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THE FREE LATTICE-ORDERED GROUP OVER A NILPOTENT GROUP

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ABSTRACT. We show that the free lattice-ordered group over a finitely generated torsionfree nilpotent group is l-solvable of some finite rank.

1. INTRODUCTION

Conrad [C] showed that for a partially ordered group G, a free lattice-ordered group over G exists if and only if the partial order of G is the intersection of right orders of G. In particular, taking G to be trivially ordered, a free lattice-ordered group $F(G)$ over the group G exists if and only if G can be right-ordered: a total order $\leq$ exists on G such that if $g \leq h$ in G, then for any $x \in G$, $gx \leq hx$.

The construction of $F(G)$ is easy to describe. Let $\{\leq_\lambda\}_\Lambda$ be the set of all right orders of G. For each $\leq_\lambda$, G acts (by multiplication on the right) as a group of order-preserving permutations of the chain $(G, \leq_\lambda)$. So by way of the right regular representation, G can be embedded into $\mathscr{A}(G, \leq_\lambda)$, the l-group of all order-preserving permutations of the chain $(G, \leq_\lambda)$. $F(G)$ is then the l-subgroup of $\prod_\Lambda \mathscr{A}(G, \leq_\lambda)$ generated by the "long constants" of $G: g \to (\dots, \overline{g}_\lambda, \dots)$. More useful in the following discussion is that if G_λ^* is the l-subgroup of $\mathscr{A}(G, \leq_\lambda)$ generated by the right regular representation of G, then $F(G)$ is the l-subgroup of $\prod_\Lambda G_\lambda^*$ generated by the long constants of G. Thus if we can demonstrate that each G_λ^* is in a variety of lattice-ordered groups, then $F(G)$ must be as well.

Very little is known at this time about what characteristics of G carry over to $F(G)$. It is well known that if G is abelian, then $F(G)$ is too. Darnel and Glass [DG] proved that if G is a torsionfree nilpotent group of class 2 (hereafter referred to as *nil*-2) generated by m elements, then $F(G)$ is l-solvable of rank $\binom{m}{2} + 1$ but may not be nilpotent.

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We will make use of the following theorem which gives a characterization for when the free lattice-ordered group over G is normal-valued. First recall that a lattice-ordered group H is *normal-valued* if for any $g, h \in H$, $|x||y| \leq |y^2||x^2|$. This is equivalent to the condition $|[x, y]| \ll |x| \bigvee |y|$ and to the condition: if $g, h \ll k$, then $gh \ll k$. These three conditions are also equivalent for right-ordered groups, and a right-ordered group satisfying any (and hence all) of the conditions is called a *Conrad* right-ordered group, more commonly known as a *c-group.* The right order is then called a *c-order.*

Theorem 1 [GHR]. *The free lattice-ordered group over a group G is normal-valued if and only if every right order of G is a c-order.*

(What was actually shown in [GHR] is that if $(G, \leq)$ is a c-group, then G^* in $\mathscr{A}(G, \leq)$ is normal-valued. Theorem 1 is then an easy consequence of this result.)

2. The l-subgroup generated by a c-ordered permutation group

Let G be a lattice-ordered group and $A \subseteq G$ be a subgroup of G. $\langle A \rangle$ will denote the l-subgroup of G generated by A. For any $a \in \langle A \rangle$, there exist finite sets I and J and $\{a_{ij}\}_{i \in I, j \in J} \subseteq A$ such that $a = \bigvee_I \bigwedge_J a_{ij} = \bigwedge_J \bigvee_{f \in I^J} a_{f(j), j}$. $G(A)$ will denote the convex l-subgroup of G generated by A; then $G(A) = \{g \in G : |g| \leq |a| \text{ for some } a \in A\}$. As in the introduction, if $(G, \leq)$ is a right-ordered group, then G^* is the l-subgroup of $\mathscr{A}(G, \leq)$ generated by the representation of G acting on the chain $(G, \leq)$ by right multiplication.

Lemma 2. *Let G be a lattice-ordered group and $A \subseteq B$ be subgroups of G such that A is normal in B and $\langle B \rangle = G$. Then $G(A)$ is normal in G.*

Moreover, if for any $b \in B \setminus A$, b is either positive or negative in G and if for any $e < b \in B \setminus A$ and any $a \in A$, $|a| \ll b$, then G is a lex extension of $G(A)$ and $G(A) = \langle A \rangle$.

Proof. Let $b \in B$ and $h \in \langle A \rangle$; then $h = \bigvee_I \bigwedge_J a_{ij}$ as above. So $b^{-1}hb = \bigvee_I \bigwedge_J (b^{-1}a_{ij}b)$ which is clearly in $\langle A \rangle$. So B normalizes $\langle A \rangle$ and hence normalizes $G(A)$. Since the normalizer of a convex l-subgroup is an l-subgroup [D2], $G = \langle B \rangle$ normalizes $G(A)$.

For the second part, we will show that any $g \in G$ is either an element of $\langle A \rangle$ or can be written in the form zb, where $z \in \langle A \rangle$ and $b \in B \setminus A$. Now $g = \bigwedge_I \bigvee_J b_{ij}$, where $b_{ij} \in B$ for all i and j. We will first assume that I is a single element and so $g = \bigvee_J b_j$.

Let $J' = \{j \in J : b_j \notin A\}$. If J' is empty, then $g \in \langle A \rangle$. So assume J' is not empty. Now if b_{i_1} and b_{i_2} are in different cosets of A, then $b_{i_1} b_{i_2}^{-1} \notin A$ and so is either positive or negative. Clearly then $b_{i_1} \vee b_{i_2}$ is the larger of the two. Also clear is the fact that if b_{i_1} is larger than b_{i_2} and b_{i_3} is in the same A-coset as b_{i_1}, then $b_{i_3} > b_{i_2}$. Thus $\bigvee_J b_j$ is the join of those b_j's in the 'highest' coset of A. Now if b_{i_1} and b_{i_2} are in this coset, then $b_{i_2} = ab_{i_1}$ and so $b_{i_1} \vee b_{i_2} = (e \vee a)b_{i_1}$. So clearly $\bigvee_J b_j$ is of the form zb, where $z \in \langle A \rangle$ and $b \in B$. Note that if $b \in A$, then this join is in $\langle A \rangle$.

Next consider $\bigwedge_I z_i b_i$, where $z_i \in \langle A \rangle$ and $b_i \in B$. Once again, if the A-coset of b_{i_1} is not that of b_{i_2}, then b_{i_1} and b_{i_2} are comparable and so $z_{i_1} b_{i_1}$ is comparable to $z_{i_2} b_{i_2}$. Thus we need consider only those $z_i b_i$'s is the 'lowest' coset of A; call this subset I'. So $\bigwedge_{I'} z_i b_i$ is of the form zb, where $z \in \langle A \rangle$ and $b \in B$.

Now let $g \in G \setminus [G(A)]$; then $g = zb$ where $z \in \langle A \rangle$ and $b \in B \setminus A$. Then b is either positive or negative and since $|b| \gg |z|$, g is positive or negative as b is. Clearly anything in the $G(A)$-coset of g is also positive or negative as g is. So G is a lex extension of $G(A)$. Equally clear now is that $G(A) = \langle A \rangle$. □

Proposition 3. *Let $(G, \leq_r)$ be a right-ordered group and let K be the convex subgroup of G generated by the derived group $G^{(1)}$. If $K \neq G$, then G^* is a lex extension of K^*.*

Proof. For any $g \in G$, let $\overline{g}$ denote the order-preserving permutation of the chain $(G, \leq_r)$ determined by multiplying elements on the right by g.

Let $e <_r g \in G \setminus K$. If $\overline{g} \not> e$ in G^*, there exists $\alpha \in G$ such that $\alpha g <_r \alpha$. But then $\alpha g \alpha^{-1} g^{-1} <_r g^{-1} <_r e$, which implies that g^{-1} and hence g is in K. So $e <_r g \in G \setminus K$ implies that $\overline{g} > e$ in G^*.

Now suppose that for some $k \in K$ and $e <_r g \in G \setminus K$, $|\overline{k}| \nless \overline{g}$. Then there exists $\alpha \in G$ such that $\alpha g <_r \alpha k$. Since K is normal in G, $\alpha k = k_1 \alpha$ and so $\alpha g <_r k_1 \alpha$ implies that $\alpha g \alpha^{-1} g^{-1} <_r k_1 g^{-1} <_r e$, a contradiction. So $\overline{g} \gg |\overline{k}|$ for all $k \in K$. Lemma 2 now applies. □

Corollary 4. *Let G be a finitely generated group and $\leq_r$ be a c-ordering of G. Let K be the convex subgroup of G generated by the commutator subgroup. Then in $\mathscr{A}(G, \leq_r)$, G^* is a lex extension of K^*.*

Proof. Let $\{a_1, \dots, a_n\}$ be generators of G. By using inverses if necessary, we can assume that $a_1 >_r \cdots >_r a_n >_r e$. Then clearly a_1 is infinitely greater than any commutator and so $K \neq G$. □

3. The free lattice-ordered group over a nilpotent group

We will call a lattice-ordered group *l-solvable of rank n* if there exists a chain of convex subgroups

$$(e) \lhd A_0 \lhd A_1 \lhd \cdots \lhd A_n = G$$

such that each quotient A_{i+1}/A_i is abelian. Smith [S] pointed out that while l-solvability of rank n implies solvability of rank n, the converse is not true. In [GHM], it was shown that $wr^n \mathscr{Z}$, the iterated ordered wreath product of the ordered group $\mathscr{Z}$ of integers with itself n times, generates the variety of l-solvable lattice-ordered groups of rank n.

Finally, let G be a finitely generated torsionfree nilpotent group. Then each subgroup $Z_i(G)$ of the ascending central series is a pure (or isolated) subgroup and so, for any i, $Z_{i+1}G/Z_i(G)$ is a finitely generated torsionfree abelian group and so is free. Thus there exists a central series

$$(e) = A_0 \triangleleft A_1 \triangleleft \cdots \triangleleft A_n = G$$

such that each A_{i+1}/A_i is free abelian of rank one. Hirsch [H] proved that any two such central series must have a common length (which is the minimum length of a central series with cyclic factors) called the *Hirsch length* or *Hirsch number* of the group. (The interested reader should also see [B, p. 184], or [Ha, Chapter 10].)

Theorem 5. *Let G be a finitely generated torsionfree nilpotent group of Hirsch length n. Then the free lattice-ordered group $F(G)$ over G is l-solvable of rank at most n.*

Proof. We of course induct on n.

If $n = 1$, then G is abelian and so $F(G)$ is abelian. So suppose $n > 1$ and that the theorem is true for all finitely generated nilpotent groups whose Hirsch length k is less than n.

Let $\leq_r$ be a right order of G and let K be the convex subgroup of G generated by the commutators. Rhemtulla [R] proved that any right order of a nilpotent group must be a c-order. By Corollary 4, G^* is a lex extension of K^*.

Now G/K is free abelian and so the Hirsch length of K is less than that of G. By induction, $F(K)$ is l-solvable and so K^* is l-solvable because K^* is an l-homomorphic image of $F(K)$. Since G^*/K^* is abelian, G^* is l-solvable of rank at most n. □

The above theorem shows that the free lattice-ordered group over a finitely generated nilpotent group is l-solvable of rank less than or equal to the Hirsch length. As mentioned above, Darnel and Glass [DG] proved that the free lattice-ordered group over a torsionfree nil-2 group generated by m elements is l-solvable of rank $\binom{m}{2} + 1$. For a comparison of these results, it is instructive to examine the free nil-2 groups of finite rank.

Let F_n be the free nil-2 group on free generators $\{a_1, \ldots, a_n\}$. Since F_n is free, for any $1 \leq i < j \leq n$, the commutator $[a_i, a_j] \neq e$. Thus the center $Z(F_n)$ is a free abelian group on the set of commutators $\{[a_i, a_j] : 1 \leq i < j \leq n\}$. Note also that any element of F_n can be written uniquely in the form

$$a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n} [a_1, a_2]^{m_{12}} \cdots [a_{n-1}, a_n]^{m_{n-1,n}}.$$

Now the Hirsch length of F_n is $n + \binom{n}{2}$, which is clearly greater than the bound of Darnel and Glass. But we can take a homomorphic image H of F_n in which $[a_i, a_j] = e$ for $1 < i < j \leq n$. The Hirsch length of H is then $2n - 1$, which is less than the bound given by Darnel and Glass.

The following theorem improves both bounds.

Theorem 6. *Let G be a finitely generated torsionfree nil-2 group and let n be the minimal number of generators for G. Then the free lattice-ordered group $F(G)$ over G is l-solvable of rank at most n.*

Proof. Let $a_1, a_2, \dots, a_r$ be elements of G such that their cosets are free generators of $G/Z(G)$; let A be the subgroup of G generated by $\{a_1, \dots, a_r\}$. If $A \neq G$, then there exist elements $b_{r+1}, \dots, b_n$ in $Z(G)$ such that, letting B be the subgroup of G generated by $\{b_{r+1}, \dots, b_n\}$, G is the direct sum of A and B.

Note that if $r \leq 1$, then G is abelian and so $F(G)$ is abelian. Clearly this is the case if $n = 1$.

So suppose $n > 1$ and that $r > 1$; further suppose that if A is generated by $k < r$ generators, the free lattice-ordered group over $A \times B$ is l-solvable of rank k.

Let $\leq_r$ be a right order of G and let K be the convex subgroup of G generated by the commutators. If K (and hence K^*) is abelian, then G^*/K^* is abelian and so G^* is l-solvable of rank 2. So suppose that K is not abelian. Then AK/K is a free abelian group of rank $m < r$. So we can assume, by rearranging and taking appropriate combinations if necessary, that $a_1, a_2, \dots, a_m$ are not in K while $a_{m+1}, \dots, a_r$ are in K. Similarly, we can choose free generators $b_{r+1}, \dots, b_n$ of B such that there exists $r+1 \leq s \leq n$ so that the set $\{Kb_{r+1}, \dots, Kb_s\}$ freely generates BK/K and for any $s < j \leq n$, $b_j \in K$. Since G is the direct sum of A and B, G/K is the direct sum of AK/K and BK/K; thus in G/K, the set $\{Ka_1, \dots, Ka_m, Kb_{r+1}, \dots, Kb_s\}$ is a free set of generators.

However, G^*/K^* is then a free abelian group with free generators $\{K^*\overline{a}_1, \dots, K^*\overline{a}_m, K^*\overline{b}_{r+1}, \dots, K^*\overline{b}_s\}$. Note that K is generated by $\{a_{m+1}, \dots, a_r\} \cup \{[a_i, a_j] : 1 \leq i \leq m,\ i+1 \leq j \leq n\} \cup \{b_{s+1}, \dots, b_n\}$. However, the elements of the last two sets generate a free abelian group. So by induction $F(K)$ is l-solvable of rank at most $r - m$, and thus K^* is as well. So G^* is l-solvable of rank $r - m + 1$ which is less than n. □

The bound in Theorem 6 is the best possible. Again consider F_n, the free nil-2 group on free generators $a_1, \dots, a_n$. One can build a total order on F_n since every element can be written uniquely in the form

$$a_1^{m_1} a_2^{m_2} [a_1, a_2]^{m_{12}} a_3^{m_3} \cdots [a_{n-2}, a_{n-1}]^{m_{n-1,n}} a_n^{m_n} [a_1, a_n]^{m_{1,n}} \cdots [a_{n-1}, a_n]^{m_{n-1,n}}.$$

Ordering lexicographically from the left defines a two-sided order on F_n under which F_n is l-solvable of rank n but not rank $n-1$.

But a better result is possible. We can actually embed the iterated small wreath product $wr^n\mathscr{Z}$ with its usual lattice ordering into the free lattice-ordered group over F_n. (This was shown in [DG] when $n = 2$.)

FIGURE 1. Graph of $\overline{a}_2$

To do this, we can alternatively write every element of F_n uniquely in the form

$$a_1^{m_1}[a_1, a_2]^{m_{12}} \cdots [a_1, a_n]^{m_{1n}} a_2^{m_2}[a_2, a_3]^{m_{23}} \cdots [a_2, a_n]^{m_{2n}} \cdots [a_{n-1}, a_n]^{m_{n-1,n}} a_n^{m_n}.$$

The positive cone of a right order $\leq_r$ on F_n can then be defined lexicographically from the left by nonnegative powers. For future reference, let H be the convex subgroup of $(F_n, \leq_r)$ generated by a_2. As before, we embed F_n into $\mathscr{A}(F_n, \leq_r)$ by its right regular action.

It is then easy to see that $\overline{a}_1$ is positive in F_n^*, as is any element of the form $[\overline{a}_i, \overline{a}_j]$, when $i < j$. However, $\alpha\overline{a}_2 >_r \alpha$ if and only if, when α is placed into the above standard form, $m_1 \geq 0$. The action of $\overline{a}_2$ is shown in Figure 1.

It is also easy to verify that $\alpha(\overline{a_1^{-1}a_2^{-1}a_1}) >_r \alpha$ if and only if, referring again to the standard form of α, $m_1 \leq 0$. Thus the support of $(\overline{a}_2 \wedge \overline{a}_1^{-1}\overline{a}_2^{-1}\overline{a}_1) \vee \overline{e}$ is $\{\alpha : m_1 = 0\}$. Furthermore, for all such α, $\alpha(a_1^{-1}a_2^{-1}a_1) >_r \alpha a_2$, and so $(\overline{a}_2 \wedge \overline{a}_1^{-1}\overline{a}_2^{-1}\overline{a}_1) \vee \overline{e}$ is just the component c_2 of $\overline{a}_2$ on $\{\alpha : m_1 = 0\}$. If for $2 < i \leq n$, we further define c_i to be the component of a_i with support $\{\alpha : m_1 = 0\}$, then the l-subgroup of F_n^* generated by $\{c_2, \ldots, c_n\}$ is l-isomorphic to the l-subgroup H^* generated by the action of H in $\mathscr{A}(H, \leq_r)$ and so by induction contains a copy of $wr^{n-1}\mathscr{Z}$.

But since for any $n \neq 0$, $\overline{a}_1^{-n}c_2\overline{a}_1^n \wedge c_2 = e$, F_n^* must contain a copy of $wr^n\mathscr{Z}$. So we have proved:

Proposition 7. *If F_n is the free nil-2 group of rank n, then $F(F_n)$ generates the l-solvable variety of rank n.*

As pointed out in [DG], this shows that the free lattice-ordered group over a nilpotent group is not nilpotent if the group is nonabelian.

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