PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 111, Number 2, February 1991

SOME ESTIMATES FOR HARMONIC MEASURES. II

JAMES A. JENKINS

(Communicated by Clifford J. Earle, Jr.)

ABSTRACT. FitzGerald, Rodin, and Warschawski proved that, for a continuum of given diameter in the closed unit disc, the harmonic measure at the center is minimized when it is an arc on the circumference. A very simple proof of this result is given, using the method of the extremal metric.

1

In [1], FitzGerald, Rodin, and Warschawski gave the solution for two extremal problems for the harmonic measure of a continuum lying in the unit disc. These results may be stated as follows, using the following notations: F is the closed unit disc $|z| \leq 1$, E the open unit disc, C a continuum in F not containing the origin, G the component of $E - C$ containing the origin, α the border entity of G determined by C, and $\omega(0, \alpha, G)$ the harmonic measure of α at 0 with respect to G.

I. *If C has diameter δ, $\omega(0, \alpha, G) \geq \frac{1}{2\pi}\theta$ where θ is the angular measure of an arc on $|z| = 1$ of diameter δ, with equality precisely in that case.*

II. *If C subtends an angle of measure θ equal at most to π at 0, $\omega(0, \alpha, G) \geq \frac{1}{2\pi}\theta$ with equality if and only if C is an arc of angular measure θ on $|z| = 1$.*

In [2], the present author pointed out that by using triad modules a very simple proof of II can be given by the method of the extremal metric. Now we will observe that I is an immediate consequence of the extremal property of the Mori extremal domain.

2

The extremal property of the Mori extremal domain [3] can be stated as follows:

III. *Let C^* be a continuum not containing the origin or the point at infinity which has two points in $|z| \leq 1$ of distance $\geq \delta > 0$. Let Γ^* be the homotopy*

Received by the editors October 3, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 31A15, 30C85.

class of rectifiable Jordan curves separating C^* *from* 0 *and* ∞. *Then the module* $m(\Gamma^*)$ *of* Γ^* *is maximal precisely when* C^* *is an arc of diameter* δ *on* $|z| = 1$.

To derive I from III, we remark fiɪst, as in [2], that we may assume C meets $|z| = 1$. Then I is equivalent to maximizing the triad module $m(0, \beta, G)$ where β is the open boundary arc of G on $|z| = 1$. Let $\widetilde{C}$ be the reflection of C in $|z| = 1$, $C^* = C \cup \widetilde{C}$. Evidently $m(0, \beta, G) = 2m(\Gamma^*)$. Now I follows at once from III.

Bibliography

1. C. FitzGerald, B. Rodin, and S. Warschawski, *Estimates of the harmonic measure of a continuum in the unit disc*, Trans. Amer. Math. Soc. **287** (1985), 681–685.
2. James A. Jenkins, *Some estimates for harmonic measures*, in Complex Analysis I (Proc. of the Special Year, University of Maryland, College Park, 1985–86), Lecture Notes in Math., vol. 1275, Springer-Verlag, 1987, pp. 210–214.
3. A. Mori, *On an absolute constant in the theory of quasiconformal mappings*, J. Math. Soc. Japan **8** (1956), 156–166.

Department of Mathematics, Washington University, St. Louis, Missouri 63130