PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 111, Number 2, February 1991

ON THE OSCILLATION OF DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

CH. G. PHILOS

(Communicated by Kenneth R. Meyer)

ABSTRACT. This paper is concerned with the oscillation of first-order linear delay differential equations in which the coefficients are periodic functions with a common period and the delays are constants and multiples of this period. A necessary and sufficient condition for the oscillation of all solutions is established.

1. INTRODUCTION

The oscillation theory of delay differential equations has been extensively developed during the past few years. We refer, for example, to the recent book by Ladde, Lakshmikantham and Zhang [3] and to the references cited therein.

Tramov [4] obtained a necessary and sufficient condition for the oscillation of all solutions of the delay differential equation

$$(\mathrm{E}_0) \qquad x'(t) + \sum_{k=1}^{m} q_k x(t - \sigma_k) = 0,$$

where q_k are positive numbers and σ_k are nonnegative numbers, $k = 1, \ldots, m$. More precisely, Tramov proved that all solutions of (E_0) are oscillatory if and only if

$$(\mathrm{C}_0) \qquad -\lambda + \sum_{k=1}^{m} q_k e^{\lambda\sigma_k} > 0, \quad \text{for all } \lambda > 0.$$

Another proof of this result appears in [2]. It is an important problem to extend the above criterion for the case of first-order linear delay differential equations with *variable coefficients*. In this paper we examine the special case where the coefficients are periodic functions with a common period and the delays are constants and multiples of this period.

Consider the delay differential equation

$$(\mathrm{E}) \qquad x'(t) + \sum_{k=1}^{m} p_k(t) x(t - \tau_k) = 0,$$

Received by the editors October 2, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 34K15.
Key words and phrases. Delay differential equation, solution, oscillation.

where p_k $(k = 1, \dots, m)$ are nonnegative continuous functions on the interval $[0, \infty)$ which are not identically zero, and τ_k $(k = 1, \dots, m)$ are nonnegative constants. It will be supposed that the coefficients p_k $(k = 1, \dots, m)$ are periodic functions with a common period $T > 0$ and that there exist nonnegative integers n_k $(k = 1, \dots, m)$ such that

$$\tau_k = n_k T \qquad (k = 1, \dots, m).$$

Let $t_0 \geq 0$. By a *solution on* $[t_0, \infty)$ of the differential equation (E) we mean a continuous function x defined on the interval $[t_0 - \tau, \infty)$, where $\tau = \max_{k=1,\dots,m} \tau_k$, which is differentiable on $[t_0, \infty)$ and satisfies (E) for all $t \geq t_0$. A solution of (E) is said to be *oscillatory* if it has arbitrary large zeros, and otherwise it is called *nonoscillatory.*

Our main result is the following theorem.

Theorem. *Set*

$$P_k = \frac{1}{T} \int_0^T p_k(t)\, dt \qquad (k = 1, \dots, m).$$

All solutions of (E) *are oscillatory if and only if*

$$\text{(C)} \qquad -\lambda + \sum_{k=1}^{m} P_k e^{\lambda \tau_k} > 0, \quad \textit{for all } \lambda > 0.$$

We observe that $\min_{\lambda > 0}(e^{\lambda\alpha}/\lambda) = e\alpha$ for every $\alpha > 0$. Thus,

$$e^{\lambda\alpha} \geq e\alpha\lambda, \quad \text{for } \lambda > 0 \text{ and } \alpha \geq 0$$

and therefore for all $\lambda > 0$

$$-\lambda + \sum_{k=1}^{m} P_k e^{\lambda\tau_k} \geq \lambda \left(-1 + e \sum_{k=1}^{m} P_k \tau_k \right),$$

where P_k $(k = 1, \dots, m)$ are defined as in the theorem. So, from our theorem it follows that all solutions of (E) are oscillatory if

$$\sum_{k=1}^{m} P_k \tau_k > \frac{1}{e}.$$

2. Proof of the theorem

First of all, we observe that the constants P_k $(k = 1, \dots, m)$ are positive, since the functions p_k $(k = 1, \dots, m)$ are nonnegative and not identically zero on the interval $[0, T]$. Set

$$\tau \equiv \max_{j=1,\dots,m} \tau_j.$$

Moreover, for any $\lambda > 0$, we define (cf. [1, p. 100])

$$f_\lambda(t) = \sum_{j=1}^{m} p_j(t) e^{\lambda\tau_j} \quad \text{for } t \geq 0.$$

Then, by taking into account the fact that the functions p_j $(j = 1, \dots, m)$ are T-periodic and that $\tau_k = n_k T$ $(k = 1, \dots, m)$, we can see that for each $\lambda > 0$

$$(1) \qquad \int_{t-\tau_k}^{t} f_\lambda(s)\,ds = \tau_k \sum_{j=1}^{m} P_j e^{\lambda \tau_j}, \quad \text{for } t \geq \tau_k \ (k = 1, \dots, m).$$

Assume first that condition (C) does not hold. We may then choose $\lambda_0 > 0$ such that

$$-\lambda_0 + \sum_{j=1}^{m} P_j e^{\lambda_0 \tau_j} = 0.$$

Hence, (1) implies that

$$(2) \qquad \int_{t-\tau_k}^{t} f_{\lambda_0}(s)\,ds = \lambda_0 \tau_k, \quad \text{for all } t \geq \tau_k \ (k = 1, \dots, m).$$

Set

$$x(t) = \exp\left[-\int_0^t f_{\lambda_0}(s)\,ds\right], \quad \text{for } t \geq 0.$$

Then, by using (2), we get for $t \geq \tau$

$$\begin{aligned}
x'(t) &+ \sum_{k=1}^{m} p_k(t) x(t - \tau_k) \\
&= -f_{\lambda_0}(t) \exp\left[-\int_0^t f_{\lambda_0}(s)\,ds\right] + \sum_{k=1}^{m} p_k(t) \exp\left[-\int_0^{t-\tau_k} f_{\lambda_0}(s)\,ds\right] \\
&= \left\{-f_{\lambda_0}(t) + \sum_{k=1}^{m} p_k(t) \exp\left[\int_{t-\tau_k}^{t} f_{\lambda_0}(s)\,ds\right]\right\} \exp\left[-\int_0^t f_{\lambda_0}(s)\,ds\right] \\
&= \left[-f_{\lambda_0}(t) + \sum_{k=1}^{m} p_k(t) e^{\lambda_0 \tau_k}\right] \exp\left[-\int_0^t f_{\lambda_0}(s)\,ds\right] \\
&= 0.
\end{aligned}$$

Thus, x is a solution on $[\tau, \infty)$ of the differential equation (E). Obviously, x is positive on $[0, \infty)$. So, (E) has a nonoscillatory solution.

Assume, conversely, that (C) is satisfied. Moreover, assume for the sake of contradiction that there is a nonoscillatory solution x on an interval $[t_0, \infty)$, $t_0 \geq 0$, of the differential equation (E). Without loss of generality, we suppose that $x(t) \neq 0$ for all $t \geq t_0 - \tau$. As the negative of a solution of (E) is also a solution of the same equation, we may (and do) assume that x is positive on $[t_0 - \tau, \infty)$. Then from (E) it follows that $x'(t) \leq 0$ for every $t \geq t_0$ and consequently x is decreasing on the interval $[t_0, \infty)$.

Define

$$\Lambda = \{\lambda > 0 : x'(t) + f_\lambda(t) x(t) \leq 0 \text{ for all large } t\}.$$

We will show that the set Λ is nonempty. We have

$$(3) \qquad x(t - \tau_k) \geq e^{P\tau_k} x(t), \quad \text{for all } t \geq t_0 + \tau + \tau_k \ (k = 1, \dots, m),$$

where

$$P \equiv \sum_{j=1}^{m} P_j > 0.$$

In fact, consider an arbitrary number $k \in \{1, \ldots, m\}$. Since for $\tau_k = 0$ the above inequality is obvious, we restrict ourselves to the case where $\tau_k > 0$. By the decreasing nature of x on the interval $[t_0, \infty)$, from (E) it follows that

$$x'(t) + \left[\sum_{j=1}^{m} p_j(t)\right] x(t) \leq 0, \quad \text{for } t \geq t_0 + \tau.$$

Thus, by using the hypothesis that the functions p_j $(j = 1, \ldots, m)$ are T-periodic and that $\tau_k = n_k T$, we obtain for every $t \geq t_0 + \tau + \tau_k$

$$\begin{aligned}
\frac{x(t-\tau_k)}{x(t)} &= \exp\left[-\ln \frac{x(t)}{x(t-\tau_k)}\right] = \exp\left[-\int_{t-\tau_k}^{t} \frac{x'(s)}{x(s)}\, ds\right] \\
&\geq \exp\left\{\int_{t-\tau_k}^{t} \left[\sum_{j=1}^{m} p_j(s)\right] ds\right\} \\
&= \exp\left[\sum_{j=1}^{m} \int_0^{\tau_k} p_j(s)\, ds\right] = \exp\left\{\tau_k \sum_{j=1}^{m} \left[\frac{1}{\tau_k}\int_0^{\tau_k} p_j(s)\, ds\right]\right\} \\
&= \exp\left\{\tau_k \sum_{j=1}^{m} \left[\frac{1}{T}\int_0^{T} p_j(s)\, ds\right]\right\} = \exp\left(\tau_k \sum_{j=1}^{m} P_j\right) = e^{P\tau_k}.
\end{aligned}$$

Now, in view of (3), from (E) we obtain for every $t \geq t_0 + 2\tau$

$$\begin{aligned}
0 &= x'(t) + \sum_{k=1}^{m} p_k(t) x(t-\tau_k) \\
&\geq x'(t) + \left[\sum_{k=1}^{m} p_k(t) e^{P\tau_k}\right] x(t) \\
&= x'(t) + f_P(t) x(t),
\end{aligned}$$

which means that $P \in \Lambda$ and so $\Lambda \neq \varnothing$. Clearly, Λ is a subinterval of $(0, \infty)$ with $\inf \Lambda = 0$.

Next, we will prove that Λ is bounded from above. When all delays τ_k $(k = 1, \ldots, m)$ are 0, condition (C) does not hold. We shall therefore assume that there exists an index $k_0 \in \{1, \ldots, m\}$ with $\tau_{k_0} > 0$. Then, since p_{k_0} is a

T-periodic function and $\tau_{k_0} = n_{k_0} T$, we get for $t \geq \tau_{k_0}$

$$\begin{aligned}\int_{t-\tau_{k_0}}^{t} p_{k_0}(s)\,ds &= \int_0^{\tau_{k_0}} p_{k_0}(s)ds \\ &= \tau_{k_0}\left[\frac{1}{\tau_{k_0}}\int_0^{\tau_{k_0}} p_{k_0}(s)\,ds\right] \\ &= \tau_{k_0}\left[\frac{1}{T}\int_0^{T} p_{k_0}(s)\,ds\right] \\ &= \tau_{k_0} P_{k_0} > 0.\end{aligned}$$

For any $t \geq \tau_{k_0}$, let $t^* = t^*(t)$ be a point in the interval $(t - \tau_{k_0}, t)$ such that

$$\begin{aligned}\int_{t-\tau_{k_0}}^{t^*} p_{k_0}(s)\,ds &= \int_{t^*}^{t} p_{k_0}(s)\,ds \\ &= \frac{1}{2}\tau_{k_0} P_{k_0}.\end{aligned} \tag{4}$$

From (E) it follows that

$$x'(t) + p_{k_0}(t)x(t - \tau_{k_0}) \leq 0, \quad \text{for } t \geq t_0.$$

Thus, since x is decreasing on the interval $[t_0, \infty)$, we obtain for $t \geq t_0 + 2\tau_{k_0}$

$$\begin{aligned}x(t^*) &\geq x(t) + \int_{t^*}^{t} p_{k_0}(s)x(s - \tau_{k_0})\,ds \\ &\geq \left[\int_{t^*}^{t} p_{k_0}(s)\,ds\right] x(t - \tau_{k_0}).\end{aligned}$$

Hence, we can use (4) to derive

$$x(t^*) \geq \frac{1}{2}\tau_{k_0} P_{k_0} x(t - \tau_{k_0}) \quad \text{for every } t \geq t_0 + 2\tau_{k_0}. \tag{5}$$

Let $\lambda \in \Lambda$. Then there exists a $t_\lambda \geq t_0$ such that for all $t \geq t_\lambda$

$$x'(t) + f_\lambda(t)x(t) = x'(t) + \left[\sum_{k=1}^{m} p_k(t)e^{\lambda\tau_k}\right] x(t) \leq 0$$

and consequently

$$x'(t) + p_{k_0}(t)e^{\lambda\tau_{k_0}} x(t) \leq 0, \quad \text{for } t \geq t_\lambda.$$

So, if we put

$$\varphi_\lambda(t) = x(t)\exp\left\{\left[\int_0^t p_{k_0}(s)\,ds\right] e^{\lambda\tau_{k_0}}\right\}, \qquad t \geq t_\lambda,$$

then we get for $t \geq t_\lambda$

$$\varphi'_\lambda(t) = \left[x'(t) + p_{k_0}(t)e^{\lambda\tau_{k_0}} x(t)\right] \exp\left\{\left[\int_0^t p_{k_0}(s)\,ds\right] e^{\lambda\tau_{k_0}}\right\} \leq 0.$$

This means that the function φ_λ is decreasing on the interval $[t_\lambda, \infty)$ and hence, for any $t \geq t_\lambda + \tau_{k_0}$, we derive

$$x(t^*) \exp\left\{\left[\int_0^{t^*} p_{k_0}(s)\,ds\right] e^{\lambda\tau_{k_0}}\right\} \equiv \varphi_\lambda(t^*) \leq \varphi_\lambda(t - \tau_{k_0})$$
$$\equiv x(t - \tau_{k_0}) \exp\left\{\left[\int_0^{t-\tau_{k_0}} p_{k_0}(s)\,ds\right] e^{\lambda\tau_{k_0}}\right\}$$

or

$$x(t^*) \leq x(t - \tau_{k_0}) \exp\left\{-\left[\int_{t-\tau_{k_0}}^{t^*} p_{k_0}(s)\,ds\right] e^{\lambda\tau_{k_0}}\right\}.$$

Therefore, by (4), we obtain

$$x(t^*) \leq x(t - \tau_{k_0}) \exp\left(-\frac{1}{2}\tau_{k_0} P_{k_0} e^{\lambda\tau_{k_0}}\right), \quad \text{for all } t \geq t_\lambda + \tau_{k_0}. \tag{6}$$

Combining (5) and (6), we conclude that

$$\frac{1}{2}\tau_{k_0} P_{k_0} \leq \exp\left(-\frac{1}{2}\tau_{k_0} P_{k_0} e^{\lambda\tau_{k_0}}\right)$$

or

$$\lambda \leq \frac{1}{\tau_{k_0}} \ln\left[\frac{2}{\tau_{k_0} P_{k_0}} \ln\left(\frac{2}{\tau_{k_0} P_{k_0}}\right)\right].$$

So, the number

$$\hat{\lambda} \equiv \frac{1}{\tau_{k_0}} \ln\left[\frac{2}{\tau_{k_0} P_{k_0}} \ln\left(\frac{2}{\tau_{k_0} P_{k_0}}\right)\right]$$

is an upper bound of the set Λ.

Now, we put $\lambda^* = \sup\Lambda$. Clearly, λ^* is a positive number. Consider an arbitrary number $\mu \in (0, \lambda^*)$ and set $r = \lambda^* - \mu$. Obviously, $0 < r < \lambda^*$ and so $r \in \Lambda$. Thus, there exists a $t_r \geq t_0$ such that

$$x'(t) + f_r(t)x(t) \leq 0 \quad \text{for all } t \geq t_r.$$

For any $k \in \{1, \ldots, m\}$ and every $t \geq t_r + \tau_k$, we obtain

$$\begin{aligned}\frac{x(t-\tau_k)}{x(t)} &= \exp\left[-\ln\frac{x(t)}{x(t-\tau_k)}\right] \\ &= \exp\left[-\int_{t-\tau_k}^{t} \frac{x'(s)}{x(s)}\,ds\right] \\ &\geq \exp\left[\int_{t-\tau_k}^{t} f_r(s)\,ds\right].\end{aligned}$$

Hence, by using (1), we have

$$x(t-\tau_k) \geq x(t)\exp\left(\tau_k \sum_{j=1}^{m} P_j e^{r\tau_j}\right), \quad \text{for } t \geq t_r + \tau_k \ (k = 1, \ldots, m).$$

Thus, from (E) we obtain for every $t \geq t_r + \tau$

$$\begin{aligned}
0 &= x'(t) + \sum_{k=1}^{m} p_k(t) x(t - \tau_k) \\
&\geq x'(t) + \left\{ \sum_{k=1}^{m} p_k(t) \exp\left[\left(\sum_{j=1}^{m} P_j e^{r\tau_j} \right) \tau_k \right] \right\} x(t) \\
&= x'(t) + f_R(t) x(t),
\end{aligned}$$

where

$$R = \sum_{j=1}^{m} P_j e^{r\tau_j}.$$

This means that $R \in \Lambda$ and consequently

$$\sum_{k=1}^{m} P_k e^{r\tau_k} \leq \lambda^* \quad \text{or} \quad \sum_{k=1}^{m} P_k e^{(\lambda^* - \mu)\tau_k} \leq \lambda^*.$$

As $\mu \to 0^+$, we obtain

$$\sum_{k=1}^{m} P_k e^{\lambda^* \tau_k} \leq \lambda^*,$$

which contradicts condition (C). The proof of the theorem is complete.

3. Remark

Consider the advanced differential equation

$$(\mathrm{E}') \qquad x'(t) - \sum_{k=1}^{m} p_k(t) x(t + \tau_k) = 0.$$

By a *solution on* $[t_0, \infty)$, where $t_0 \geq 0$, of the differential equation (E') we mean a differentiable function x on the interval $[t_0, \infty)$ which satisfies (E') for all $t \geq t_0$.

Our theorem has a straightforward extension to the advanced equation (E'). By similar arguments we can establish the following dual result.

Let P_k $(k = 1, \ldots, m)$ *be defined as in our theorem. All solutions of* (E') *are oscillatory if and only if* (C) *holds.*

If $\lambda_0 > 0$ satisfies $-\lambda_0 + \sum_{k=1}^{m} P_k e^{\lambda_0 \tau_k} = 0$, then (E') has the nonoscillatory solution

$$x(t) = \exp\left[\int_0^t f_\lambda(s)\, ds \right], \qquad t \geq 0.$$

Also, if x is a positive solution on $[t_0, \infty)$, $t_0 \geq 0$, of the differential equation (E'), then the set Λ must be defined by

$$\Lambda = \{\lambda > 0 : x'(t) - f_\lambda(t) x(t) \geq 0 \text{ for all large } t\}.$$

We omit the details.

References

1. L. E. El'sgol'ts and S. B. Norkin, *Introduction to the theory and application of differential equations with deviating arguments*, Academic Press, New York, 1973.
2. G. Ladas, Y. G. Sficas, and I. P. Stavroulakis, *Necessary and sufficient conditions for oscillations*, Amer. Math. Monthly **90** (1983), 637–640; see also *Necessary and sufficient conditions for oscillations of higher order delay differential equations*, Trans. Amer. Math. Soc. **285** (1984), 81–90.
3. G. S. Ladde, V. Lakshmikantham, and B. G. Zhang, *Oscillation theory of differential equations with deviating arguments*, Marcel Dekker, New York, 1987.
4. M. I. Tramov, *Conditions for oscillatory solutions of first-order differential equations with a delayed argument*, Izv. Vyssh. Uchebn. Zaved. Mat. **19** (1975), 92–96.

Department of Mathematics, University of Ioannina, P. O. Box 1186, 451 10 Ioannina, Greece