ON THE WEIGHTED APPROXIMATION OF CONTINUOUSLY DIFFERENTIABLE FUNCTIONS

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Abstract. This note is an improvement of the available methods for getting results on the weighted approximation of continuously differentiable functions.

We shall present what we believe to be a simplified version of the reasoning to establish some results concerning weighted approximation of continuously differentiable scalar functions on \( \mathbb{R}^n \) (see [3]). For references to weighted approximation of continuous scalar functions on \( \mathbb{R}^n \) (see [1] and [3]). Lemma 1 reduces the search for sufficient conditions in order that a weight on \( \mathbb{R} \) be \( C^m \)-fundamental to the finding of sufficient conditions for a weight on \( \mathbb{R} \) to be \( C \)-fundamental. Similarly, Lemma 2 reduces the finding of sufficient conditions for a weight on \( \mathbb{R}^n \) to be \( C^m \)-fundamental to the search of sufficient conditions for a weight on \( \mathbb{R} \) to be \( C^m \)-fundamental. We then apply Lemmas 1 and 2 together to obtain Propositions 2, 4, and 5.

Fix integers \( n \in \mathbb{N}, n \geq 1 \), and \( m \in \mathbb{N} \) (the case \( m = \infty \) will be excluded since it follows easily from all \( m \in \mathbb{N} \)). Let \( K \) denote either \( \mathbb{R} \) or \( \mathbb{C} \). Consider the algebras \( \mathcal{P}(\mathbb{R}^n) \) of all \( K \)-valued polynomials on \( \mathbb{R}^n \) and \( C^m(\mathbb{R}^n) \) of all continuously \( m \)-differentiable \( K \)-valued functions on \( \mathbb{R}^n \). Write \( \mathbb{N}^m_n = \{ \alpha \in \mathbb{N}^n : |\alpha| \leq m \} \) where \( |\alpha| = \alpha_1 + \cdots + \alpha_n \) if \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \). Let \( D^\alpha f \) be the \( \alpha \)th partial derivative of \( f \in C^m(\mathbb{R}^n) \) for \( \alpha \in \mathbb{N}^m_n \). Denote by \( \mathcal{D}^m(\mathbb{R}^n) \) the subalgebra of \( C^m(\mathbb{R}^n) \) of all functions with compact support.

A \( C^m \)-weight on \( \mathbb{R}^n \) is a family \( v = \{ v_\alpha : \alpha \in \mathbb{N}^m_n \} \) of upper semicontinuous functions \( v_\alpha \geq 0 \) on \( \mathbb{R}^n \). Such a weight \( v \) defines the vector space \( C^m v_\infty(\mathbb{R}^n) \) of all \( f \in C^m(\mathbb{R}^n) \) such that \( v_\alpha D^\alpha f \) tends to zero at infinity for every \( \alpha \in \mathbb{N}^m_n \). Set \( \|f\|_{v_\alpha} = \sup\{v_\alpha(t) \cdot |D^\alpha f(t)| : t \in \mathbb{R}^n \} \) to get a seminorm \( f \in C^m v_\infty(\mathbb{R}^n) \rightarrow \|f\|_{v_\alpha} \in \mathbb{R}_+ \) for every \( \alpha \in \mathbb{N}^m_n \). The finite family of such seminorms makes \( C^m v_\infty(\mathbb{R}^n) \) into a seminormable space, actually a

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seminormed space if for instance we use the seminorm \( f \in C^m v_\infty(R^n) \mapsto \| f \|_v = \sup \{ \| f \|_{v_\alpha} : \alpha \in \mathbb{N}_m^n \} = \sup \{ v_\alpha(t) \cdot |D^\alpha f(t)| : t \in R^n, \alpha \in \mathbb{N}_m^n \} \in \mathbb{R}_+ \).

We now have that \( \mathcal{D}^m(R^n) \subset C^m v_\infty(R^n) \). It is known that \( \mathcal{D}^m(R^n) \) is dense in \( C^m v_\infty(R^n) \) if the weight \( v \) is decreasing in the sense that, for every \( \alpha, \beta \in \mathbb{N}_m^n \) with \( \beta \leq \alpha \), there exists \( C_{\alpha \beta} \geq 0 \) such that \( v_\alpha \leq C_{\alpha \beta} v_\beta \) ([3, Lemma 1]). \( \mathcal{D}^m(R^n) \) may be dense in \( C^m v_\infty(R^n) \) even if \( v \) is not decreasing, and it may fail to be dense as well. The weight \( v \) is said to be rapidly decreasing if \( \mathcal{D}(R^n) \subset C^m v_\infty(R^n) \). If, moreover, \( \mathcal{P}(R^n) \) is dense in \( C^m v_\infty(R^n) \), then \( v \) is called \( C^m \)-fundamental.

**Lemma 1.** Let \( m \in \mathbb{N}, v_i \geq 0, i = 0, \ldots, m, \) and \( u \geq 0 \) be upper semicontinuous on \( R \). Consider the \( C^m \)-weight \( v = (v_0, \ldots, v_m) \) on \( R \) which is assumed to be decreasing. Let \( u \) be \( C \)-fundamental on \( R \) such that \( v_i \leq u, i = 0, \ldots, m \). Then \( v \) is a \( C^m \)-fundamental weight on \( R \).

**Proof.** The lemma is true if \( m = 0 \) because then \( v_0 \leq u \) and hence \( v_0 \) is \( C \)-fundamental along with \( u \). Assume now that \( m \geq 1 \) and that the lemma is true for \( m - 1 \). Notice that \( \mathcal{P}(R) \subset C^m v_\infty(R) \) because \( \mathcal{P}(R) \subset C u_\infty(R) \) and \( v_i \leq u, i = 0, \ldots, m \). Hence \( v \) is rapidly decreasing. Fix any \( f \in \mathcal{D}^m(R) \) and \( \varepsilon > 0 \). Then \( f^{(m)} \in \mathcal{D}(R) \subset C u_\infty(R) \) and there is a \( P^{(m)} \in \mathcal{P}(R) \) so that

\[
(1) \quad u(t) \cdot |P^{(m)}(t) - f^{(m)}(t)| \leq \varepsilon, \quad t \in R.
\]

Choose \( P \in \mathcal{P}(R) \) whose \( m \)th derivative is \( P^{(m)} \) such that \( P^{(i)}(0) = f^{(i)}(0), i = 0, \ldots, m - 1 \). We claim that

\[
(2) \quad v_i(t) \cdot |P^{(i)}(t) - f^{(i)}(t)| \leq \varepsilon, \quad t \in R, \quad i = 0, \ldots, m.
\]

In fact, for \( i = m \) this follows from (1) and \( v_m \leq u \). For every \( t \in R, i = 0, \ldots, m - 1 \), there is an \( s \in R, |s| \leq |t| \) such that \( |P^{(i)}(t) - f^{(i)}(t)| \leq |t|^{m-i} \cdot |P^{(m)}(s) - f^{(m)}(s)| \) by the mean value theorem applied \( m - i \) times. Notice that \( |t|^{m-i} v_i(t) \leq u(t) \leq u(s) \). Hence (1) with \( s \) in place of \( t \) shows that (2) is true for \( i = 0, \ldots, m - 1 \) as well as for \( i = m \). It follows that \( \mathcal{D}^m(R) \) is contained in the closure of \( \mathcal{P}(R) \) in \( C^m v_\infty(R) \). Therefore, \( \mathcal{P}(R) \) is dense in \( C^m v_\infty(R) \) along with \( \mathcal{D}^m(R) \). Thus \( v \) is \( C^m \)-fundamental.

Let \( v \geq 0 \) be upper semicontinuous on \( R \). We say that \( v \) is an analytic weight on \( R \) when there exist \( C > 0, c > 0 \) such that \( v(t) \leq Ce^{-ct}, t \in R \).

It is then known that \( v \) is a \( C \)-fundamental weight on \( R \) ([2, §28, Lemma 2]). More generally, if \( v \geq 0 \) is upper semicontinuous on \( R^n \), we say that \( v \) is an analytic weight on \( R^n \) when there exist \( C > 0, c > 0 \) such that \( v(t) \leq Ce^{-c(|t_1| + \cdots + |t_n|)}, t \in R^n \). It is then known that \( v \) is \( C \)-fundamental on \( R^n \).

**Proposition 2.** Let \( m \in \mathbb{N}, v_i \geq 0, i = 0, \ldots, m, \) be upper semicontinuous on \( R \). Consider the \( C^m \)-weight \( v = (v_0, \ldots, v_m) \) on \( R \) which is assumed to be decreasing. Let each \( v_i, i = 0, \ldots, m, \) be an analytic weight. Then \( v \) is a \( C^m \)-fundamental weight on \( R \).
Proof. Assume that $v_i(t) \leq C e^{-c|t|}$, $t \in \mathbb{R}$, $i = 0, \ldots, m$, for some $C > 0$, $c > 0$. Choose $D > 0$, $0 < d < c$ so that, if $u(t) = De^{-d|t|}$, $t \in \mathbb{R}$, all assumptions in Lemma 1 are satisfied. □

Lemma 3. Let $n \in \mathbb{N}$, $n \geq 1$, $m \in \mathbb{N}$, $v_\alpha \geq 0$, $\alpha \in \mathbb{N}_m^n$, be upper semicontinuous on $\mathbb{R}$. Consider the $C^m$-weight $v = (v_\alpha; \alpha \in \mathbb{N}_m^n)$ on $\mathbb{R}^n$ which is assumed to be decreasing. Let $u_{ij} \geq 0$, $i = 1, \ldots, n$, $j = 0, \ldots, m$, be upper semicontinuous on $\mathbb{R}$. Consider the $C^m$-weights $u_i = (u_{i0}, \ldots, u_{im})$, $i = 1, \ldots, n$, on $\mathbb{R}$ which are supposed to be decreasing and $C^m$-fundamental. Assume

$$v_\alpha(t) \leq u_{1\alpha_1}(t_1) \cdots u_{n\alpha_n}(t_n), \quad t \in \mathbb{R}^n, \alpha \in \mathbb{N}_m^n.$$  

Then $v$ is $C^m$-fundamental on $\mathbb{R}^n$.

Proof. Consider the $n$-linear mapping $\pi$ that, with every

$$(f_1, \ldots, f_n) \in C^m(u_1)_\infty(\mathbb{R}) \times \cdots \times C^m(u_n)_\infty(\mathbb{R})$$

associates $f_1 \otimes \cdots \otimes f_n \in C^m v_\infty(\mathbb{R}^n)$ where $(f_1 \otimes \cdots \otimes f_n)(t) = f_1(t_1) \cdots f_n(t_n)$ for $t \in \mathbb{R}^n$. The assumptions make it sure that $\pi$ is well defined and continuous because

$$\|f_1 \otimes \cdots \otimes f_n\|_{v_\alpha} \leq \|f_1\|_{u_{1\alpha_1}} \cdots \|f_n\|_{u_{n\alpha_n}}.$$  

By hypothesis, $\mathcal{P}(\mathbb{R})$ is dense in $C^m(u_1)_\infty(\mathbb{R})$ so $\mathcal{D}^m(\mathbb{R})$ is contained in the closure of $\mathcal{P}(\mathbb{R})$ in $C^m(u_1)_\infty(\mathbb{R})$, $i = 1, \ldots, n$. Thus $\pi[\mathcal{D}^m(\mathbb{R}) \times \cdots \times \mathcal{D}^m(\mathbb{R})]$ is contained in the closure of $\pi[\mathcal{P}(\mathbb{R}) \times \cdots \times \mathcal{P}(\mathbb{R})]$ in $C^m v_\infty(\mathbb{R}^n)$. Hence the vector subspace $\mathcal{D}^m(\mathbb{R}) \otimes \cdots \otimes \mathcal{D}^m(\mathbb{R})$ generated by $\pi[\mathcal{D}^m(\mathbb{R}) \times \cdots \times \mathcal{D}^m(\mathbb{R})]$ is contained in the closure in $C^m v_\infty(\mathbb{R}^n)$ of the vector subspace $\mathcal{P}(\mathbb{R}^n) = \mathcal{P}(\mathbb{R}) \otimes \cdots \otimes \mathcal{P}(\mathbb{R})$. It is known that $\mathcal{D}^m(\mathbb{R}) \otimes \cdots \otimes \mathcal{D}^m(\mathbb{R})$ is dense in $\mathcal{D}^m(\mathbb{R}^n)$ in the natural inductive limit topology of $\mathcal{D}^m(\mathbb{R}^n)$, hence in the coarser topology on $\mathcal{D}^m(\mathbb{R}^n)$ defined by the norm $f \in \mathcal{D}^m(\mathbb{R}^n) \mapsto \sup \{|D^m f(t)|: t \in \mathbb{R}^n, |a| \leq m \in \mathbb{R}_+\}$, hence (because all $v_\alpha$ are upper bounded along with all $u_{ij}$) in the even coarser topology that the natural topology of $C^m v_\infty(\mathbb{R}^n)$ induces on $\mathcal{D}^m(\mathbb{R}^n)$. Since $\mathcal{D}^m(\mathbb{R}^n)$ is dense in $C^m v_\infty(\mathbb{R}^n)$, then $\mathcal{D}^m(\mathbb{R}) \otimes \cdots \otimes \mathcal{D}^m(\mathbb{R})$ is dense in $C^m v_\infty(\mathbb{R}^n)$. The fact that $\mathcal{D}^m(\mathbb{R}) \otimes \cdots \otimes \mathcal{D}^m(\mathbb{R})$ is contained in the closure of $\mathcal{P}(\mathbb{R}^n)$ in $C^m v_\infty(\mathbb{R}^n)$ implies then that $\mathcal{P}(\mathbb{R}^n)$ is dense in $C^m v_\infty(\mathbb{R}^n)$. Thus $v$ is $C^m$-fundamental. □

Proposition 4. Let $n \in \mathbb{N}$, $n \geq 1$, $m \in \mathbb{N}$, $v_\alpha \geq 0$, $\alpha \in \mathbb{N}_m^n$, be upper semicontinuous on $\mathbb{R}^n$. Consider the $C^m$-weight $v = (v_\alpha; \alpha \in \mathbb{N}_m^n)$ on $\mathbb{R}^n$ which is assumed to be decreasing. Assume that each $v_\alpha$, $\alpha \in \mathbb{N}_m^n$, is an analytic weight. Then $v$ is $C^m$-fundamental on $\mathbb{R}^n$.

Proof. Let $v_\alpha(t) \leq C e^{-c(|t_1| + \cdots + |t_n|)}$, $t \in \mathbb{R}^n$, $\alpha \in \mathbb{N}_m^n$, for suitable $C > 0$, $c > 0$. Choose $u_{ij}(t_i) = C^{1/n} e^{-c|t_i|}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, so that all assumptions in Lemma 3 are satisfied. □
Let \( v \geq 0 \) be upper semicontinuous on \( \mathbb{R} \) and rapidly decreasing. Put 
\[
M_m = \sup \{|r^n| \cdot v(t) : t \in \mathbb{R}\} \in \mathbb{R}_+ , \quad m = 0, 1, \ldots
\]
We say that \( v \) is a quasi-analytic weight on \( \mathbb{R} \) when 
\[
\sum_{m=1}^{\infty} \left( 1/ \sqrt{M_m} \right) = +\infty.
\]
It is then known that \( v \) is a \( C \)-fundamental weight on \( \mathbb{R} \) and that every analytic weight on \( \mathbb{R} \) is quasi-analytic (see [2, §29, Lemma 2]). More generally, if \( v \geq 0 \) is upper semicontinuous on \( \mathbb{R}^n \), we say that \( v \) is a quasi-analytic weight on \( \mathbb{R}^n \) if there are quasi-analytic weights \( v_1, \ldots, v_n \) on \( \mathbb{R} \) such that 
\[
v(t) \leq v_1(t_1) \cdots v_n(t_n), \quad t \in \mathbb{R}^n.
\]
It is then known that \( v \) is \( C \)-fundamental on \( \mathbb{R}^n \), and that every analytic weight on \( \mathbb{R}^n \) is quasi-analytic.

**Proposition 5.** Let \( n \in \mathbb{N}, \quad n \geq 1, \quad m \in \mathbb{N}, \quad v_0 \geq 0, \quad \alpha \in \mathbb{N}_m^n, \) be upper semicontinuous on \( \mathbb{R}^n \). Consider the \( C^m \)-weight 
\[
v = (v_\alpha ; \alpha \in \mathbb{N}_m^n)
\]
which is supposed to be decreasing. Assume that there are quasi-analytic weights \( v_i, \) \( i = 1, \ldots, n, \) on \( \mathbb{R} \) such that 
\[
v_\alpha(t) \leq v_1(t_1) \cdots v_n(t_n), \quad t \in \mathbb{R}^n, \quad \alpha \in \mathbb{N}_m^n.
\]
Then \( v \) is a \( C^m \)-fundamental weight on \( \mathbb{R}^n \).

**Proof.** All assumptions of Lemma 3 apply. \( \square \)

**Lemma 6.** The linear mapping \( D : f \in \mathcal{D}^1(\mathbb{R}) \mapsto f' \in \mathcal{H}(\mathbb{R}) = \mathcal{D}^0(\mathbb{R}) \) is injective. Its image is \( \mathcal{D}_0(\mathbb{R}) = \{ g \in \mathcal{H}(\mathbb{R}) : \int g = 0 \}. \) If \( u \geq 0 \) is upper semicontinuous on \( \mathbb{R} \), this image is dense in \( C_{u\infty}(\mathbb{R}) \) if and only if \( \int 1/u = +\infty \).

**Proof.** Only the final part of the lemma requires a proof. Consider the linear form \( I : f \in \mathcal{H}(\mathbb{R}) \mapsto \int f \in K. \) Assume \( \int 1/u = +\infty \). We claim that \( I \) is not continuous on \( \mathcal{H}(\mathbb{R}) \) for the seminorm induced by \( C_{u\infty}(\mathbb{R}) \). In fact, given any \( c \geq 0, \) there is an \( f \in \mathcal{H}(\mathbb{R}) \) such that \( 0 \leq f \leq 1/u, \) \( \int f \geq c \). Therefore \( \|f\|_u \leq 1 \) and \( I(f) \geq c \) show that \( I \) is not continuous on \( \mathcal{H}(\mathbb{R}) \) for the seminorm induced by \( C_{u\infty}(\mathbb{R}) \). It follows that \( I^{-1}(0) = \mathcal{D}_0(\mathbb{R}) \) is dense in \( C_{u\infty}(\mathbb{R}) \). Conversely let \( c = \int 1/u < +\infty \). Then the set where \( u \) vanishes has a void interior. It follows that the seminorm on \( C_{u\infty}(\mathbb{R}) \) is actually a norm. We claim that \( |I(f)| \leq c \cdot \|f\|_u \) for \( f \in \mathcal{H}(\mathbb{R}) \). This is clear if \( \|f\|_u = 0 \). If \( \|f\|_u > 0 \) then \( u(t)|f(t)| \leq \|f\|_u \) implies \( |f(t)| \leq \|f\|_u/u(t) \) for \( t \in \mathbb{R} \), hence \( |I(f)| \leq c\|f\|_u \) for \( f \in \mathcal{H}(\mathbb{R}) \) as asserted. Thus \( I \) is continuous on \( \mathcal{H}(\mathbb{R}) \) for the norm induced by \( C_{u\infty}(\mathbb{R}) \) and \( I \) extends uniquely to a continuous linear form \( I \) on \( C_{u\infty}(\mathbb{R}) \), since \( \mathcal{H}(\mathbb{R}) \) is dense in \( C_{u\infty}(\mathbb{R}) \). We know that \( I \neq 0 \) because \( I \) does not vanish on \( \mathcal{H}(\mathbb{R}) \), but \( I \) does vanish on \( \mathcal{D}_0(\mathbb{R}) \). It follows that \( \mathcal{D}_0(\mathbb{R}) \) is not dense in \( C_{u\infty}(\mathbb{R}) \). \( \square \)

**Example 7.** Consider \( v_0 = 0, v_1 \geq 0 \) upper semicontinuous on \( \mathbb{R} \), and the \( C^1 \)-weight \( v = (v_0, v_1) \) on \( \mathbb{R} \). (Notice that \( v \) is not decreasing unless \( v_1 = 0 \).)
Then \( \mathcal{D}^1(\mathbb{R}) \) is dense in \( C^1v_{\infty}(\mathbb{R}) \) if and only if \( \int 1/v_1 = +\infty \).

**Proof.** Since \( v_0 = 0 \) we see that \( \mathcal{D}^1(\mathbb{R}) \) is dense in \( C^1v_{\infty}(\mathbb{R}) \) if and only if \( \mathcal{D}^1(R) \) is dense in \( C(v_1)_\infty(\mathbb{R}) \). It remains to apply Lemma 6 with \( v_1 \) in place of \( u \). \( \square \)
References


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