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ON THE WEIGHTED APPROXIMATION OF CONTINUOUSLY DIFFERENTIABLE FUNCTIONS

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ABSTRACT. This note is an improvement of the available methods for getting results on the weighted approximation of continuously differentiable functions.

We shall present what we believe to be a simplified version of the reasoning to establish some results concerning weighted approximation of continuously differentiable scalar functions on $\mathbf{R}^n$ (see [3]). For references to weighted approximation of continuous scalar functions on $\mathbf{R}^n$ (see [1] and [3]). Lemma 1 reduces the search for sufficient conditions in order that a weight on $\mathbf{R}$ be C^m-fundamental to the finding of sufficient conditions for a weight on $\mathbf{R}$ to be C-fundamental. Similarly, Lemma 2 reduces the finding of sufficient conditions for a weight on $\mathbf{R}^n$ to be C^m-fundamental to the search of sufficient conditions for a weight on $\mathbf{R}$ to be C^m-fundamental. We then apply Lemmas 1 and 2 together to obtain Propositions 2, 4, and 5.

Fix integers $n \in \mathbf{N}$, $n \geq 1$, and $m \in \mathbf{N}$ (the case $m = \infty$ will be excluded since it follows easily from all $m \in \mathbf{N}$). Let $\mathbf{K}$ denote either $\mathbf{R}$ or $\mathbf{C}$. Consider the algebras $\mathscr{P}(\mathbf{R}^n)$ of all $\mathbf{K}$-valued polynomials on $\mathbf{R}^n$ and $C^m(\mathbf{R}^n)$ of all continuously m-differentiable $\mathbf{K}$-valued functions on $\mathbf{R}^n$. Write $\mathbf{N}_m^n = \{\alpha \in \mathbf{N}^n : |\alpha| \leq m\}$ where $|a| = \alpha_1 + \cdots + \alpha_n$ if $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$. Let $D^\alpha f$ be the αth partial derivative of $f \in C^m(\mathbf{R}^n)$ for $\alpha \in \mathbf{N}_m^n$. Denote by $\mathscr{D}^m(\mathbf{R}^n)$ the subalgebra of $C^m(\mathbf{R}^n)$ of all functions with compact support.

A C^m-weight on $\mathbf{R}^n$ is a family $v = \{v_\alpha : \alpha \in \mathbf{N}_m^n\}$ of upper semicontinuous functions $v_\alpha \geq 0$ on $\mathbf{R}^n$. Such a weight v defines the vector space $C^m v_\infty(\mathbf{R}^n)$ of all $f \in C^m(\mathbf{R}^n)$ such that $v_\alpha D^\alpha f$ tends to zero at infinity for every $\alpha \in \mathbf{N}_m^n$. Set $\|f\|_{v_\alpha} = \sup\{v_\alpha(t) \cdot |D^\alpha f(t)| : t \in \mathbf{R}^n\}$ to get a seminorm $f \in C^m v_\infty(\mathbf{R}^n) \mapsto \|f\|_{v_\alpha} \in \mathbf{R}_+$ for every $\alpha \in \mathbf{N}_m^n$. The finite family of such seminorms makes $C^m v_\infty(\mathbf{R}^n)$ into a seminormable space, actually a

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seminormed space if for instance we use the seminorm $f \in C^m v_\infty(\mathbf{R}^n) \mapsto \|f\|_v = \sup\{\|f\|_{v_\alpha} : \alpha \in \mathbf{N}_m^n\} = \sup\{v_\alpha(t) \cdot |D^\alpha f(t)| : t \in \mathbf{R}^n, \alpha \in \mathbf{N}_m^n\} \in \mathbf{R}_+$. We now have that $\mathscr{D}^m(\mathbf{R}^n) \subset C^m v_\infty(\mathbf{R}^n)$. It is known that $\mathscr{D}^m(\mathbf{R}^n)$ is dense in $C^m v_\infty(\mathbf{R}^n)$ if the weight v is decreasing in the sense that, for every α, $\beta \in \mathbf{N}_m^n$ with $\beta \leq \alpha$, there exists $C_{\alpha\beta} \geq 0$ such that $v_\alpha \leq C_{\alpha\beta} v_\beta$ ([3, Lemma 1]). $\mathscr{D}^m(\mathbf{R}^n)$ may be dense in $C^m v_\infty(\mathbf{R}^n)$ even if v is not decreasing, and it may fail to be dense as well. The weight v is said to be rapidly decreasing if $\mathscr{P}(\mathbf{R}^n) \subset C^m v_\infty(\mathbf{R}^n)$. If, moreover, $\mathscr{P}(\mathbf{R}^n)$ is dense in $C^m v_\infty(\mathbf{R}^n)$, then v is called C^m-fundamental.

Lemma 1. *Let $m \in \mathbf{N}$, $v_i \geq 0$, $i = 0, \dots, m$, and $u \geq 0$ be upper semicontinuous on $\mathbf{R}$. Consider the C^m-weight $v = (v_0, \dots, v_m)$ on $\mathbf{R}$ which is assumed to be decreasing. Let u be C-fundamental on $\mathbf{R}$ such that s, $t \in \mathbf{R}$, $|s| \leq |t|$ imply $u(t) \leq u(s)$ and $|t|^{m-i} v_i(t) \leq u(t)$, $v_i(t) \leq u(t)$ for $t \in \mathbf{R}$, $i = 0, \dots, m$. Then v is a C^m-fundamental weight on $\mathbf{R}$.*

Proof. The lemma is true if $m = 0$ because then $v_0 \leq u$ and hence v_0 is C-fundamental along with u. Assume now that $m \geq 1$ and that the lemma is true for $m-1$. Notice that $\mathscr{P}(\mathbf{R}) \subset C^m v_\infty(\mathbf{R})$ because $\mathscr{P}(\mathbf{R}) \subset C u_\infty(\mathbf{R})$ and $v_i \leq u$, $i = 0, \dots, m$. Hence v is rapidly decreasing. Fix any $f \in \mathscr{D}^m(\mathbf{R})$ and $\varepsilon > 0$. Then $f^{(m)} \in \mathscr{D}(\mathbf{R}) \subset C u_\infty(\mathbf{R})$ and there is a $P^{(m)} \in \mathscr{P}(\mathbf{R})$ so that

$$u(t) \cdot |P^{(m)}(t) - f^{(m)}(t)| \leq \varepsilon, \qquad t \in \mathbf{R}. \tag{1}$$

Choose $P \in \mathscr{P}(\mathbf{R})$ whose mth derivative is $P^{(m)}$ such that $P^{(i)}(0) = f^{(i)}(0)$, $i = 0, \dots, m-1$. We claim that

$$v_i(t) \cdot |P^{(i)}(t) - f^{(i)}(t)| \leq \varepsilon, \qquad t \in \mathbf{R},\ i = 0, \dots, m. \tag{2}$$

In fact, for $i = m$ this follows from (1) and $v_m \leq u$. For every $t \in \mathbf{R}$, $i = 0, \dots, m-1$, there is an $s \in \mathbf{R}$, $|s| \leq |t|$ such that $|P^{(i)}(t) - f^{(i)}(t)| \leq |t|^{m-i} \cdot |P^{(m)}(s) - f^{(m)}(s)|$ by the mean value theorem applied $m - i$ times. Notice that $|t|^{m-i} v_i(t) \leq u(t) \leq u(s)$. Hence (1) with s in place of t shows that (2) is true for $i = 0, \dots, m-1$ as well as for $i = m$. It follows that $\mathscr{D}^m(\mathbf{R})$ is contained in the closure of $\mathscr{P}(\mathbf{R})$ in $C^m v_\infty(\mathbf{R})$. Therefore, $\mathscr{P}(\mathbf{R})$ is dense in $C^m v_\infty(\mathbf{R})$ along with $\mathscr{D}^m(\mathbf{R})$. Thus v is C^m-fundamental. □

Let $v \geq 0$ be upper semicontinuous on $\mathbf{R}$. We say that v is an analytic weight on $\mathbf{R}$ when there exist $C > 0$, $c > 0$ such that $v(t) \leq Ce^{-c|t|}$, $t \in \mathbf{R}$. It is then known that v is a C-fundamental weight on $\mathbf{R}$ ([2, §28, Lemma 2]). More generally, if $v \geq 0$ is upper semicontinuous on $\mathbf{R}^n$, we say that v is an analytic weight on $\mathbf{R}^n$ when there exist $C > 0$, $c > 0$ such that $v(t) \leq Ce^{-c(|t_1|+\cdots+|t_n|)}$, $t \in \mathbf{R}^n$. It is then known that v is C-fundamental on $\mathbf{R}^n$.

Proposition 2. *Let $m \in \mathbf{N}$, $v_i \geq 0$, $i = 0, \dots, m$, be upper semicontinuous on $\mathbf{R}$. Consider the C^m-weight $v = (v_0, \dots, v_m)$ on $\mathbf{R}$ which is assumed to be decreasing. Let each v_i, $i = 0, \dots, m$, be an analytic weight. Then v is a C^m-fundamental weight on $\mathbf{R}$.*

Proof. Assume that $v_i(t) \le Ce^{-c|t|}$, $t \in \mathbf{R}$, $i = 0, \dots, m$, for some $C > 0$, $c > 0$. Choose $D > 0$, $0 < d < c$ so that, if $u(t) = De^{-d|t|}$, $t \in \mathbf{R}$, all assumptions in Lemma 1 are satisfied. □

Lemma 3. *Let $n \in \mathbf{N}$, $n \ge 1$, $m \in \mathbf{N}$, $v_\alpha \ge 0$, $\alpha \in \mathbf{N}_m^n$, be upper semicontinuous on $\mathbf{R}$. Consider the C^m-weight $v = (v_\alpha; \alpha \in \mathbf{N}_m^n)$ on $\mathbf{R}^n$ which is assumed to be decreasing. Let $u_{ij} \ge 0$, $i = 1, \dots, n$, $j = 0, \dots, m$, be upper semicontinuous on $\mathbf{R}$. Consider the C^m-weights $u_i = (u_{i0}, \dots, u_{im})$, $i = 1, \dots, n$, on $\mathbf{R}$ which are supposed to be decreasing and C^m-fundamental. Assume*

$$v_\alpha(t) \le u_{1\alpha_1}(t_1) \dots u_{n\alpha_n}(t_n), \qquad t \in \mathbf{R}^n, \alpha \in \mathbf{N}_m^n.$$

Then v is C^m-fundamental on $\mathbf{R}^n$.

Proof. Consider the n-linear mapping π that, with every

$$(f_1, \dots, f_n) \in C^m(u_1)_\infty(\mathbf{R}) \times \dots \times C^m(u_n)_\infty(\mathbf{R})$$

associates $f_1 \otimes \dots \otimes f_n \in C^m v_\infty(\mathbf{R}^n)$ where $(f_1 \otimes \dots \otimes f_n)(t) = f_1(t_1) \cdots f_n(f_n)$ for $t \in \mathbf{R}^n$. The assumptions make it sure that π is well defined and continuous because

$$\|f_1 \otimes \dots \otimes f_n\|_{v_\alpha} \le \|f_1\|_{u_{1\alpha_1}} \dots \|f_n\|_{u_{n\alpha_n}}.$$

By hypothesis, $\mathscr{P}(\mathbf{R})$ is dense in $C^m(u_i)_\infty(\mathbf{R})$ so $\mathscr{D}^m(\mathbf{R})$ is contained in the closure of $\mathscr{P}(\mathbf{R})$ in $C^m(u_i)_\infty(\mathbf{R})$, $i = 1, \dots, n$. Thus $\pi[\mathscr{D}^m(\mathbf{R}) \times \dots \times \mathscr{D}^m(\mathbf{R})]$ is contained in the closure of $\pi[\mathscr{P}(\mathbf{R}) \times \dots \times \mathscr{P}(\mathbf{R})]$ in $C^m v_\infty(\mathbf{R}^n)$. Hence the vector subspace $\mathscr{D}^m(\mathbf{R}) \otimes \dots \otimes \mathscr{D}^m(\mathbf{R})$ generated by $\pi[\mathscr{D}^m(\mathbf{R}) \times \dots \times \mathscr{D}^m(\mathbf{R})]$ is contained in the closure in $C^m v_\infty(\mathbf{R}^n)$ of the vector subspace $\mathscr{P}(\mathbf{R}^n) = \mathscr{P}(\mathbf{R}) \otimes \dots \otimes \mathscr{P}(\mathbf{R})$ generated by $\pi[\mathscr{P}(\mathbf{R}) \times \dots \times \mathscr{P}(\mathbf{R})]$. It is known that $\mathscr{D}^m(\mathbf{R}) \otimes \dots \otimes \mathscr{D}^m(\mathbf{R})$ is dense in $\mathscr{D}^m(\mathbf{R}^n)$ in the natural inductive limit topology of $\mathscr{D}^m(\mathbf{R}^n)$, hence in the coarser topology on $\mathscr{D}^m(\mathbf{R}^n)$ defined by the norm $f \in \mathscr{D}^m(\mathbf{R}^n) \mapsto \sup\{|D^\alpha f(t)|: t \in \mathbf{R}^n, |\alpha| \le m\} \in \mathbf{R}_+$, hence (because all v_α are upper bounded along with all u_{ij}) in the even coarser topology that the natural topology of $C^m v_\infty(\mathbf{R}^n)$ induces on $\mathscr{D}^m(\mathbf{R}^n)$. Since $\mathscr{D}^m(\mathbf{R}^n)$ is dense in $C^m v_\infty(\mathbf{R}^n)$, then $\mathscr{D}^m(\mathbf{R}) \otimes \dots \otimes \mathscr{D}^m(\mathbf{R})$ is dense in $C^m v_\infty(\mathbf{R}^n)$. The fact that $\mathscr{D}^m(\mathbf{R}) \otimes \dots \otimes \mathscr{D}^m(\mathbf{R})$ is contained in the closure of $\mathscr{P}(\mathbf{R}^n)$ in $C^m v_\infty(\mathbf{R}^n)$ implies then that $\mathscr{P}(\mathbf{R}^n)$ is dense in $C^m v_\infty(\mathbf{R}^n)$. Thus v is C^m-fundamental. □

Proposition 4. *Let $n \in \mathbf{N}$, $n \ge 1$, $m \in \mathbf{N}$, $v_\alpha \ge 0$, $\alpha \in \mathbf{N}_m^n$, be upper semicontinuous on $\mathbf{R}^n$. Consider the C^m-weight $v = (v_\alpha; \alpha \in \mathbf{N}_m^n)$ on $\mathbf{R}^n$ which is assumed to be decreasing. Assume that each v_α, $\alpha \in \mathbf{N}_m^n$, is an analytic weight. Then v is C^m-fundamental on $\mathbf{R}^n$.*

Proof. Let $v_\alpha(t) \le Ce^{-c(|t_1| + \dots + |t_n|)}$, $t \in \mathbf{R}^n$, $\alpha \in \mathbf{N}_m^n$, for suitable $C > 0$, $c > 0$. Choose $u_{ij}(t_i) = C^{1/n} e^{-c|t_i|}$, $i = 1, \dots, n$, $j = 1, \dots, m$, so that all assumptions in Lemma 3 are satisfied. □

Let $v \geq 0$ be upper semicontinuous on $\mathbf{R}$ and rapidly decreasing. Put $M_m = \sup\{|t^m| \cdot v(t) : t \in \mathbf{R}\} \in \mathbf{R}_+$, $m = 0, 1, \ldots$. We say that v is a quasi-analytic weight on $\mathbf{R}$ when $\sum_{m=1}^{\infty} (1/\sqrt[m]{M_m}) = +\infty$. It is then known that v is a C-fundamental weight on $\mathbf{R}$ and that every analytic weight on $\mathbf{R}$ is quasi-analytic (see [2, §29, Lemma 2]). More generally, if $v \geq 0$ is upper semicontinuous on $\mathbf{R}^n$, we say that v is a quasi-analytic weight on $\mathbf{R}^n$ if there are quasi-analytic weights $v_1, \ldots, v_n$ on $\mathbf{R}$ such that $v(t) \leq v_1(t_1) \cdots v_n(t_n)$, $t \in \mathbf{R}^n$. It is then known that v is C-fundamental on $\mathbf{R}^n$, and that every analytic weight on $\mathbf{R}^n$ is quasi-analytic.

Proposition 5. *Let $n \in \mathbf{N}$, $n \geq 1$, $m \in \mathbf{N}$, $v_\alpha \geq 0$, $\alpha \in \mathbf{N}_m^n$, be upper semicontinuous on $\mathbf{R}^n$. Consider the C^m-weight $v = (v_\alpha; \alpha \in \mathbf{N}_m^n)$ which is supposed to be decreasing. Assume that there are quasi-analytic weights v_i, $i = 1, \ldots, n$, on $\mathbf{R}$ such that $v_\alpha(t) \leq v_1(t_1) \cdots v_n(t_n)$, $t \in \mathbf{R}^n$, $\alpha \in \mathbf{N}_m^n$. Then v is a C^m-fundamental weight on $\mathbf{R}^m$.*

Proof. All assumptions of Lemma 3 apply. □

Lemma 6. *The linear mapping $D: f \in \mathscr{D}^1(\mathbf{R}) \mapsto f' \in \mathscr{K}(\mathbf{R}) = \mathscr{D}^0(\mathbf{R})$ is injective. Its image is $D\mathscr{D}^1(\mathbf{R}) = \{g \in \mathscr{K}(\mathbf{R}); \int g = 0\}$. If $u \geq 0$ is upper semicontinuous on $\mathbf{R}$, this image is dense in $Cu_\infty(\mathbf{R})$ if and only if $\int 1/u = +\infty$.*

Proof. Only the final part of the lemma requires a proof. Consider the linear form $I: f \in \mathscr{K}(\mathbf{R}) \mapsto \int f \in \mathbf{K}$. Assume $\int 1/u = +\infty$. We claim that I is not continuous on $\mathscr{K}(\mathbf{R})$ for the seminorm induced by $Cu_\infty(\mathbf{R})$. In fact, given any $c \geq 0$, there is an $f \in \mathscr{K}(\mathbf{R})$ such that $0 \leq f \leq 1/u$, $\int f \geq c$. Therefore $\|f\|_u \leq 1$ and $I(f) \geq c$ show that I is not continuous on $\mathscr{K}(\mathbf{R})$ for the seminorm induced by $Cu_\infty(\mathbf{R})$. It follows that $I^{-1}(0) = D\mathscr{D}^1(\mathbf{R})$ is dense in $Cu_\infty(\mathbf{R})$. Conversely let $c = \int 1/u < +\infty$. Then the set where u vanishes has a void interior. It follows that the seminorm on $Cu_\infty(\mathbf{R})$ is actually a norm. We claim that $|I(f)| \leq c \cdot \|f\|_u$ for $f \in \mathscr{K}(\mathbf{R})$. This is clear if $\|f\|_u = 0$. If $\|f\|_u > 0$ then $u(t)|f(t)| \leq \|f\|_u$ implies $|f(t)| \leq \|f\|_u/u(t)$ for $t \in \mathbf{R}$, hence $|I(f)| \leq c\|f\|_u$ for $f \in \mathscr{K}(\mathbf{R})$ as asserted. Thus I is continuous on $\mathscr{K}(\mathbf{R})$ for the norm induced by $Cu_\infty(\mathbf{R})$ and I extends uniquely to a continuous linear form I on $Cu_\infty(\mathbf{R})$, since $\mathscr{K}(\mathbf{R})$ is dense in $Cu_\infty(\mathbf{R})$. We know that $I \neq 0$ because I does not vanish on $\mathscr{K}(\mathbf{R})$, but I does vanish on $D\mathscr{D}^1(\mathbf{R})$. It follows that $D\mathscr{D}^1(\mathbf{R})$ is not dense in $Cu_\infty(\mathbf{R})$. □

Example 7. Consider $v_0 = 0$, $v_1 \geq 0$ upper semicontinuous on $\mathbf{R}$, and the C^1-weight $v = (v_0, v_1)$ on $\mathbf{R}$. (Notice that v is not decreasing unless $v_1 = 0$.) Then $\mathscr{D}^1(\mathbf{R})$ is dense in $C^1v_\infty(\mathbf{R})$ if and only if $\int 1/v_1 = +\infty$.

Proof. Since $v_0 = 0$ we see that $\mathscr{D}^1(\mathbf{R})$ is dense in $C^1v_\infty(\mathbf{R})$ if and only if $D\mathscr{D}^1(\mathbf{R})$ is dense in $C(v_1)_\infty(\mathbf{R})$. It remains to apply Lemma 6 with v_1 in place of u. □

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