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MAXIMAL TRIANGULAR SUBALGEBRAS NEED NOT BE CLOSED

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ABSTRACT. We give an example to show that a maximal triangular subalgebra need not be closed.

INTRODUCTION

Let $\mathfrak{A}$ be a C^*-algebra and $\mathscr{D}$ a maximal abelian self-adjoint subalgebra (masa) of $\mathfrak{A}$. A subalgebra $\mathscr{T}$ of $\mathfrak{A}$ is said to be triangular (with respect to $\mathscr{D}$) if $\mathscr{T} \cap \mathscr{T}^* = \mathscr{D}$. A triangular subalgebra $\mathscr{T}$ is said to be maximal if the only triangular subalgebra containing $\mathscr{T}$ is $\mathscr{T}$ itself. Studies in triangular subalgebras can be traced back to the works of Kadison and Singer [6] and Arveson [1–3]. In [7], Muhly, Saito and Solel gave a detailed study of σ-weakly closed triangular subalgebras in a von Neumann algebra and they raised the question of whether a maximal triangular subalgebra in a von Neumann algebra whose diagonal is a Cartan subalgebra must be σ-weakly closed. In this note, we will show that the answer to this question is negative. Recently, triangular subalgebras of AF algebras have been the subject of study of many authors, e.g. Powers [13–15], Baker [4], Peters, Poon and Wagner [10], Peters and Wagner [11], Thelwal [17, 18], Ventura [19, 20] and Poon [12]. In these works, the maximal triangular subalgebras are usually assumed to be norm closed. This leads to the following question: Must a maximal triangular subalgebra in an AF algebra be norm closed? Or, equivalently, must the norm closure of a triangular subalgebra be triangular? This question was first raised by Kadison and Singer (see Question 3 and the references in Erdos [5]). Arveson [2] gave the first example of a norm closed triangular subalgebra that is irreducible and showed in [3] that if $\mathfrak{A}$ is the algebra of all bounded operators on a Hilbert space then every triangular transitive subalgebra is strongly dense in $\mathfrak{A}$. (See [2] and [3] for the definitions of *irreducible* and *transitive*.) AF algebras are a special type of groupoid C^*-algebras developed by Renault [16]. In [8], Muhly and Solel laid down the foundation for the study of triangular subalgebras of groupoid

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C^*-algebras. In this note, we are going to use the results developed in [8] to give an example of a triangular AF algebra $\mathcal{T}$ such that the norm closure of $\mathcal{T}$ is not triangular. A similar example has been constructed earlier by John Orr [9] but the proof in the crucial step (the first in Proposition 2) given here is much simpler.

1. Preliminaries

We use the definitions in [8] for groupoid C^*-algebras and triangular subalgebras. The triangular subalgebra we are going to construct is defined by an equivalence relation.

For each subset E of the unit interval $[0, 1]$, χ_E will denote the characteristic function on E. Let $\mathcal{Q}$ be the set of rational numbers and $\mathcal{D}$ the C^*-algebra generated by $\{\chi_{[0,s)} : 0 < s < 1, s \text{ in } \mathcal{Q}\} \cup \{\chi_{[0,1]}\}$ with supremum norm. Then the maximal ideal space of $\mathcal{D}$ is a compact, totally disconnected space X containing $[0, 1]$. Specifically, every rational number s with $0 < s < 1$ is replaced by two points s_L, s_R and the order on X is inherited from that of $[0, 1]$ with the additional order $s_L \le s_R$ for every rational number $0 < s < 1$. For $x, y \in X$ with $x \le y$, $[x, y]$ will denote the interval defined by the order $\le$. Writing 0_R for 0 and 1_L for 1, we define $X_L = \{s_L : 0 < s \le 1, s \in \mathcal{Q}\}$ and $X_R = \{s_R : 0 \le s < 1, s \in \mathcal{Q}\}$. For each $x \in X_R$, $y \in X_L$, $[x, y]$ is a closed and open (clopen) set in X and the topology on X is generated by all these clopen sets. The addition of a number $0 \le s \le 1$ and a rational number t such that $0 \le s + t \le 1$ can be extended to X by $s_R + t = (s+t)_R$ and $(s_L + t) = (s+t)_L$ when s is rational. Since we will only consider intervals of the form $[s_R, t_L]$ in X, we will denote it by $[s, t]$ for simplicity of notations. Finally, we define an equivalence relation $\sim$ on X by $x \sim y$ if $x = y + t$ for some $t \in \mathcal{Q}$. Let $G = \{(x, y) \in X \times X : x \sim y\}$. The topology on G is generated by taking the sets $\{(x, x+r) \in X \times X : x \in [s, t]\}$, $r \in \mathcal{Q}$, $s \in X_R$, $t \in X_L$, $s \le t$, as a basis of clopen sets. This relation is a groupoid of the type discussed by Muhly and Solel in [8].

We first recall some basic facts from [8]. Let $C_c(G)$ be the space of continuous functions on G with compact support. Given $f, g \in C_c(G)$, $f * g$ and f^* are defined by the formulas

$$f * g(x, y) = \sum_z f(x, z) g(z, y), \quad \text{and}$$
$$f^*(x, y) = \overline{f(y, x)} \quad .$$

Under a suitable norm, the completion of $C_c(G)$ is a C^*-algebra denoted by $C^*(G)$. The following proposition is a collection of results in [8].

Proposition 1. *Let $C^*(G)$ be as defined above; then*

(1) *Every element in $C^*(G)$ can be represented by a continuous function on G.*

(2) *If f, g are continuous functions on G representing elements in $C^*(G)$, then the product $f \cdot g$ in $C^*(G)$ is represented by the function $f * g$.*
(3) *By identifying functions in $\mathscr{D} \cong C(X)$ as functions on $G^0 = \{(x, x) : x \in X\}$, $\mathscr{D}$ is a masa of $C^*(G)$.*
(4) *Given a closed subset $Q \subseteq G$, define*

$$I(Q) = \{a \in C^*(G) : a = 0 \quad \text{on} \quad Q\}.$$

Then $I(Q)$ is a norm closed $\mathscr{D}$-bimodule of $C^*(G)$ and if $\mathscr{B}$ is a norm closed $\mathscr{D}$-bimodule of $C^*(G)$, then $\mathscr{B} = I(Q(B))$ where

$$Q(B) = \{(x, y) \in G : a(x, y) = 0 \quad \text{for all} \quad a \in B\}.$$

2. Construction

The idea is to construct a subalgebra $\mathscr{B}$ of $C^*(G)$ satisfying
(1) $\mathscr{B} \cap \mathscr{D} = \{0\} = \mathscr{B} \cap \mathscr{B}^*$ and
(2) $\bar{\mathscr{B}}$, the norm closure of $\mathscr{B}$, contains $\mathscr{D}$ and $\bar{\mathscr{B}} \cap \bar{\mathscr{B}}^* = \mathscr{D}$.

Then the algebra

$$\mathscr{T} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{pmatrix} a, d \in \mathscr{D} + \mathscr{B} \\ c, b \in \mathscr{B} \end{pmatrix} \right\}$$

is a triangular subalgebra of $M_2 \otimes C^*(G)$ (where M_2 is the algebra of 2×2 matrices) because (1) implies $\mathscr{T} \cap \mathscr{T}^* = \mathscr{D}$. But (2) gives $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \in \bar{\mathscr{T}} \cap \bar{\mathscr{T}}^*$. Hence, $\bar{\mathscr{T}}$ is not triangular.

Let p_n be the nth prime number. For each n, let $E_0 = G^0$, $E_n = \{(x, x + 1/p_n) \in G : x \in [0, 1 - 1/p_n]\}$ and for $n \geq 0$, $a_n = \chi_{E_n}$ is the characteristic function on E_n. Since E_n is clopen, $a_n \in C^*(G)$. Let $a = \sum_{n=0}^{\infty} a_n/2^n$. Then $a \in C^*(G)$. We have

Proposition 2. *Let $\mathscr{B}$ be the algebra generated by $\{fag : f, g \in \mathscr{D}\}$. Then conditions (1) and (2) are satisfied.*

Proof. (1) For each $m \geq 1$, define $\mathscr{B}_m = \{\sum_{i=1}^k f_0^i a f_1^i a \dots a f_m^i : k \geq 1, f_j^i \in \mathscr{D}\}$. Suppose $b \in \mathscr{B} \cap \mathscr{D}$ and $b(x_0, x_0) \neq 0$ for some $x_0 \in X$. Let $b = \sum_{m=1}^M b_m$ for some $b_m \in \mathscr{B}_m$. Since $b(x_0, x_0) = \sum_{m=1}^M b_m(x_0, x_0) \neq 0$, there exists m such that $b_m(x_0, x_0) \neq 0$. Without loss of generality, we may assume that x_0 is irrational and $b_m(x_0, x_0) = 1$. We are going to find $y_0 \neq x_0$ such that $b(x_0, y_0) = b_m(x_0, y_0) \neq 0$, which is a contradiction to $b \in \mathscr{D}$.

Let $b_m = \sum\limits_{i=1}^{k} f_0^i a f_1^i \dots a f_m^i$, $k \geq 1$, $f_j^i \in \mathscr{D}$. Then there exists a clopen set V containing x_0 such that

$$\left| \prod_{j=0}^{m} f_j^i(y_j) - \prod_{j=0}^{m} f_j^i(x_j) \right| < \frac{1}{2k}$$

for all $1 \le i \le k$ and $(x_j), (y_j) \in V^{m+1}$. Choose n large enough so that $x_0 + i/p_n \in V$ for $0 \le i \le m$. For $f_0, \dots, f_k \in \mathscr{D}$, $k \ge 1$, we have that $(f_0 a f_1 a \dots a f_k) \times (x_0, x_0 + m/p_n) = 0$ if $k \ne m$ and that $(f_0 a f_1 a \dots a f_m) \times (x_0, x_0 + m/p_n) = 2^{-mn} \prod_{j=0}^{m} f_j(x_0 + j/p_n)$. Thus, taking $y_0 = x_0 + m/p_n$, we have

$$\begin{aligned} |b(x_0, y_0) - 2^{-mn}| &= |b_m(x_0, y_0) - 2^{-mn} b_m(x_0, x_0)| \\ &= \left| 2^{-mn} \sum_{i=1}^{k} \left(\prod_{j=0}^{m} f_j^i \left(x_0 + \frac{j}{p_m} \right) - \prod_{j=0}^{m} f_j^i(x_0) \right) \right| \\ &< 2^{-mn-1}. \end{aligned}$$

Hence, $b(x_0, y_0) \ne 0$.

So we have $\mathscr{B} \cap \mathscr{D} = \{0\}$. Since $a_n(x, y) = 0$ for all $y \lneq x$, we have $\mathscr{B} \cap \mathscr{B}^* \subset \mathscr{D}$. Hence, $\mathscr{B} \cap \mathscr{B}^* = \mathscr{B} \cap \mathscr{B}^* \cap \mathscr{D} = \{0\}$.

(2) Given a subset $E \subset G$, let $E^{-1} = \{(x, y) : (y, x) \in E\}$, we have

$$\begin{aligned} & G \backslash G^0 \supseteq Q(\bar{\mathscr{B}}) \supseteq P = \{(x, y) : y \lneq x\} \\ & \Rightarrow G \backslash G^0 \supseteq Q(\bar{\mathscr{B}}) \cup Q(\bar{\mathscr{B}}^*) \supseteq P \cup P^{-1} = G \backslash G^0 \\ & \Rightarrow Q(\bar{\mathscr{B}} \cap \bar{\mathscr{B}}^*) = Q(\bar{\mathscr{B}}) \cup Q(\bar{\mathscr{B}}^*) = G \backslash G^0 \\ & \Rightarrow \bar{\mathscr{B}} \cap \bar{\mathscr{B}}^* = I(G \backslash G^0) = \mathscr{D}. \end{aligned}$$

3. Some remarks

3.1. Let $\mathscr{S}$ be a maximal triangular subalgebra in $M_2 \otimes C^*(G)$ containing the triangular subalgebra $\mathscr{T}$ defined in §2. Then $\mathscr{S}$ does not contain $\bar{\mathscr{T}}$. Hence, $\mathscr{S}$ is not closed.

3.2. Let $\mathscr{T}$ be a norm closed triangular subalgebra of a groupoid C^*-algebra $C^*(G)$. We say that $\mathscr{T}$ is norm-maximal if the only norm closed triangular subalgebra containing $\mathscr{T}$ is $\mathscr{T}$ itself. Then it can be shown from Proposition 1 that every norm closed triangular subalgebra is contained in a norm-maximal one.

3.3. The groupoid G has a σ-finite measure μ formed in [7] by the Lebesgue measure on the unit interval and the counting measure on the rational numbers. Let $\mathscr{H}$ be the Hilbert space $L^2(G, \mu) \oplus L^2(G, \mu)$ and $\mathscr{B}(\mathscr{H})$ the algebra of all bounded operators on $\mathscr{H}$. A variation of the above construction gives an example of a maximal triangular subalgebra of $\mathscr{B}(\mathscr{H})$ which is not norm closed and consequently not σ-weakly closed, thus answering the question raised by Muhly, Saito and Solel in [7].

3.4. An earlier version of [10] contains the incorrect statement that the norm closure of a triangular subalgebra in an AF algebra is also triangular. It can be shown that $C^*(G)$ is an AF algebra (actually a UHF algebra) and $\mathscr{D}$ is a masa of the type described in [10]. Hence, $\mathscr{T}$ is a triangular subalgebra of the AF algebra $M_2 \otimes C^*(G)$ such that $\bar{\mathscr{T}}$ is not triangular.

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References

1. W. B. Arveson, *Analyticity in operator algebras*, Amer. J. Math. **89**(1967), 578–642.
2. ——, *Operator algbebras and measure preserving automorphisms*, Acta Math. **188**(1967), 95–109.
3. ——, *A density theorem for operator algebras*, Duke Math. J. **23**(1967), 635–648.
4. R. L. Baker, *Triangular UHF algebras*, J. Func. Anal. (to appear).
5. J. A. Erdos, *Some questions concerning triangular operator algebras*, Proc. Roy Irish Acad. Sec. A **74**, 223–232.
6. R. Kadison and I. M. Singer, *Triangular operator algebras*, Amer. J. Math. **82**(1960), 227–259.
7. P. S. Muhly, K. -S. Saito, and B. Solel. *Coordinates for triangular operator algebras*, Ann. of Math. **127**(1988), 245–278.
8. P. S. Muhly and B. Solel, *Subalgebras of groupoid C^* algebras*, J. Reine Angew. Math. **402**(1989), 41–75.
9. J. L. Orr, *On the closure of triangular algebras*, Amer. J. Math. (to appear).
10. J. R. Peters, Y. T. Poon and B. H. Wagner, *Triangular AF algebras*, J. Operator Theory (to appear).
11. J. R. Peters and B. H. Wagner, *Triangular AF algebras and nest subalgebras of UHF algebras*, J. Operator Theory (to appear).
12. Y. T. Poon, *A complete isomorphism invariant for a class of traingular UHF algebras*, preprint.
13. S. C. Power, *On ideals of nest subalgebras of C^*-algebras*, Proc. London Math. Soc. (3) **50** (1985), 314–332.
14. S. C. Power, *Classification of tensor products of triangular operator algebras*, Proc. London Math. Soc. (to appear).
15. S. C. Power, *The classification of triangular subalgebras of AF C^*-algebras*, Bull. London Math. Soc. (to appear).
16. J. Renault, A groupoid approach to C^*-algebras, Lecture Notes in Math., vol. 793, Springer-Verlag, 1980.
17. M. A. Thelwall, *Maximal triangular subalgebras of AF algebras*, preprint.
18. ——, *Dilation theory for subalgebras of AF algebras*, preprint.
19. B. A. Ventura, *Strongly maximal triangular UHF algebras*, preprint.
20. ——, *A note on subdiagonality for triangular algebras*, preprint.

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