COMPARISON THEOREMS FOR THE $\nu$-ZEROES OF LEGENDRE FUNCTIONS $P^m_\nu(z_0)$ WHEN $-1 < z_0 < 1$

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Abstract. We consider the problem of ordering the elements of $\{\nu^m_j(z_0)\}$, the set of $\nu$-zeroes of Legendre functions $P^m_\nu(z_0)$ for $m = 0, 1, \ldots$ and $z_0 \in (-1, 1)$. In general, we seek to determine conditions on $(m, j)$ and $(n, i)$ under which we can assert that $\nu^m_j < \nu^n_i$. A number of such results were established in [2] for $z_0 \in [0, 1)$, and in the work that we present here we extend a number of these to the case $z_0 \in (-1, 1)$. In addition, we prove $\nu^{m+2}_j < \nu^m_j$ for $z_0 \in (-1, 0)$ and $\nu^3_j < \nu^6_1$ for $z_0 \in (-1, 1)$. Using the results established here and in [2], we are able to determine the ordering of the first eleven $\nu$-zeroes of $P^m_\nu(z_0)$ for $0 < z_0 < 1$ and show that the twelfth $\nu$-zero is not necessarily distinct.

1. Introduction

For a fixed $z_0 \in (-1, 1)$ and $m = 0, 1, \ldots$, we will let $\{\nu^m_j(z_0)\}$ denote the set of positive $\nu$-zeroes of Legendre functions $P^m_\nu(z_0)$. The principal goal is determine conditions on $(m, j)$ and $(n, i)$ under which we can assert that $\nu^m_j < \nu^n_i$. One such result which follows from the Sturm-Liouville theory is that

\begin{equation}
\nu^m_j(z_0) < \nu^{m+1}_j(z_0) < \nu^m_{j+1}(z_0), \quad -1 < z_0 < 1
\end{equation}

(See [10].) The problem of ordering the $\nu^m_j$'s when $0 \leq z_0 < 1$ was first considered in [2], where the following results were established:

\begin{equation}
\nu^{m+2}_j(z_0) < \nu^m_{j+1}(z_0), \quad 0 < z_0 < 1,
\end{equation}

\begin{equation}
\nu^{m+2}_j(0) = \nu^m_{j+1}(0),
\end{equation}

\begin{equation}
\nu^0_2(z_0) < \nu^3_1(z_0), \quad \nu^1_2(z_0) < \nu^4_1(z_0), \quad \nu^0_3(z_0) < \nu^5_1(z_0), \quad 0 < z_0 < 1.
\end{equation}

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Combining (1.1)–(1.4), it follows that the first ten \( \nu \)-zeroes are
\[
(1.5) \quad \nu_1^0 < \nu_1^1 < \nu_2^0 < \nu_1^3 < \nu_1^2 < \nu_2^1 < \nu_1^4 < \nu_2^2 < \nu_3^0 < \nu_1^5, \quad 0 < z_0 < 1.
\]
The ordering in (1.5) is unique.

The above results lead to several additional questions: Which are the next \( \nu \)-zeroes in the chain of inequalities in (1.5)? How do \( \nu_{j+1}^m \) and \( \nu_{j+2}^m \) compare for \( z_0 \in (-1, 1) \)? In the following, we will consider these and related questions. The main results of this paper are contained in §3.

On the basis of numerical calculations, it was conjectured in [2] that the \( \nu \)-zero which followed \( \nu_1^5 \) in (1.5) was \( \nu_2^3 \). An analytical proof of this follows from the inequality,
\[
(1.6) \quad \nu_2^3(z_0) < \nu_1^6(z_0), \quad -1 < z_0 < 1,
\]
which is established in Theorem 1. In Lemma 2, we show that
\[
(1.7) \quad \nu_{j+1}^m(z_0) < \nu_j^{m+2}(z_0), \quad -1 < z_0 < 0.
\]

Theorem 3 combines (1.1)–(1.3) with Lemma 2 and gives the relative ordering of \( \nu_i^m(z_0), \nu_{j+1}^m(z_0), \) and \( \nu_{k+2}^m(z_0) \) for \( -1 < z_0 < 1 \).

In addition, we will show that the inequalities in (1.4) hold for all \( z_0 \in (-1, 1) \). (See Theorem 2.) Although (1.1)–(1.6) imply that the first eleven \( \nu \)-zeroes of \( P_{\nu}^m(z) \) are distinct for all \( 0 < z_0 < 1 \), in §4 we show that the twelfth \( \nu \)-zero is not necessarily distinct. Moreover, Theorem 3 shows that the ordering in (1.5) is not preserved for \( -1 < z_0 \leq 0 \).

2. Preliminaries

In this section, we present some properties of the Legendre functions and their zeroes which will be needed in §3. For convenience, here and in the following sections \( m \) will denote a nonnegative integer and \( n, i, j, k \) will denote positive integers, unless otherwise stated.

The solution \( y = P_{\nu}^m(z) \) that satisfies
\[
(2.1a) \quad \frac{d}{dz} \left( (1 - z^2) \frac{d}{dz} y \right) + \left( \nu(\nu + 1) - \frac{m^2}{1 - z^2} \right) y = 0, \quad -1 < z < 1,
\]
\[
(2.1b) \quad y(1) \text{ is bounded},
\]
is called a Legendre function of the first kind of degree \( \nu \) and order \( m \). (See [4] for a general discussion of the Legendre functions and their properties.) If \( -1 < z \leq 1 \) and \( \nu > 0 \), then \( P_{\nu}^m(z) \) can be expressed [3, p. 148] as
\[
(2.2) \quad P_{\nu}^m(z) = \frac{(-1)^m \Gamma(\nu + m + 1)}{2^m m! \Gamma(\nu - m + 1)} (1 - z^2)^{m/2} \sum_{n=0}^{\infty} \frac{(1 + m + \nu)_n (m - \nu)_n}{(m + 1)_n n! 2^n} (1 - z)^n,
\]
where \( \Gamma(z) \) is the gamma function and \( (a)_n \) denotes the Pochhammer symbol,
\[
(a)_0 = 1,
\]
\[
(a)_n = a(a + 1) \ldots (a + n - 1).
\]
Later, we will need the following identity:

\[(2.3) \quad P_{\nu}^{m+2}(z) + 2(m+1)z(1-z^2)^{-1/2}P_{\nu}^{m+1}(z) + (\nu-m)(\nu+m+1)P_{\nu}^{m}(z) = 0\]

(See [3, p. 161].) For a fixed \(z_0 \in (-1,1)\), the pairs \((\nu(\nu+1), y(z))\) satisfying (2.1) and \(y(z_0) = 0\) will be denoted by

\[(2.4) \quad (\nu_j^m(\nu_j^m + 1), P_{\nu_j^m}^{m}(z))\]

where \(P_{\nu_j^m}^{m}(z_0) = 0\), \(m = 0, 1, \ldots\), and \(j = 1, 2, \ldots\).

The next result summarizes a few of the important properties of the \(\nu_j^m\)'s. Its proof follows the arguments in [2, Lemma 1] with some minor modifications.

**Lemma 1.** Let \(m\) be a nonnegative integer. There exists a unique sequence \(\{\nu_j^m(\tau)\} \ni j = 1, 2, \ldots\) such that for every \(j\), the function \(\nu = \nu_j^m(\tau)\) satisfies

\[(*) \quad P_{\nu}^{m}(\tau) = 0, \quad \text{for all } \tau \in (-1, 1).\]

Moreover, each \(\nu_j^m(\tau)\) is analytic and strictly increasing as a function of \(\tau\) for \(\tau \in (-1, 1)\).

It will be convenient at times to consider \(P_{\nu}^{m}(\cos \phi)\) for \(0 < \phi \leq \phi_0 < \pi\) where \(z = \cos \phi\) and \(z_0 = \cos \phi_0\). As a function of \(\phi_0\), we see from Lemma 1 that \(\nu_j^m(\phi_0)\) is decreasing for \(\phi_0 \in (0, \pi)\). It will be clear from the context whether \(\nu_j^m\) is to be considered as a function of \(z_0\) or as a function of \(\phi_0 = \arccos(z_0)\).

A straightforward calculation shows that if \(P_{\nu}^{m}(z)\) is a solution of (2.1), then \(u = \sqrt{\sin \phi}P_{\nu}^{m}(\cos \phi)\) satisfies

\[(2.5) \quad u'' + \left(\frac{\nu^2 + 1}{2} + \frac{1-4m^2}{4\sin^2 \phi}\right) u = 0.\]

From [7, p. 17], we see that \(v = \sqrt{\phi}J_m((\nu + \frac{1}{2})\phi)\) is a solution of

\[v'' + \left(\nu + \frac{1}{2} + \frac{1-4m^2}{4\phi^2}\right) v = 0,\]

where \(J_m\) is the Bessel function of order \(m\). From the Sturm-Liouville theory (See [10, Chapter 7]), \(P_{\nu_j^m}^{m}(z)\) has \(j - 1\) \(z\)-zeroes on \((z_0, 1)\). Moreover, for a general \(\nu\) (not necessarily one of the \(\nu_j^m\)'s), we see from (2.5) that there are \([\nu - m]\) \(z\)-zeroes of \(P_{\nu}^{m}(z)\) on \((-1, 1)\), where \([x] = n\) for \(n \leq x < n + 1\). \(P_{n}^{m}(z)\) has exactly \(n - m\) \(z\)-zeroes on \((-1, 1)\) and \(P_{n+1}^{m}\) has exactly \(n - m + 1\) \(z\)-zeroes on \((-1, 1)\). (See [6, p. 246].) To see this, suppose \(\nu = \nu^*\) and \(n < \nu^* < n + 1\). By applying the Sturm Comparison Theorem [5] to the solutions of (2.5) for \(\nu = n, \nu^*\) and \(n + 1\), respectively, we see that \(P_{\nu}^{m}(z)\) must have at least \(n - m\) \(z\)-zeroes on \((-1, 1)\) and at most \(n - m + 1\) \(z\)-zeroes on \((-1, 1)\). We conclude that \(P_{\nu}^{m}(z)\) has exactly \([\nu^* - m]\) \(z\)-zeroes on \((-1, 1)\).
For fixed \( \nu \), we will denote the \( z \)-zeroes of \( P^m_\nu(z) \) that are between \(-1\) and \(1\) by \( z^m_{\nu,i} \), where \( z^m_{\nu,i} > z^m_{\nu,i+1} \). By definition of the \( \nu_j^m(z_0) \)'s, it follows that

\[
(2.6) \quad \nu_j(z^m_{\nu,j}) = \nu.
\]

We define \( \phi^m_{\nu,j} \) to be the solution of

\[
(z^m_{\nu,j} = \cos(\phi^m_{\nu,j})
\]

such that \( 0 < \phi^m_{\nu,j} < \pi \). It follows that \( \phi^m_{\nu,j} < \phi^m_{\nu,j+1} \).

From (2.2), we see that

\[
(2.7) \quad P^m_m(z_0) = \frac{(-1)^m \Gamma(2m+1)}{2^m m!} (1 - z_0^2)^{m/2}.
\]

From Rodrigue's formula [6, p. 174] and Rolle's Theorem, it follows that \( P^m_{m+2j-1}(z) \) has \( j - 1 \) \( z \)-zeroes on \((0, 1)\) and \( \nu_j^m = m + 2j - 1 \) when \( z_0 = 0 \). Similarly, \( P^m_{m+j-1}(z) \) has \( j - 1 \) \( z \)-zeroes on \((-1, 1)\) and

\[
\lim_{z_0 \to -1+} \nu_j^m(z_0) = m + j - 1.
\]

Since \( \nu_j^m(z_0) \) is increasing on \((-1, 1)\), we see that

\[
(2.8) \quad m + j - 1 < \nu_j^m(z_0) < m + 2j - 1, \quad -1 < z_0 < 0.
\]

From [8], we have

\[
(2.9) \quad \frac{1}{3} < \frac{1}{\sin^2 \phi} - \frac{1}{\phi^2} < \alpha(\phi), \quad 0 < \phi < \bar{\phi} \leq \pi/2,
\]

where \( \alpha(\phi) = \sin^{-2} \phi - \phi^{-2} \). Note that \( \lim_{\phi \to 0^+} \alpha(\phi) = 1/3 \). From (2.9), we see that

\[
(2.10) \quad \frac{\phi^2}{1 + \alpha(\phi) \phi^2} < \sin^2 \phi < \frac{\phi^2}{1 + \frac{1}{3} \phi^2}, \quad 0 < \phi < \bar{\phi} \leq \pi/2.
\]

Multiplying (2.9) by \( 1 - 4m^2 \) with \( m \geq 1 \), we obtain

\[
(2.11) \quad \frac{1 - 4m^2}{4 \phi^2} > \frac{h^2}{4} > \frac{1 - 4m^2}{4 \sin^2 \phi} > \frac{1 - 4m^2}{4 \phi^2} - \frac{k^2(\bar{\phi})}{4}, \quad 0 < \phi < \bar{\phi} \leq \pi/2,
\]

where \( h^2 = (4m^2 - 1)/3 \) and \( k^2(\bar{\phi}) = (4m^2 - 1)\alpha(\bar{\phi}) \).

Next, we consider the following pair of differential equations and their re-
spective solutions which are related to Bessel's equation [7, p. 17],

\[ U'' + \left( \left( \nu + \frac{1}{2} \right)^2 - \frac{h^2}{4} + \frac{1 - 4m^2}{4\phi^2} \right) U = 0, \quad 0 < \phi < \phi, \]

\[ U = \sqrt{\phi} J_m \left( \sqrt{\left( \nu + \frac{1}{2} \right)^2 - \frac{h^2}{4} \phi} \right), \]

\[ V'' + \left( \left( \nu + \frac{1}{2} \right)^2 - \frac{k^2(\phi)}{4} + \frac{1 - 4m^2}{4\phi^2} \right) V = 0, \quad 0 < \phi < \phi, \]

\[ V = \sqrt{\phi} J_m \left( \sqrt{\left( \nu + \frac{1}{2} \right)^2 - \frac{k^2}{4} \phi} \right). \]

Let \( u \) be a solution of (2.5). From (2.11) and the Sturm Comparison Theorem, it follows that the \( k \)th zero of \( U \) occurs before the \( k \)th zero of \( u \) and the \( k \)th zero of \( u \) occurs before the \( k \)th zero of \( V \). In particular, we have for \( m \geq 1 \),

\[(2.12a) \quad \frac{J_m^k}{\sqrt{(\nu + \frac{1}{2})^2 - \frac{h^2}{4}}} < \phi_{\nu,k}^m < \frac{J_m^k}{\sqrt{(\nu + \frac{1}{2})^2 - \frac{k^2}{4}}}, \]

where \( J_m^k \) is the \( k \)th positive zero of \( J_m(z) \). If \( m = 0 \), we find that

\[(2.12b) \quad \frac{J_0^k}{\sqrt{(\nu + \frac{1}{2})^2 + \frac{\nu}{4}}} < \phi_{\nu,k}^0 < \frac{J_0^k}{\sqrt{(\nu + \frac{1}{2})^2 + \frac{1}{12}}}. \]

3. Ordering the \( \nu \)-zeroes of Legendre functions

This section contains the principle results of this paper. We begin with a comparison of \( \nu_2^3 \) and \( \nu_1^6 \):

**Theorem 1.** \( \nu_2^3(z_0) < \nu_1^6(z_0) \) for all \(-1 < z_0 < 1\).

**Proof.** Here, it will be convenient to let \( z_0 = \cos \phi_0 \) and to consider \( \nu_1^6 \) and \( \nu_2^3 \) as functions of \( \phi_0 \). First, we will show that if \( \nu = \nu_1^6 = \nu_2^3 \), then \( \nu > 7 \). Then, we will show that \( \nu = \nu_1^6 = \nu_2^3 \) is impossible if \( \nu > 7 \).

**Part 1.** Since \( \nu_2^3(\phi_0) \) and \( \nu_1^6(\phi_0) \) are decreasing in \( \phi_0 \), by (2.8), we have

\[ \lim_{\phi_0 \to \pi^-} \nu_2^3(\phi_0) = 4, \quad \lim_{\phi_0 \to \pi^-} \nu_1^6(\phi_0) = 6. \]

Since \( \nu_1^6(\phi_0) > 6 \) for \( \phi_0 \in (0, \pi) \) and \( \nu_2^3(\phi_0) < 6 \) for \( \phi_0 \in (\pi/2, \pi) \), we see that if \( \nu = \nu_2^3(\phi_0) = \nu_1^6(\phi_0) \) for some \( \phi_0 \), then \( \nu \geq 6 \) and \( 0 < \phi_0 < \pi/2 \).

Next, suppose that \( \nu = \nu_2^3 = \nu_1^6 \) for some \( z_0 = \cos \phi_0 \) and \( 0 < \phi_0 < \pi/2 \).

From (2.3) we see that

\[ (A) \quad \left( \begin{array}{c} 10z_0(1 - z_0^2)^{-1/2} \\ 1 \\ \end{array} \right) \left( \begin{array}{c} (\nu - 4)(\nu + 5) \\ 8z_0(1 - z_0^2)^{-1/2} \end{array} \right) \left( \begin{array}{c} P_5^\nu(z_0) \\ P_6^\nu(z_0) \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right). \]
Since by (1.1), $P^5_\nu(z_0)$ and $P^4_\nu(z_0)$ cannot vanish simultaneously, we see that the determinant of the $2 \times 2$ matrix in (A) must be zero, and we are led to the condition that

$$z_0^2 = \frac{\nu(\nu + 1) - 20}{\nu(\nu + 1) + 60}.$$  

From (B), we see that for $6 \leq \nu < 7$, $\sqrt{1151} \leq z_0 < \sqrt{929}$. On the other hand, if we set $\phi = \text{arc} \cos \sqrt{1151}$, $m = 6$, $k = 1$ and $j_1^6 \doteq 9.9361$ in (2.12a), we find that for such a $\phi_0$ and $\nu$,

$$z_0 = \cos \phi_0 = \cos(\phi_{\nu, 1}^6) < \cos \left( \frac{j_1^6}{\sqrt{(7.5)^2 - 143/12}} \right) \doteq 0.0784,$$

which is a contradiction. Thus, we have shown that if $\nu = \nu_1^6 = \nu_2^3$, then $\nu \geq 7$. Moreover, we have also shown that if $\nu_1^6 = \nu_2^3$ for some $\phi_0$, then $\phi_0 \leq \text{arc} \cos \sqrt{929}$.

**Part 2.** By the relationship in (B), we are motivated to define a function $z_0(\nu)$ for $\nu \geq 7$ as follows:

$$(3.1) \quad z_0(\nu) = \left( \frac{\nu(\nu + 1) - 20}{\nu(\nu + 1) + 60} \right)^{1/2}.$$  

For such a $z_0(\nu)$, we also define $\phi_0(\nu) = \text{arc} \cos(z_0(\nu))$ and observe that

$$(3.2) \quad \sin^2(\phi_0(\nu)) = \frac{80}{\nu(\nu + 1) + 60}.$$  

From (2.12a), (2.10), and (3.2), with $m = 6$, $k = 1$ and $\phi = \text{arc} \cos \sqrt{929}$, we see that for all $\nu \geq 7$,

$$\sin^2(\phi_{\nu, 1}^6) \geq \sin^2 \left( \frac{j_1^6}{\sqrt{(\nu + 1/2)^2 - \frac{b^2}{4}}} \right)$$

$$\geq \frac{(j_1^6)^2}{\nu(\nu + 1) + \frac{b^2}{4} + \alpha(\phi)(j_1^6)^2}$$

$$= \frac{80}{\nu(\nu + 1) + 60}$$

$$= \sin^2(\phi_0(\nu)), $$

where $j_1^6 \doteq 9.9361$. To complete the proof, we observe that if $\nu^* = \nu_1^6 = \nu_2^3$ for some $\nu^* \geq 7$, then necessarily we must have $\phi_{\nu^*, 1}^6 = \phi_{\nu^*, 2}^3 = \phi_0(\nu^*)$.

However, from (3.3) we see that $\phi_0(\nu) < \phi_{\nu, 1}^6$ for all $\nu \geq 7$. It follows that $\nu_1^6 \neq \nu_2^3$ for $0 < \phi_0 < \pi$. Since $\nu_2^3 < \nu_1^6$ for $\pi/2 \leq \phi_0 < \pi$, we conclude that $\nu_2^3 < \nu_1^6$ for all $0 < \phi_0 < \pi$ (or equivalently, for all $z_0 \in (-1, 1)$).
As a consequence of (1.1)–(1.5) and Theorem 1, we see that the first eleven $\nu$-zeroes are

(3.4)
\[

v_0^0 < v_1^1 < v_2^2 < v_3^3 < v_4^4 < v_5^5 < v_6^6 < v_7^7 < v_8^8 < v_9^9 < v_{10}^0 < 1,
\]

and that this ordering is unique. The inequalities $v_0^0 < v_1^1$, $v_2^2 < v_3^3$, and $v_4^4 < v_1^1$ were established in [2] for $0 < z_0 < 1$. By applying (2.8) and arguing as we did at the beginning of the proof of Theorem 1, these inequalities can be shown to hold for $-1 < z_0 < 0$ as well. In particular, we have the following:

**Theorem 2.** $v_2^0 < v_1^1$, $v_2^1 < v_1^3$, $v_3^0 < v_1^5$ for all $z_0 \in (-1, 1)$.

The inequality $v_j^{m+2} < v_j^m$ for $0 < z_0 < 1$ was established in [2]. Next, we consider the case $-1 < z_0 < 0$.

**Lemma 2.** If $-1 < z_0 < 0$, then $v_j^m < v_{j+1}^m$.

**Proof.** The $v_j^m$'s are simple zeroes of $P^m_{\nu}(z_0)$. (See [2].) From (1.1), we have that $v_j^m, v_{j+1}^m \in (\nu_j^{m+1}, \nu_{j+1}^{m+1})$. Suppose that $v_j^{m+2}(z_0) < v_{j+1}^m(z_0)$ for some $z_0 \in (-1, 0)$. From (2.7)–(2.8), we see that

\[

\text{sign}(P_{\nu}^m(z_0)) = (-1)^{m+j}, \quad v_j^m < v < v_{j+1}^m, \quad z_0 \in (-1, 0).
\]

The signs of $P_{\nu}^{m+1}(z_0)$ and $P_{\nu}^{m+2}(z_0)$ can also be determined in this way. We are led to the results summarized in Table 1.

**Table 1.** Suppose $v_j^{m+2}(z_0) \leq v_j^m(z_0)$ for $z_0 \in (-1, 0)$

<table>
<thead>
<tr>
<th>$z_0 \in (-1, 0)$</th>
<th>$v \in (v_j^{m+1}, v_j^{m+2})$</th>
<th>$v \in (v_j^{m+2}, v_{j+1}^m)$</th>
<th>$v \in (v_{j+1}^m, v_{j+1}^{m+1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>\text{sign($P_{\nu}^m(z_0)$)}</td>
<td>$(-1)^{m+j}$</td>
<td>$(-1)^{m+j}$</td>
<td>$(-1)^{m+j+1}$</td>
</tr>
<tr>
<td>\text{sign($P_{\nu}^{m+1}(z_0)$)}</td>
<td>$(-1)^{m+j+1}$</td>
<td>$(-1)^{m+j+1}$</td>
<td>$(-1)^{m+j+1}$</td>
</tr>
<tr>
<td>\text{sign($P_{\nu}^{m+2}(z_0)$)}</td>
<td>$(-1)^{m+j+1}$</td>
<td>$(-1)^{m+j+2}$</td>
<td>$(-1)^{m+j+2}$</td>
</tr>
</tbody>
</table>

The second column in Table 1 contradicts (2.3), and we conclude that $v_j^{m+2} > v_j^m$ for all $z_0 \in (-1, 0)$.

From (1.1)–(1.3) and Lemma 2, we are led to the following:

**Theorem 3.** If $v_j^m(z_0)$ is a solution of (*), then

(i) $v_j^{m+1} < v_j^{m+2} < v_j^{m+1}$, \quad $-1 < z_0 < 0$,

(ii) $v_j^{m+2} = v_j^{m+1} = m + 2j - 1$, \quad $z_0 = 0$,

(iii) $v_j^{m+1} < v_j^{m+2} < v_j^m < v_j^{m+1}$, \quad $0 < z_0 < 1$.

4. **Concluding remarks**

Since the zeroes of the Bessel functions are distinct [9, p. 484] the elements of $\mathcal{J} = \{j_k^m\}$ can be arranged as an increasing sequence. In particular, we can
define integer-valued functions \( m(i) \), \( k(i) \) so that \( j_{k(i)}^m \) denotes the \( i \)th element in the sequence \( \mathcal{J} \). Clearly, there is no such ordering of all the elements of \( \mathcal{N}_{\phi} = \{ \nu_j^m(\phi_0) \} \) that is independent of \( \phi_0 \). On the other hand, if we let 
\[
\overline{\phi} = \phi_0 = \nu_{\nu,k}^m \quad \text{and} \quad \nu = \nu_k^m(\phi_0) \quad \text{in (2.12), we see that}
\]
\[
(4.1) \quad \lim_{\phi_0 \to 0^+} \phi_0 \left( \nu_k^m(\phi_0) + \frac{1}{2} \right) = j_k^m.
\]

The limit in (4.1) is related to the well-known result, \( \lim_{n \to \infty} \phi_0^0, k(n + \frac{1}{2}) = j_k^0 \) (see [8]) and implies that for \( \phi_0 \) sufficiently small, \( \nu_{k(i)}^m(\phi_0) \) is the \( i \)th element in the sequence \( \mathcal{N}_{\phi} = \{ \nu_{j(i)}^m(\phi_0) \} \).

In view of (1.6) and the first two inequalities in (1.4), it is natural to con-
jecture if there is an inequality that relates \( \nu_{j+1}^m \) and \( \nu_{j+3}^m \) for \( \phi_0 \in (0, \pi/2) \). Such an inequality is not possible. From [1], we see that 
\[
\begin{align*}
j_{k(18)}^m &= j_{k(18)}^8 = 12.225, \quad \text{and} \quad j_{k(19)}^m = j_5^5 = 12.338. \quad \text{Since} \quad \nu_1^8(\pi/2) = 9, \quad \nu_2^6(\pi/2) = 8, \quad \text{and} \quad j_1^8 < j_2^5, \quad \text{we conclude that} \quad \nu_1^8(\phi_0) = \nu_2^6(\phi_0) \quad \text{for some} \quad \phi_0 \in (0, \pi/2). \quad \text{Numerical calculations indicate that} \quad \nu_1^8 = \nu_2^5 = 26.706 \quad \text{when} \quad \phi_0 = 26.134^\circ.
\end{align*}
\]

Although (1.5) and Theorem 1 demonstrate that the first eleven \( \nu \)-zeros of 
\( P_n^m(\cos \phi_0) \) are distinct for \( 0 < \phi_0 < \pi/2 \), the twelfth \( \nu \)-zero is not necessarily distinct. Since \( \nu_1^3(\pi/2) = 6, \quad \nu_1^5(\pi/2) = 7, \quad \text{and} \quad j_{k(12)}^m = j_6^6 = 9.936, \quad \text{and} \quad j_{k(13)}^m = j_1^3 = 10.173, \) from (4.1), we see that \( \nu_1^6(\phi_0) = \nu_1^3(\phi_0) \) for some \( \phi_0 \in (0, \pi/2). \) Numerics indicate \( \nu_1^6 = \nu_3^1 = 15.780 \) when \( \phi_0 = 35.821^\circ \) (see [2]).

REFERENCES

2. F. Baginski, Ordering the zeroes of the Legendre functions \( P_n^m(z_0) \) when considered as a function of \( \nu \), J. Math. Anal. Appl. 147 (1990), 296–308.