PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 111, Number 2, February 1991

ZEROS OF SOLUTIONS OF A SECOND ORDER NONLINEAR DIFFERENTIAL INEQUALITY

FU-HSIANG WONG

(Communicated by Kenneth R. Meyer)

ABSTRACT. Under suitable assumptions on r, g, and F, we show that every zero of a solution of the nonlinear differential inequality

$$(r(t)y'(t))' + g(t)F(y(t)) \leq 0 (\geq 0)$$

is simple.

Recently, Kwong [1] proved the following very interesting result:

Theorem A. *If $y(t)$ is positive (negative) in an interval (a, b) with $y(a) = 0$ or $y(b) = 0$ and it satisfies the inequality*

$$y''(t) + f(t)y'(t) + g(t)y(t) \leq 0 (\geq 0) \text{ in } (a, b), \tag{1}$$

where f and g are continuous functions, then $y'(a) \neq 0$ or $y'(b) \neq 0$; that is, the given zeros of the solution are simple.

Kwong's proof is based on the formula of variation of constants.

The purpose of this article is to extend Theorem A to a nonlinear case by using the following LaSalle inequality [2]:

Theorem B. *Let*

(C_1) $\overline{F} \in C([0, c]; [0, \infty))$ *be positive and nondecreasing on* $(0, c)$, *for some* $c > 0$,

(C_2) $h \in L^1(\mathbf{R}; [0, \infty))$,

(C_3) $y \in C([a, b]; [0, c))$.

Then the inequalities

$$y(t) \leq \int_a^t h(s)\overline{F}(y(s))\, ds \quad \text{for } t \in [a, b] \tag{2}$$

Received by the editors December 5, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 34C10.
Key words and phrases. Simple zero, LaSalle's inequality.

and

(3) $$y(t) \le \int_t^b h(s)\overline{F}(y(s))\,ds \quad \textit{for } t \in [a, b]$$

imply that

(4) $$\int_0^{y(t)} \frac{ds}{\overline{F}(s)} \le \int_a^t h(s)\,ds \quad \textit{for } t \in [a, b]$$

and

(5) $$\int_0^{y(t)} \frac{ds}{\overline{F}(s)} \le \int_t^b h(s)\,ds \quad \textit{for } t \in [a, b],$$

respectively. In addition, if

(C_4) $\displaystyle\int_0^c \frac{ds}{\overline{F}(s)} = \infty$, *then* $y(t) \equiv 0$ *on* $[a, b]$.

Theorem 1. *Assume that*

(C_5) $r \in L^1([a, b]; (0, \infty))$ *and* $g \in L^1([a, b]; \mathbf{R})$,

(C_6) $F \in L^1(\mathbf{R}; \mathbf{R})$ *and there exist a positive constant* c *and a function* $\overline{F}$ *satisfying* (C_1), (C_4), *and* $|F(y)| \le \overline{F}(|y|)$ *for all* $y \in (-c, c)$.

Suppose that $y(t)$ *is positive (negative) in the interval* (a, b) *with* $y(a) = 0$ *or* $y(b) = 0$ *and it satisfies*

(6) $$(r(t)y'(t))' + g(t)F(y(t)) \le 0 (\ge 0).$$

Then $y'(a) \ne 0$ *or* $y'(b) \ne 0$.

Proof. We prove only the case $y(a) = 0$. Suppose to the contrary that $y'(a) = 0$.

Case (1). Suppose that $y(t) > 0$ on (a, b). It follows from (6) that

$$r(t)y'(t) \le -\int_a^t g(s)F(y(s))\,ds, \quad \text{for } t \in (a, b),$$

which implies

$$y(t) \le -\int_a^t \left\{ \frac{1}{r(s)} \int_a^s g(u)F(y(u))\,du \right\} ds.$$

Then

$$\begin{aligned} y(t) &\le \int_a^t m|g(s)|\,|F(y(s))|\,ds \\ &\le \int_a^t m|g(s)|\,|\overline{F}(y(s))|\,ds, \quad \text{for } t \in [a, d], \end{aligned}$$

where $m := \int_a^b \frac{ds}{r(s)}$ and $d \in (a, b]$ such that $|y(t)| < c$ for $t \in [a, d]$. Hence, by LaSalle's inequality, we see that $y(t) \equiv 0$ on $[a, d]$. This contradicts the hypothesis $y(t) > 0$ on (a, b).

Case (2). Suppose that $y(t) > 0$ on (a, b). It follows from (6) that

$$r(t)y'(t) \geq -\int_a^t g(s)F(y(s))\,ds, \quad \text{for } t \in (a, b),$$

which implies that

$$y(t) \geq -\int_a^t \left\{\frac{1}{r(s)} \int_a^s g(u)F(y(u))\,du\right\} ds.$$

Then

$$\begin{aligned} |y|(t)| &\leq \int_a^t m|g(s)|\,|F(y(s))|\,ds \\ &\leq \int_a^t m|g(s)|\,|\overline{F}(y(s))|\,ds, \quad \text{for } t \in [a, d], \end{aligned}$$

where m and d are defined as in Case (1). Hence, by LaSalle's inequality, we see that $y(t) \equiv 0$ on $[a, d]$. This contradicts the hypothesis $y(t) < 0$ on (a, b).

Theorem 2. *Assume that*

(C_7) $\quad r \in C([a, b]; (0, \infty))$ *and* $g \in C([a, b]; \mathbf{R})$,

(C_8) $\quad F \in L^1(\mathbf{R}; \mathbf{R})$ *such that* $yF(y) > 0$ *on* $(-c, c) - \{0\}$ *for some* $c > 0$, *and*

$$\int_0^c \frac{ds}{(\int_0^s F(s)\,ds)^{1/2}} = \infty.$$

Suppose that $y(t)$ *is positive (negative) in the interval* (a, b) *with* $y(a)$ *or* $y(b) = 0$ *and it satisfies* (6). *Then* $y'(a) \neq 0$ *or* $y'(b) \neq 0$.

Proof. Using the classical Liouville transformation, we can always get rid of the coefficient $r(t)$. So we may assume without loss of generality that the inequality we are studying is

$$y''(t) + g(t)F(y(t)) \leq 0. \tag{7}$$

Let $C > 0$ be a large constant, larger than the maximum of $|g(t)|$ over $[a, b]$. Then

$$y''(t) - CF(y(t)) \leq -(C + |g(t)|)F(y(t)) \leq 0, \tag{8}$$

at least in a neighborhood of a when $|y(t)|$ is still less than c. We can now work with (7), which has a constant coefficient C. Multiplying (8) by $y'(t)$ and integrating it from a to t (near a), we obtain that

$$y'^2(t) \leq 2c \int_0^{y(t)} F(s)\,ds,$$

which implies that

$$y'(t) \leq \left\{ 2c \int_0^{y(t)} F(s)\, ds \right\}^{1/2}.$$

Integrating the above inequality from a to t (near a), one obtains that

$$y(t) \leq \sqrt{2c} \int_a^t \left\{ \int_0^{y(s)} F(u)\, du \right\}^{1/2} ds.$$

Hence, by LaSalle's inequality, we see that $y(t) \equiv 0$ in a neighborhood of a. This contradicts the hypothesis $y(t) > 0$ on (a, b).

Acknowledgment

The author would like to express his gratitude to the referee for a number of helpful suggestions.

References

1. M. K. Kwong, *Uniqueness of positive solutions of* $\Delta u - u + u^p = 0$ *in* $\mathbf{R}^n$, Arch. Rational Mech. Anal. **105** (1989), 243–266.
2. J. P. LaSalle, *Uniqueness theorems and successive approximations*, Ann. of Math. **50** (1949), 722–730.

Department of Mathematics, National Central University, Chung-Li, Taiwan, Republic of China