem.

Let X be a subset of a topological vector space E and let G be a set-valued map from X into E such that for each finite subset $\{x_1, \dots, x_n\}$ of X, $\mathrm{co}\{x_1, \dots, x_n\} \subset \bigcup_{i=1}^n Gx_i$ and for each $x \in X$, Gx is finitely closed, i.e., for any finite-dimensional subspace L of E, $Gx \cap L$ is closed in the Euclidean topology of L. Then $\{Gx: x \in X\}$ has the finite intersection property.

By relaxing, among others, the condition that X is a subset of E, we obtain a further generalization of the theorem and show some of its applications.

1. INTRODUCTION

In 1961 Fan [5] showed an infinite-dimensional version of the classical Knaster–Kuratowski–Mazurkiewicz theorem [13]. We can find some versions of Fan's theorem in [2, 7, 14, 18]. One of them is Theorem A which was proved by Dugundji and Granas [2]. First we state some definitions. Let X be a subset of a vector space E. A set-valued map G from X into E is called a Knaster–Kuratowski–Mazurkiewicz map or simply a KKM map if for each finite subset $\{x_1, \dots, x_n\}$ of X,

$$\mathrm{co}\{x_1, \dots, x_n\} \subset \bigcup_{i=1}^{n} Gx_i.$$

We can find many examples of KKM maps in [10, 14]. A subset of a vector space E is called finitely closed if its intersection with each finite-dimensional linear space $L \subset E$ is closed in the Euclidean topology of L.

Theorem A (Dugundji and Granas). *Let E be a vector space, X an arbitrary subset of E, and $G: X \to 2^E$ a KKM map such that each Gx is finitely closed. Then the family $\{Gx: x \in X\}$ of sets has the finite intersection property.*

The object of this paper is to obtain a generalization of Theorem A by relaxing, among others, the condition that X is a subset of E. The idea is inspired

Received by the editors November 8, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47H10, 58C30.
Key words and phrases. Fixed point, KKM-map, minimax theorem.

by Theorem 3 in [11] which is a simple result of our theorem. Our main result is Theorem 1 and its proof relies on Górniewicz's fixed point theorem in [8]. As applications of our theorem, we show two system theorems concerning inequalities and a minimax theorem. In [16, 17], Simons classifies minimax theorems into two groups. One is of a fixed point type and the other is of a Hahn–Banach type. Theorem B, which is Theorem 1.4 in [17], is a minimax theorem of a typical fixed point type. Recall that a function $f: X \to R$ is called quasi-convex if for any real number α, $\{x \in X: f(x) \leq \alpha\}$ is convex, where X is a convex subset of a vector space. f is called quasi-concave if $-f$ is quasi-convex.

Theorem B (Simons). *Let X be a nonempty convex subset of a topological vector space, let Y be a nonempty compact convex subset of a topological vector space, let $f: X \times Y \to R$ be quasi-concave in its first variable and lower semicontinuous in its second variable, let $g: X \times Y \to R$ be upper semicontinuous in its first variable and quasi-convex in its second variable, and let $f \leq g$ on $X \times Y$. Then*

$$\min_{y\in Y}\sup_{x\in X} f(x, y) \leq \sup_{x\in X}\inf_{y\in Y} g(x, y).$$

Simons deduced this theorem from the KKM theorem. Theorem 6 in this paper is a minimax theorem of a typical Hahn–Banach type. Simons deduced Theorem 6 from the Hahn–Banach theorem. We deduce Theorem 6 from our main result, Theorem 1. Since Theorem 1 is a generalization of the KKM theorem, minimax theorems of both types are easily obtained from Theorem 1.

2. Main results

In this paper all topological structures are implicitly assumed to satisfy the Hausdorff separation axiom and by homology, we understand the Čech homology [4, 9] with rational coefficients. Let G be a set-valued map from a set X into a set Y; we denote by $G(A)$ the subset $\bigcup_{x\in A} Gx$ of Y. Let D be a subset of a vector space; we denote by $\mathrm{co}\, D$ the convex hull of D.

Let X and Y be topological spaces. A nonempty set-valued map G from X into Y is said to be upper semicontinuous if for any point $x_0 \in X$ and any open set V in Y containing Gx_0, there is a neighborhood U of x_0 in X such that $Gx \subset V$ for all $x \in U$. If Y is compact and G is a compact set-valued map then G is upper semicontinuous if and only if the graph $\{(x, y) \in X \times Y: y \in Gx\}$ of G is closed in $X \times Y$. If X is compact and G is a compact set-valued map then G is upper semicontinuous if and only if the graph $\{(x, y) \in X \times Y: y \in Gx\}$ of G is closed in $X \times Y$ and $G(X)$ is compact.

We start with the following lemma which is crucial to prove Theorem 1.

Lemma 1 (Eilenberg and Montgomery, Górniewicz). *Let Z be an n-simplex with the Euclidean simplex topology and let W be a compact space. Let p be a single-valued continuous map from W into Z and let T be a set-valued upper semicontinuous map from Z into W such that for each $x \in Z$, Tx is* *a nonempty compact acyclic subset of* W. *Then there is a point* x_0 *in* Z *such that* $x_0 \in p(Tx_0)$.

For a proof, see the argument in [3, §3] with Theorem 6.3 in [8] instead of the coincidence theorem that is used in [3, §3].

Theorem 1. *Let* X *be a subset of a vector space* E *and let* Y *be a topological space. Let* G *be a set-valued map from* X *into* Y *and let* T *be a set-valued map from* $\operatorname{co} X$ *into* Y *such that*

(i) *for each* $x \in \operatorname{co} X$, Tx *is a nonempty compact acyclic subset of* Y;
(ii) *for each finite subset* $\{x_1, \dots, x_n\}$ *of* X, $T(\operatorname{co}\{x_1, \dots, x_n\}) \subset \bigcup_{i=1}^n Gx_i$;
(iii) *for each finite subset* $\{x_1, \dots, x_n\}$ *of* X, $T|_Z : Z \to 2^Y$ *is upper semicontinuous, where* $Z = \operatorname{co}\{x_1, \dots, x_n\}$ *and* Z *is endowed with the Euclidean simplex topology; and*
(iv) *for each finite subset* $\{x_1, \dots, x_n\}$ *of* X, $Gx_i \cap T(Z)$ *is relatively closed in* $T(Z)$ *for all* $i = 1, \dots, n$, *where* $Z = \operatorname{co}\{x_1, \dots, x_n\}$.

Then the family $\{Gx : x \in X\}$ *of sets has the finite intersection property. More explicitly, for any finite subset* $\{x_1, \dots, x_n\}$ *of* X,

$$\left(\bigcap_{i=1}^n Gx_i\right) \cap T(\operatorname{co}\{x_1, \dots, x_n\}) \neq \varnothing.$$

Remark. If $E = Y$ and T is the identity mapping, the condition (ii) implies that G is a KKM map. If G is a closed set-valued map then the condition (iv) holds trivially. The condition (iii) is useful when E is not endowed with topology. In fact, prove Theorem 5 in [11], which is one of Fan's general fixed point theorems [6], as Ha did in [11].

Proof. Let $\{x_1, \dots, x_n\}$ be any finite subset of X and let $Z = \operatorname{co}\{x_1, \dots, x_n\}$ which is endowed with the Euclidean simplex topology. Suppose that $(\bigcap_{i=1}^n Gx_i) \cap T(Z) = \varnothing$ then $\{T(Z)\backslash Gx_1, \dots, T(Z)\backslash Gx_n\}$ is an open covering of $T(Z)$. Since T is upper semicontinuous on Z and compact valued, $T(Z)$ is compact. Hence there is a partition of unity $\{\alpha_1, \dots, \alpha_n\}$ corresponding to the covering. Let p be the function from $T(Z)$ into Z defined by

$$p(y) = \sum_{i=1}^n \alpha_i(y) x_i, \qquad y \in T(Z).$$

For each $y \in T(Z)$, let $I_y = \{i : \alpha_i(y) > 0\}$. If $i \in I_y$ then $\alpha_i(y) > 0$ and so $y \notin Gx_i$. Hence the condition (ii) implies that

$$(1) \qquad y \notin T(\operatorname{co}\{x_i : i \in I_y\}) \quad \text{for any } y \in T(Z).$$

On the other hand by Lemma 1, there is a point $x_0 \in Z$ such that $x_0 \in p(Tx_0)$. Let y_0 be a point in $T(Z)$ such that $y_0 \in Tx_0$ and $p(y_0) = x_0$. Since $p(y_0) = \sum_{i \in I_{y_0}} \alpha_i(y_0) x_i$, x_0 is contained in $\operatorname{co}\{x_i : i \in I_{y_0}\}$. Hence $y_0 \in T(\operatorname{co}\{x_i : i \in I_{y_0}\})$. This contradicts (1). □

We show some conditions which guarantee the whole intersection property, and hence we show some generalizations of KKM theorems in [5, 7, 14, 18].

Theorem 2. *Let X be a subset of a vector space E and let Y be a topological space. Let G be a set-valued map from X into Y and T be a set-valued map from* $\operatorname{co} X$ *into Y such that*

(i) *for each $x \in \operatorname{co} X$, Tx is a nonempty compact acyclic subset of Y and for each $x \in X$, Gx is a closed subset of Y;*
(ii) *for each finite subset $\{x_1, \dots, x_n\}$ of X, $T(\operatorname{co}\{x_1, \dots, x_n\}) \subset \bigcup_{i=1}^n Gx_i$; and*
(iii) *for each finite subset $\{x_1, \dots, x_n\}$ of X, $T|_Z : Z \to 2^Y$ is upper semicontinuous, where $Z = \operatorname{co}\{x_1, \dots, x_n\}$ and Z is endowed with the Euclidean simplex topology.*

Furthermore suppose that there exists a compact subset K of Y such that $T(X) \subset K$. Then $(\bigcap_{x\in X} Gx) \cap K \neq \varnothing$.

Proof. By Theorem 1, $\{Gx \cap K : x \in X\}$ is a family of subsets of K that has the finite intersection property. Since K is compact, the conclusion holds. □

Theorem 3. *Let X be a subset of a topological vector space E and let Y be a topological space. Let G be a set-valued map from X into Y and T be an upper semicontinuous set-valued map from* $\operatorname{co} X$ *into Y such that*

(i) *for each $x \in \operatorname{co} X$, Tx is a nonempty compact acyclic subset of Y and for each $x \in X$, Gx is a closed subset of Y; and*
(ii) *for each finite subset $\{x_1, \dots, x_n\}$ of X, $T(\operatorname{co}\{x_1, \dots, x_n\}) \subset \bigcup_{i=1}^n Gx_i$.*

Furthermore suppose that one of the following conditions is satisfied:

(a) *X is compact or*
(b) *X is convex or closed and there exists a compact convex subset X_0 of X such that $\bigcap_{x\in X_0} Gx$ is compact.*

Then $\bigcap_{x\in X} Gx \neq \varnothing$.

Proof. First suppose that the condition (a) is satisfied. Since X is compact and T is upper semicontinuous, $T(X)$ is compact. By Theorem 2, $(\bigcap_{x\in X} Gx) \cap T(X) \neq \varnothing$ and hence the conclusion holds.

Next suppose that the condition (b) is satisfied. Since X_0 is compact, $\bigcap_{z\in X_0} Gz \neq \varnothing$. We show that the family $\{Gx \cap (\bigcap_{z\in X_0} Gz) : x \in X\}$ has the finite intersection property. For any finite subset $\{x_1, \dots, x_n\}$ of $X \backslash X_0$, if X is convex then let $X_1 = \operatorname{co}\{X_0 \cup \{x_1, \dots, x_n\}\}$; otherwise let $X_1 = X \cap \operatorname{co}\{X_0 \cup \{x_1, \dots, x_n\}\}$. Since X_1 is compact, $\bigcap_{x\in X_1} Gx \neq \varnothing$ and hence $\bigcap_{i=1}^n (Gx_i \cap (\bigcap_{z\in X_0} Gz)) \neq \varnothing$. Therefore the conclusion holds. □

The following corollary is a slight generalization of Ha's theorem [11] and the theorem of Ben-El-Mechaiekh, Deguire, and Granas [1]. We can see some of its applications in [11] and [12].

Corollary 1 (Ha). *Let E, F be topological vector spaces, $X \subset E$, $Y \subset F$ be nonempty convex subsets and let $A \subset B \subset C$ be subsets of $X \times Y$ such that*

(i) *for each $x \in X$, the set $\{y \in Y: (x, y) \in C\}$ is closed in Y;*
(ii) *for each $x \in X$, the set $\{y \in Y: (x, y) \notin B\}$ is empty or convex;*
(iii) *A is closed in $X \times Y$; and*
(iv) *there exists a compact subset K of Y such that for each $x \in X$, the set $\{y \in K: (x, y) \in A\}$ is nonempty and convex.*

Then there exists a point $y_0 \in K$ such that $X \times \{y_0\} \subset C$.

Proof. For each $x \in X$, let $Tx = \{y \in K: (x, y) \in A\}$, let $Gx = \{y \in Y: (x, y) \in B\}$ and let $Hx = \{y \in Y: (x, y) \in C\}$. Then for each $x \in X$, Hx is closed and Tx is nonempty, compact, and convex. And also for each finite subset $\{x_1, \dots, x_n\}$ of X, $T(Z) \subset \bigcup_{i=1}^n Gx_i \subset \bigcup_{i=1}^n Hx_i$ and $T|_Z: Z \to 2^Y$ is upper semicontinuous, where $Z = \operatorname{co}\{x_1, \dots, x_n\}$. Furthermore, there is the compact set $K \subset Y$ such that $T(X) \subset K$. Hence by Theorem 2, $(\bigcap_{x \in X} Hx) \cap K \neq \varnothing$. This implies that there exists a point $y_0 \in K$ such that $X \times \{y_0\} \subset C$. $\square$

3. Some applications

We show two theorems concerning inequalities and a minimax theorem. First, the following theorem is a special case of Theorem 1 in [15], but the proof is simpler than that in [15]. For each $n \in N$, we define sets W_n and S_n in R^n as

$$W_n = \{e_j = (0, \dots, \overset{j}{1}, \dots, 0) \in R^n: j = 1, \dots, n\};$$

$$S_n = \operatorname{co} W_n = \left\{\alpha = \sum_{j=1}^n \alpha_j e_j \in R^n: \sum_{j=1}^n \alpha_j = 1,\ \alpha_j \geq 0\right\}.$$

Theorem 4. *Let X be a nonempty compact convex subset of a topological vector space. Let $f_1, \dots, f_n$ be lower semicontinuous convex functions on X with values in $(-\infty, +\infty]$. Then the following are equivalent:*

(i) *The system of convex inequalities*

$$f_i(x) \leq 0 \quad \text{for all } i \in I \tag{2}$$

is consistent on X; that is, there exists an $x \in X$ satisfying (2).

(ii) *For any $\alpha \in S_n$, there exists an $x \in X$ such that*

$$\sum_{i=1}^n \alpha_i f_i(x) \leq 0.$$

Proof. It is clear that (i) implies (ii). We prove that (ii) implies (i). We define set-valued maps $T: S_n \to 2^X$ and $G: W_n \to 2^X$ by

$$T\alpha = \left\{x \in X: \sum_{i=1}^n \alpha_i f_i(x) \leq 0\right\} \qquad (\alpha \in S_n)$$

and

$$Ge_i = \{x \in X\colon f_i(x) \le 0\} \qquad (e_i \in W_n).$$

It is easy to see that for each $\alpha \in S_n$, $T\alpha$ is a nonempty compact convex subset of X and for each $i = 1, \ldots, n$, Ge_i is closed. For any finite subset $\{e_{i_1}, \ldots, e_{i_m}\}$ of W_n,

$$\left\{x \in X\colon \text{there exists a } \beta \in S_m \text{ such that } \sum_{j=1}^{m} \beta_j f_{i_j}(x) \le 0\right\} \subset \bigcup_{j=1}^{m} Ge_{i_j};$$

that is,

$$T(\mathrm{co}\{e_{i_1}, \ldots, e_{i_m}\}) \subset \bigcup_{j=1}^{m} Ge_{i_j}.$$

By Lemma 1 in [15], it is also easy to see that T is upper semicontinuous. Hence, by Theorem 1, there is a point $x_0 \in X$ such that

$$x_0 \in \bigcap_{i=1}^{n} Ge_i.$$

Therefore we have

$$f_i(x_0) \le 0 \quad \text{for all } i \in I. \quad \square$$

Next we show Takahashi's system theorem [18] concerning inequalities. Let X and Y be arbitrary sets. A function $f\colon X \times Y \to R$ is convexlike in its second variable if for any $y_1, y_2 \in Y$ and $0 < a < 1$, there exists a $y_0 \in Y$ such that

$$f(x, y_0) \le af(x, y_1) + (1-a)f(x, y_2)$$

for all $x \in X$. Also, a function $g\colon X \times Y \to R$ is concavelike in its first variable if for any $x_1, x_2 \in X$ and $0 < a < 1$, there exists an $x_0 \in X$ such that

$$g(x_0, y) \ge ag(x_1, y) + (1-a)g(x_2, y)$$

for all $y \in Y$.

Theorem 5 (Takahashi). *Let X be a set, let $f_1, \ldots, f_n$ be real valued functions on X and suppose that the function F on $I \times X$, defined by $F(i, x) = f_i(x)$ for $x \in X$ and $i \in I$, is convexlike in its second variable, where $I = \{1, \ldots, n\}$. Let $c \in R$. If for any $\alpha \in S_n$, there exists an $x_0 \in X$ such that $\sum_{i=1}^{n} \alpha_i f_i(x_0) \le c$, then*

$$\inf_{x \in X} \max_{i \in I} f_i(x) \le c.$$

Proof. Let $d = \inf_{x\in X} \max_{i\in I} f_i(x)$. If $d = -\infty$ the conclusion holds trivially. So we may assume that d is a real number. Choose any finite subset $\{x_1, \ldots, x_m\}$ of X and let $f(\alpha, x) = \sum_{i=1}^{n} \alpha_i f_i(x)$ for any $x \in X$ and $\alpha \in S_n$. We define set-valued maps $T\colon S_m \to 2^{S_n}$ and $G\colon W_m \to 2^{S_n}$ by

$$T\beta = \left\{\alpha \in S_n\colon \sum_{j=1}^{m} \beta_j f(\alpha, x_j) \ge d\right\} \qquad (\beta \in S_m)$$

and

$$Ge_j = \{\alpha \in S_n : f(\alpha, x_j) \geq d\} \qquad (e_j \in W_m).$$

Since it is easy to check that the conditions of Theorem 1 hold, we have $\bigcap_{j=1}^m Ge_j \neq \varnothing$. Hence, by compactness of S_n,

$$\bigcap_{x \in X} \{\alpha \in S_n : f(\alpha, x) \geq d\} \neq \varnothing;$$

that is, there exists an $\dot{\alpha} \in S_n$ such that

$$f(\dot{\alpha}, x) \geq d \quad \text{for all } x \in X.$$

From the hypothesis, this implies

$$\inf_{x \in X} \max_{i \in I} f_i(x) \leq c. \quad \square$$

Finally, we show Simons's minimax theorem [16]. Takahashi obtained a similar result in [18]. Compare the proof of the next theorem with those of Simons and Takahashi.

Theorem 6 (Simons). *Let X and Y be arbitrary sets and let f, g be real valued functions on $X \times Y$ satisfying*

(i) *$f(x, y) \leq g(x, y)$ for each $(x, y) \in X \times Y$;*
(ii) *f is convexlike in its second variable; and*
(iii) *g is concavelike in its first variable.*

Then for any nonempty finite subset X_0 of X,

$$\inf_{y \in Y} \max_{x \in X_0} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} g(x, y).$$

Proof. Let X_0 be a finite subset $\{x_1, \dots, x_n\}$ of X and

$$d = \inf_{y \in Y} \max_{x \in X_0} f(x, y).$$

If $d = -\infty$ the conclusion holds trivially. So we may assume that d is a real number. Choose any finite subset $\{y_1, \dots, y_m\}$ of Y. We define set-valued maps $T: S_m \to 2^{S_n}$ and $G: W_m \to 2^{S_n}$ by

$$T\beta = \left\{\alpha \in S_n : \sum_{j=1}^m \beta_j \sum_{i=1}^n \alpha_i f(x_i, y_j) \geq d\right\} \qquad (\beta \in S_m)$$

and

$$Ge_j = \left\{\alpha \in S_n : \sum_{i=1}^n \alpha_i f(x_i, y_j) \geq d\right\} \qquad (e_j \in W_m).$$

Since it is easy to check that the conditions of Theorem 1 hold, we have $\bigcap_{j=1}^m Ge_j \neq \varnothing$. Hence, by compactness of S_n,

$$\bigcap_{y \in Y} \left\{\alpha \in S_n : \sum_{i=1}^n \alpha_i f(x_i, y) \geq d\right\} \neq \varnothing;$$

that is, there exists an $\dot{\alpha} \in S_n$ such that

$$d \leq \sum_{i=1}^{n} \dot{\alpha}_i f(x_i, y) \leq \sum_{i=1}^{n} \dot{\alpha}_i g(x_i, y) \quad \text{for all } y \in Y.$$

From hypothesis (iii), there exists an $x_0 \in X$ such that

$$d \leq g(x_0, y) \quad \text{for all } y \in Y.$$

Therefore we have

$$\inf_{y\in Y} \max_{x\in X_0} f(x, y) \leq \sup_{x\in X} \inf_{y\in Y} g(x, y). \quad \square$$

ACKNOWLEDGMENT

The author would like to express his hearty thanks to Professor Wataru Takahashi for encouragement and many helpful comments.

REFERENCES

1. H. Ben-El-Mechaiekh, P. Deguire, and A. Granas, *Une alternative non linéaire en analyse convexe et applications*, C. R. Acad. Sci. Paris Sér. I **295** (1983), 257–259.
2. J. Dugundji and A. Granas, *KKM maps and variational inequalities*, Ann. Scoula Norm. Sup. Pisa Cl. Sci. **5** (1978), 679–682.
3. S. Eilenberg and D. Montgomery, *Fixed point theorems for multi-valued transformations*, Amer. J. Math. **58** (1946), 214–222.
4. S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton Univ. Press, Princeton, NJ, 1952.
5. K. Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Ann. **142** (1961), 305–310.
6. ——, *A minimax inequality and applications*, Inequalities III (O. Shisha, ed.), Academic Press, 1972, pp. 103–110.
7. ——, *Some properties of convex sets related to fixed point theorems*, Math. Ann. **266** (1984), 519–537.
8. L. Górniewicz, *A Lefschetz-type fixed point theorem*, Fund. Math. **LXXXVIII** (1975), 103–115.
9. ——, *Homological methods in fixed point theory of multi-valued maps*, Dissertationes Math. **129** (1976), 1–66.
10. A Granas, *KKM-maps and their applications to nonlinear problems*, The Scottish Book: Mathematics from the Scottish Café (R. D. Mauldin, ed.), Birkhäuser, Basel, Boston, 1982, pp. 45–61.
11. C. W. Ha, *Minimax and fixed point theorems*, Math. Ann. **248** (1980), 73–77.
12. ——, *On a minimax inequality of Ky Fan*, Proc. Amer. Math. Soc. **99** (1987), 680–682.
13. B. Knaster, C. Kuratowski, and S. Mazurkiewicz, *Ein Beweis des Fixpunktsatzes für n-dimensional simplexe*, Fund. Math. **XIV** (1929), 132–137.
14. M. Lassonde, *On the use of KKM multifunctions in fixed point theory and related topics*, J. Math. Anal. Appl. **97** (1983), 151–201.

15. N. Shioji and W. Takahashi, *Fan's theorem concerning systems of convex inequalities and its applications*, J. Math. Anal. Appl. **135** (1988), 383–398.

16. S. Simons, *Minimax and variational inequalities. Are they of fixed point or Hahn-Banach type?*, Game Theory and Mathematical Economics, North-Holland, 1981, pp. 379–387.

17. ——, *Two-function minimax theorems and variational inequalities for functions on compact and noncompact sets, with some comments on fixed-point theorems*, Proc. Sympos. Pure Math. **45** (1986), 377–392.

18. W. Takahashi, *Fixed point, minimax, and Hahn–Banach theorems*, Proc. Sympos. Pure Math. **45** (1986), 419–427.

DEPARTMENT OF INFORMATION SCIENCE, TOKYO INSTITUTE OF TECHNOLOGY, OOKAYAMA, MEGUROKU, TOKYO 152, JAPAN