LEVEL CROSSINGS
OF A RANDOM TRIGONOMETRIC POLYNOMIAL

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Abstract. This paper provides an asymptotic estimate for the expected number of \( K \)-level crossings of the random trigonometric polynomial \( g_1 \cos x + g_2 \cos 2x + \ldots + g_n \cos nx \), where \( g_j \) (\( j = 1, 2, \ldots, n \)) are independent normally distributed random variables with mean \( \mu \) and variance one. It is shown that the result for \( K = \mu = 0 \) remains valid for any finite constant \( \mu \) and any \( K \) such that \( (K^2/n) \rightarrow 0 \) as \( n \rightarrow \infty \).

1. Introduction

Let

\[
T(x) \equiv T_n(x, \omega) = \sum_{j=1}^{n} g_j(\omega) \cos jx
\]

where \( g_1(\omega), g_2(\omega), \ldots, g_n(\omega) \) is a sequence of independent random variables defined on a probability space \( (\Omega, \mathcal{A}, P) \), each normally distributed with finite mathematical expectation \( \mu \) and variance one. The set of equations \( y = T(x) \) represents a family of curves in the \( xy \)-plane. Some years ago Dunnage \cite{2} showed that the number of times all save a certain exceptional set of these curves crosses the \( x \)-axis in the interval \( 0 \leq x \leq 2\pi \) is \( (2n)/\sqrt{3} + O\{n^{11/13}(\log n)^{3/13}\} \). The measure of his exceptional set does not exceed \( (\log n)^{-1} \). Later Sambandham and Renganathan \cite{7} and Farahmand \cite{3} considered the case when \( \mu \neq 0 \) and found that the expected number of crossings of this family with the \( x \)-axis is asymptotic to \( 2n\sqrt{3} \). This asymptotic number of crossings remains invariant in the work of Farahmand \cite{4} when, for \( \mu = 0 \), he considered the crossings of this family with the line \( y = K \) (i.e., with level \( K \)), where \( K \) is any real value constant such that \( (K^2/n) \rightarrow 0 \) as \( n \rightarrow \infty \).

Here we consider the effect of \( \mu \) being nonzero on the \( K \)-level crossings of this family of curves, and show even in this case the above asymptotic number
of crossings persists. Denote by $N(\alpha, \beta) \equiv N_K(\alpha, \beta)$ the number of times that the family of curves $y = T(x)$ crosses the line $y = K$ on the interval $\alpha \leq x \leq \beta$, and let $EN(\alpha, \beta)$ be its expectation. We prove the following.

**Theorem.** If the coefficients of $T(x)$ in (1.1) are independent normally distributed random variables with mean $\mu$ and variance one, then for all sufficiently large $n$ and any constant $K$ the expected number of real roots of the equation $T(x) = K$ satisfies

$$EN(0, 2\pi) = (2n)/\sqrt{3} + O(n^{3/4}) \quad \text{if} \quad K = O(n^{3/8})$$

and

$$EN(0, 2\pi) = (2n)/\sqrt{3} + o(n) \quad \text{if} \quad K = o(\sqrt{n}).$$

2. **A Formula for the Number of Crossings**

Let

$$\Phi(t) = (2\pi)^{-1/2} \int_{-\infty}^{t} \exp(-y^2/2) \, dy$$

and

$$\phi(t) = \Phi'(t) = (2\pi)^{-1/2} \exp(-t^2/2),$$

then by using the expected number of level crossings given by Cramer and Leadbetter [1, p. 285] for our equation $T(x) - K = 0$ we can obtain

$$(2.1) \quad EN(\alpha, \beta) = \int_{\alpha}^{\beta} \left( (B/A)(1-\lambda^2)^{1/2}\phi(\xi/A)[2\phi(\eta) + \eta\{2\Phi(\eta) - 1\}] \right) \, dx$$

where

$$A^2 = \text{Var}\{T(x) - K\}, \quad B^2 = \text{Var}\{T'(x)\},$$

$$\lambda = (AB)^{-1} \text{cov}\{\{T(x) - K\}, T'(x)\}, \quad \xi = E\{T(x) - K\},$$

$$\eta = B^{-1}(1-\lambda^2)^{-1/2}(\gamma - B\lambda\xi/A) \quad \text{and} \quad \gamma = E\{T'(x)\}.$$

Since the coefficients of $T(x)$ are independent normal random variables with mean $\mu$ and variance one we can easily find that

$$\xi = \mu \sum_{j=1}^{n} \cos jx - K = m_1 - K, \quad \text{say},$$

$$\gamma = -\mu \sum_{j=1}^{n} j \sin jx = -m_2, \quad \text{say}$$

$$A^2 = \sum_{j=1}^{n} \cos^2 jx, \quad B^2 = \sum_{j=1}^{n} j^2 \sin^2 jx,$$

$$C = \sum_{j=1}^{n} j \sin jx \cos jx, \quad \lambda = -C/AB,$$

$$\eta = \{C(m_1 - K) - A^2m_2\}/A\Delta \quad \text{and} \quad \Delta^2 = A^2B^2 - C^2.$$
Hence from (2.1) and since $\Phi(t) = 1/2 + (\pi)^{-1/2} \text{erf}(t/\sqrt{2})$, we have the extension of the Kac-Rice formula [5].

$$EN(\alpha, \beta) = \int_\alpha^\beta (\Delta/\pi A^2)^{\alpha} \exp\left\{-B^2(m_1 - K)^2 - A^2 m_2 + 2m_2 C(m_1 - K)\right\}/(2\Delta^2) \, dx$$

$$= \int_\alpha^\beta (\Delta/\pi A^2)^{\alpha} \exp\left\{-B^2(m_1 - K)^2 - A^2 m_2 + 2m_2 C(m_1 - K)\right\}/(2\Delta^2) \, dx$$

$$+ \int_\alpha^\beta \left(\sqrt{2/\pi}\right) A^{-3} |C(m_1 - K) - A^2 m_2| \exp\left\{-(m_1 - K)^2/2A^2\right\} \times \text{erf}\{|C(m_1 - K) - A^2 m_2|/\sqrt{2A}\} \, dx$$

$$= \int_\alpha^\beta I_1(x) \, dx + \int_\alpha^\beta I_2(x) \, dx.$$

3. Overview of the proof of the theorem and some approximations

In order to evaluate the expected number of real roots we divide the real roots into two groups: (i) those lying in the intervals $(0, \varepsilon)$, $(\pi - \varepsilon, \pi + \varepsilon)$ and $(2\pi - \varepsilon, 2\pi)$ and (ii) those lying in the intervals $(\varepsilon, \pi - \varepsilon)$ and $(\pi + \varepsilon, 2\pi - \varepsilon)$. For the roots (i) which, it so happens, are negligible, we need some modification to apply Dunnage's [2] approach, which is based on an application of Jensen's theorem [8, p. 125] or [6, p. 332]. For roots (ii) we use the Kac-Rice formula (2.2). The $\varepsilon$ should be chosen positive and small enough to facilitate handling type (i) roots, but large enough to allow for the approximation needed to discuss type (ii) roots. We choose $\varepsilon = n^{-1/4}$.

The function

$$S(x) = \sin(2n + 1)x/\sin x$$

and, defined by continuity at $x = j\pi$, will occur frequently and so it is convenient to collect together some related inequalities. Since for $\varepsilon \leq x \leq \pi - \varepsilon$ or $\pi + \varepsilon \leq x \leq 2\pi - \varepsilon$ we have $|S(x)| < 1/\sin \varepsilon$, we can obtain

$$S(x) = O(1/\varepsilon).$$

Further

$$S'(x) = (2n + 1) \cos(2n + 1)x/\sin x - \cot x S(x) = O(n/\varepsilon)$$

and

$$S''(x) = -(2n + 1)^2 S(x) - (2n + 1) \cos x \cos(2n + 1)x \sin^{-2} x$$

$$- \cot x S'(x) + \cosec^2 x S(x) = O(n^2/\varepsilon).$$

Hence from (3.1) and since by expanding $\sin x(1 + 2 \sum_{j=1}^n \cos 2jx)$, we can show that

$$\sum_{j=1}^n \cos 2jx = (1/2)\{S(x) - 1\}$$
for this range of \( x \) we have

\[
(3.5) \quad A^2 = (1/2) \sum_{j=1}^{n} (1 + \cos 2jx) = n/2 + (1/4)\{S(x) - 1\} = n/2 + O(1/e).
\]

Also from (3.3) and since from (3.4)

\[
S''(x) = -8 \sum_{j=1}^{n} j^2 \cos 2jx = 4 \sum_{j=1}^{n} j^2 (2 \sin^2 jx - 1)
\]

we have

\[
(3.6) \quad B^2 = n(n + 1)(2n + 1)/12 + (1/8)S''(x) = n^3/6 + O(n^2/e).
\]

From (3.2) we obtain

\[
(3.7) \quad C = -(1/2) \frac{d}{dx}(A^2) = -(1/8)S'(x) = O(n/e).
\]

Also from (3.1), (3.2), and (3.4) we can easily obtain the two following estimates:

\[
(3.8) \quad m_1 = (\mu/2)\{S(x/2) - 1\} = O(1/e)
\]

and

\[
(3.9) \quad m_2 = (-\mu/4)S'(x/2) = O(n/e).
\]

Then (3.5), (3.6), and (3.7) give

\[
(3.10) \quad \Delta^2 = n^4/12 + O(n^3/e).
\]

4. Proof of the theorem

From (2.2) and (3.5)–(3.10) we can obtain

\[
(4.1) \quad \int_{\pi - \epsilon}^{\pi} I_1(x) \, dx = (n/\sqrt{3})\{1 + O(\epsilon)\} \exp\{-K^2/n + O(1/ne^2)\}
\]

and

\[
(4.2) \quad \int_{\pi - \epsilon}^{\pi} I_2(x) \, dx = O\{(n^{1/2}/e) \exp(-1/ne^2 - K^2/n)\}.
\]

Hence from (2.2), (4.1), and (4.2) for \( K = O(n^{3/8}) \), we have

\[
(4.3) \quad EN(\epsilon, \pi - \epsilon) = EN(\pi + \epsilon, 2\pi - \epsilon) = n/\sqrt{3} + O(n^{3/4})
\]

and for \( K = o(\sqrt{n}) \)

\[
(4.4) \quad EN(\epsilon, \pi - \epsilon) = EN(\pi + \epsilon, 2\pi - \epsilon) = n/\sqrt{3} + o(n).
\]

Now we show that the expected number of real roots in the intervals \((0, \epsilon)\), \((\pi - \epsilon, \pi + \epsilon)\), and \((2\pi - \epsilon, 2\pi)\) is negligible. The period of \( T(x) \) is \( 2\pi \) and so the number of zeros in \((0, \epsilon)\) and \((2\pi - \epsilon, 2\pi)\) is the same as the number in \((-\epsilon, \epsilon)\). We shall therefore confine ourselves to this last interval; the interval
\((\pi - \varepsilon, \pi + \varepsilon)\) can be treated in exactly the same way to give the same result. We consider the random integral function of the complex variable \(z\),

\[ T(z, \omega) - K = \sum_{j=1}^{n} g_j(\omega) \cos jz - K. \]

We are seeking an upper bound to the number of real roots in the segment of the real axis joining the points ±\(e\), and this certainly does not exceed the number of real roots in the circle \(|z| < e\). Let \(N(r) \equiv N(r, \omega, K)\) denote the number of real roots of \(T(z, \omega) - K = 0\) in \(|z| < r\). By Jensen's theorem [8, p. 125] or [6, p. 332]

\[
\int_{e}^{2e} r^{-1} N(r) \, dr \leq \int_{0}^{2e} r^{-1} N(r) \, dr
\]

\[= (2\pi)^{-1} \int_{0}^{2\pi} \log \|\{T(2ee^{ix}, \omega) - K\}/\{T(0) - K\}\| \, dx, \]

assuming that \(T(0) \neq K\), from which we have

\[
N(e) \log 2 \leq (2\pi)^{-1} \int_{0}^{2\pi} \log \|\{T(2ee^{ix}, \omega) - K\}/\{T(0) - K\}\| \, dx.
\]

Now since the distribution function of \(T(0, \omega) = \sum_{j=1}^{n} g_j(\omega)\) is

\[ G(x) = (2\pi n)^{-1/2} \int_{-\infty}^{x} \exp\{-(t - \mu)^2/2n\} \, dt \]

we can see that for any positive \(\nu\)

\[
\text{Prob}(-e^{-\nu} \leq T(0) - K < e^{-\nu}) = (2\pi n)^{-1/2} \int_{K-e^{-\nu}}^{K+e^{-\nu}} \exp\{-(t - \mu)^2/2n\} \, dt
\]

\[
< (2/\pi)^{1/2} e^{-\nu}.
\]

Also we have

\[
|T(2ee^{ix})| = \left| \sum_{j=1}^{n} g_j \cos(2jxe^{ix}) \right| \leq ne^{2\nu} \max |g_j|
\]

where the maximum is taken over \(1 \leq j \leq n\). The distribution function of \(|g_j|\) is

\[ F(x) = \begin{cases} \sqrt{2}/\pi \int_{0}^{x} \exp\{-(t - \mu)^2/2\} \, dt, & x \geq 0, \\ 0, & x < 0, \end{cases} \]

and so for any positive \(\nu\) and all sufficiently large \(n\)

\[
\text{Prob}\{\max |g_j| > ne^{-\nu}\} \leq n \text{Prob}\{|g_1| > ne^{-\nu}\}
\]

\[
= n\sqrt{2}/\pi \int_{ne^{-\nu}}^{\infty} \exp\{- (t - \mu)^2/2\} \, dt
\]

\[
\sim \sqrt{2}/\pi \exp\{-\nu - (ne^{-\nu} - \mu)^2/2\}.
\]
Therefore from (4.7) and (4.8) except for sample functions in an \( \omega \)-set of measure not exceeding \( (2/\pi)^{1/2} \exp\{-\nu - (ne^\nu - \mu)^2/2\} \),
\begin{equation}
|r(2\epsilon e^{ix})| < n^2 \exp(2\pi e + \nu).
\end{equation}

Hence from (4.6), (4.9) and since
\begin{equation*}
|n^2 \exp(2\pi e + \nu) - K| < 2n^2 \exp(2\pi e + \nu)
\end{equation*}
if \( K = o(\sqrt{n}) \) or if \( K = O(n^{3/8}) \), we obtain
\begin{equation}
\frac{|\{ T(2\epsilon e^{ix}, \omega) - K \} / \{ T(0, \omega) - K \} |}{\leq e^\nu |2n^2 \exp(2\pi e + \nu) - K|} \leq 2n^2 \exp(2\pi e + 2\nu)
\end{equation}
except for sample functions in an \( \omega \)-set of measure not exceeding \( (2/\pi_n)^{1/2} e^{-\nu} + (2/\pi)^{1/2} \exp\{-\nu - (ne^\nu - \mu)^2/2\} \).

Therefore from (4.5) and (4.10) we can show that outside the exceptional set
\begin{equation}
N(e) \leq (\log 2 + 2 \log n + 2\pi e + 2\nu)/\log 2.
\end{equation}

Because \( e = n^{-1/4} \), it follows from (4.11) and for all sufficiently large \( n \)
\begin{equation}
\text{Prob}\{N(e) > 3n\pi + 2\nu\} \leq (2/\pi_n)^{1/2} e^{-\nu} + (2/\pi)^{1/2} \exp\{-\nu - (ne^\nu - \mu)^2/2\}.
\end{equation}

Let \( n' = [3n^{3/4}] \) be the greatest integer less than or equal to \( 3n^{3/4} \); then from (4.12) and for \( n \) large enough we obtain
\begin{equation}
EN(e) = \sum_{j > 0} \text{Prob}\{N(e) \geq j\}
= \sum_{1 \leq j \leq n'} \text{Prob}\{N(e) > j\} + \sum_{j \geq 1} \text{Prob}\{N(e) > n' + j\}
\leq n' + (2/\pi_n)^{1/2} \sum_{j \geq 1} e^{-j/2} + (2/\pi)^{1/2} \sum_{j \geq 1} \exp\{-j/2 - (ne^j - \mu)^2/2\}
\end{equation}

\begin{equation}
= O(n^{3/4}).
\end{equation}

Finally (4.3), (4.4), and (4.13) complete proof of the theorem.

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**REFERENCES**


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