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ON $p - C^*$ SUMMING OPERATORS

KRZYSZTOF NOWAK

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ABSTRACT. We prove that every bounded linear operator $T : \mathscr{A} \to C_p(H)$ such that $i \circ T : \mathscr{A} \to B(H)$ is positive (where $\mathscr{A}$ is a unital C^*-algebra, $C_p(H)$ a Schatten class, i the identity map from $C_p(H)$ into $B(H)$) is $p - C^*$ summing. This permits us to characterize $p - C^*$ summing operators in some classes of multipliers.

INTRODUCTION

Gilles Pisier introduced the notion of $p - C^*$ summing operator in order to prove Grothendieck's inequality for noncommutative C^*-algebras (see [5]). In fact he used, in his proof, only 4 and $2 - C^*$ summing operators. In this paper we prove that every bounded operator $T : \mathscr{A} \to C_p(H)$ such that $i \circ T$ is positive (where $\mathscr{A}$ is a unital C^*-algebra, $C_p(H)$ a Schatten class, i the canonical embedding of $C_p(H)$ in $B(H)$) is $p - C^*$ summing. We remark that, for $1 \le p < 2$, the assumption "$i \circ T$ is positive" cannot be omitted. Using this result, we give the characterisation of $p - C^*$ summing operators in the class of multiplier operators on $B(H)$ and positive Herz-Schur multipliers.

1. ON POSITIVE $p - C^*$ SUMMING OPERATORS

A linear map T from a C^*-algebra $\mathscr{A}$ into a Banach space X is $p - C^*$ summing (we assume $p \ge 1$) if there is a constant c such that, for any finite sequence

$$\{x_i\}_{i=1}^N \subset \mathscr{A}^h = \{x \in \mathscr{A} : x^* = x\},$$

the following condition holds:

$$\left(\sum_{i=1}^N \|Tx_i\|^p \right)^{1/p} \le c \left\| \sum_{i=1}^N |x_i|^p \right\|^{1/p},$$

where

$$|x| = (x^* x)^{1/2}.$$

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The least constant c for which this condition is satisfied is denoted by $c_p(T)$. It is shown in [5] that T is $p - C^*$ summing if and only if there is a constant c and a state φ on $\mathscr{A}$ such that, for all x in $\mathscr{A}^h$,

$$\|Tx\| \le c\varphi(|x|^p)^{1/p}.$$

The least of those constants is equal to $c_p(T)$.

Let $\mathscr{A}$ and $\mathscr{B}$ be two C^*-algebras and $T : \mathscr{A} \to \mathscr{B}$ a linear map. T is called positive if Tx is positive in $\mathscr{B}$ for all positive x in $\mathscr{A}$ and completely positive if $T \otimes i_n : \mathscr{A} \otimes M_n \to \mathscr{B} \otimes M_n$ is positive for all natural n.

We will use the following notation:

$B(H)$—algebra of all bounded linear operators on the Hilbert space H equipped with operator norm,

$C(H)$—the ideal of compact operators on H,

$C_p(H)$—Schatten class, i.e., operators in $C(H)$ of the form

$$\sum_i \lambda_i \varphi_i \otimes \bar{\psi}_i = \sum_i \lambda_i \langle \cdot , \psi_i \rangle \varphi_i ,$$

where $\{\varphi_i\}$, $\{\psi_i\}$ are orthonormal sets in H and $\sum_i |\lambda_i|^p < \infty$ with the norm $\| \sum_i \lambda_i \varphi_i \otimes \bar{\psi}_i \|_p = (\sum_i |\lambda_i|^p)^{1/p}$.

We will need

Lemma 1.1. [7, p. 95]. *Let H be a Hilbert space, and let A, B be positive operators in $B(H)$. If $p \ge 2$ and $B \in C_p(H)$ then $\|AB\|_p \le \|A^{p/2}B^{p/2}\|_2^{2/p}$.*

Theorem 1.2. *Let $\mathscr{A}$ be a unital C^*-algebra and H be a Hilbert space. If $T : \mathscr{A} \to C_p(H)$ $(p \ge 1)$ is a bounded linear map such that $i \circ T : \mathscr{A} \to B(H)$ is positive (where $i : C_p(H) \to B(H)$ denotes the identity map), then T is $p - C^*$ summing and $c_p(T) \le \|T\|$.*

Proof. Let x be a hermitian element of $\mathscr{A}$, and let $\mathscr{B}$ be a unital C^*-algebra generated by x. Since $\mathscr{B}$ is commutative, $T : \mathscr{B} \to B(H)$ is completely positive and, by Stinespring's theorem, may be represented in the form $Ty = V^*\pi(y)V$ for all $y \in \mathscr{B}$, where $\pi : \mathscr{B} \to B(R)$ is a unital *-representation on a Hilbert space R and $V : H \to R$ is a bounded linear operator (see [1]).

Setting $V_1 = \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$, an element of $B(R \oplus H)$, and $\pi_1(y) = \begin{bmatrix} \pi(y) & 0 \\ 0 & 0 \end{bmatrix}$, a *-representation of $\mathscr{B}$ in $R \oplus H$, we have $\pi_1(|x|) = |\pi_1(x)| = U^*\pi_1(x)$ for a unitary $U \in B(R \oplus H)$ which commutes with $\pi_1(x)$, and we may write

$$\begin{aligned}
\|Tx\|_p &= \|V^*\pi(x)V\|_p = \|V_1^*\pi_1(x)V_1\|_p \\
&= \|V_1^*U\pi_1(|x|^{1/2})\pi_1(|x|^{1/2})V_1\|_p \\
&\le \|V_1^*U\pi_1(|x|^{1/2})\|_{2p}\|\pi_1(|x|^{1/2})V_1\|_{2p} \\
&= \|V_1^*\pi_1(|x|^{1/2})U\|_{2p}\|\pi_1(|x|^{1/2})V_1\|_{2p} \\
&\le \|\pi_1(|x|^{1/2})V_1\|_{2p}^2 \le \|\pi_1(|x|^{1/2})|V_1^*|\,\|_{2p}^2 .
\end{aligned}$$

An application of Lemma 1.1 gives

$$\begin{aligned}\|Tx\|_p &\le \|\pi_1(|x|^{p/2})|V_1^*|^p\|_2^{2/p}\\ &= (\mathrm{tr}|V_1^*|^p\pi_1(|x|^p)|V_1^*|^p)^{1/p}\\ &= (\mathrm{tr}(V_1V_1^*)^p\pi_1(|x|^p))^{1/p}.\end{aligned}$$

Since $(V_1V_1^*)^{m/n} = V_1(V_1^*V_1)^{\frac{m}{n}-1}V_1^*$ for all natural m, n such that $m \ge n$ (to see this take the nth power of both sides), we have, by continuity argument, $(V_1V_1^*)^p = V_1(V_1^*V_1)^{p-1}V_1^*$, and it follows that

$$\begin{aligned}\|Tx\|_p &\le (\mathrm{tr}\, V_1(V_1^*V_1)^{p-1}V_1^*\pi_1(|x|^p))^{1/p}\\ &= (\mathrm{tr}(V_1^*V_1)^{p-1}V_1^*\pi_1(|x|^p)V_1)^p\\ &= (\mathrm{tr}(V^*V)^{p-1}V^*\pi(|x|^p)V)^{1/p}\\ &= (\mathrm{tr}(Te)^{p-1}T(|x|^p))^{1/p}.\end{aligned}$$

It is easily seen that the functional $x \mapsto \mathrm{tr}(Te)^{p-1}T(x)$ is positive on $\mathscr{A}$ and its norm equals $\|Te\|_p^p$, so the proof is complete. □

Remark 1.3. Proposition 2.3 shows that, for $1 \le p < 2$, the assumption $i \circ T$ is positive is essential.

Corollary 1.4. *Let $\mathscr{A}$ be a C^*-algebra with the unit e, and let H be a Hilbert space. If $T : \mathscr{A} \to B(H)$ is a positive linear map and $Te \in C_p(H)$ $(p \ge 1)$, then T is $p-C^*$ summing, $c_p(T) \le \|Te\|_p$, $T(\mathscr{A}) \subset C_p(H)$ and T is bounded as the map from $\mathscr{A}$ into $C_p(H)$ with the norm $\|Te\|_p$.*

Proof. Taking unitary element u instead of hermitian x in the first part of the proof of Theorem 1.2, we get $\|Te\|_p = \|V^*V\|_p = \|V_1^*V_1\|_p$, so $V_1 \in C_{2p}(R\oplus H)$ and $\|V_1\|_{2p}^2 = \|Te\|_p$, $\|Tu\|_p = \|V_1^*\pi_1(u)V_1\|_p = \|\pi_1(u)\|\,\|V_1\|_{2p}^2 \le \|Te\|_p$. Since $\sup\{\|Tx\|_p : \|x\| \le 1\} = \sup\{\|Tu\|_p : u\text{-unitary}\}$ (see [3]), we know that T is bounded as the map from $\mathscr{A}$ into $C_p(H)$ and its norm equals $\|Te\|_p$. The rest immediately follows from Theorem 1.2. □

We will demonstrate that, in some cases, the assumption $Te \in C_p(H)$ of Corollary 1.4 is not only sufficient but also necessary.

2. Application to multiplier operators

Let $B \in B(H)$ and T_B be a mapping from $B(H)$ into $B(H)$ defined by $T_B(A) = BAB^*$.

Proposition 2.1. *Let $p \ge 1$. Then T_B is a $p - C^*$ summing operator if and only if B^*B belongs to $C_p(H)$; moreover, $c_p(T) = \|B^*B\|_p$.*

Proof. Let us assume that T_B is a $p - C^*$ summing operator and $\{\varphi_i\}$ an arbitrary orthonormal basis in H. We may write

$$\begin{aligned}\sum_{i=1}^{n}\langle B^*B\varphi_i, \varphi_i\rangle^p &= \sum_{i=1}^{n}\|(\varphi_i\otimes\bar{\varphi}_i)B^*B(\varphi_i\otimes\bar{\varphi}_i)\|^p \\ &= \sum_{i=1}^{n}\|B(\varphi_i\otimes\bar{\varphi}_i)B^*\|^p \le c_p^p(T_B).\end{aligned}$$

Let $T = B^*B = \int_0^{\|T\|}\lambda\, dE(\lambda)$ be a spectral decomposition of T. We have

$$T \ge \int_{\varepsilon}^{\|T\|}\lambda\, dE(\lambda) \ge \varepsilon E((\varepsilon, \|T\|)),$$

and, since

$$\sum_i \langle T\varphi_i, \varphi_i\rangle^p \le c_p^p(T_B),$$

for any orthonormal basis $\{\varphi_i\}$, we get that the operator $E((\varepsilon, \|T\|))$ is of finite rank. Hence T is compact and may be represented in the following form: $T = \sum_i \lambda_i \psi_i \otimes \bar{\psi}_i$, $\{\psi_i\}$ is an orthonormal basis in H. Since

$$\sum_i \lambda_i^p = \sum_i \langle T\psi_i, \psi_i\rangle^p \le c_p^p(T_B),$$

we infer that B^*B belongs to $C_p(H)$, $\|B^*B\|_p \le c_p(T_B)$.

The converse is an immediate consequence of Corollary 1.4. □

Now we consider the left regular representation on $B(H)$. For $B \in B(H)$, we define $L_B : B(H) \to B(H)$ by the formula $L_B(A) = BA$.

Proposition 2.2. *Let $p \ge 2$. Then L_B is $p - C^*$ summing if and only if B belongs to $C_p(H)$; moreover, $c_p(L_B) = \|B\|_p$.*

Proof. Let us assume that L_B is $p - C^*$ summing and that $\{\varphi_i\}$ is an arbitrary orthonormal basis in H. Then

$$\begin{aligned}\sum_{i=1}^{n}\langle B^*B\varphi_i, \varphi_i\rangle^{p/2} &= \sum_{i=1}^{n}\|(\varphi_i\otimes\bar{\varphi}_i)B^*B(\varphi_i\otimes\bar{\varphi}_i)\|^{p/2} \\ &= \sum_{i=1}^{n}\|B(\varphi_i\otimes\bar{\varphi}_i)\|^p \le c_p^p(L_B).\end{aligned}$$

Following the reasoning from Proposition 2.1, we state that $B^*B \in C_{p/2}(H)$ and $\|B^*B\|_{p/2} \le c_p^2(L_B)$, hence $\|B\|_p \le c_p(L_B)$. If $B \in C_p(H)$ then $\|BA\| = \|BA^2B^*\|^{1/2} \le c_{p/2}^{1/2}(T_B)\varphi(|A|^p)^{1/2}$, where φ is a state on $B(H)$. We have $c_p(L_B) \le c_{p/2}^{1/2}(T_B) = \|B^*B\|_{p/2}^{1/2} = \|B\|_p$. □

Proposition 2.3. *If $1 \le p < 2$, $B \in B(H)$ and $B \ne 0$, then L_B is not $p - C^*$ summing.*

Proof. To see this, let us show first that, for any $\varphi \in H$, $\|\varphi\| = 1$, $L_{\varphi\otimes\bar{\varphi}}$ is not $p - C^*$ summing. For all natural n we can find orthonormal vectors $\{\varphi_i\}_{i=1}^{n}$

such that

$$\varphi = \sum_{i=1}^{n} \frac{1}{\sqrt{n}} \varphi_i$$

and

$$\sum_{i=1}^{n} \|(\varphi \otimes \bar{\varphi})(\varphi_i \otimes \bar{\varphi}_i)\|^p = \sum_{i=1}^{n} \langle \varphi , \varphi_i \rangle^p = n^{(2-p)/2} ,$$

so the operator $L_{\varphi \otimes \varphi}$ is not $p - C^*$ summing.

Let us assume that $B \in B(H)$, $B \neq 0$ and L_B is $p - C^*$ summing. We can find $\xi \in H$, $\|\xi\| = 1$, such that

$$(\xi \otimes \bar{\xi})B = \xi \otimes \overline{B^* \xi} \neq 0 .$$

We see that $L_{\xi \otimes \overline{B^* \xi}}$ is $p - C^*$ summing, so

$$L_{B^* \xi \otimes \overline{B^* \xi}} = L_{(\xi \otimes \overline{B^* \xi})^* (\xi \otimes \overline{B^* \xi})}$$

is non-zero and $p - C^*$ summing, and we are done. □

Let H be a Hilbert space and $\{\varphi_i\}$ an orthonormal basis of H. Let $M = (m_{ij})$ be a positive Herz-Schur multiplier, i.e., for every A in $B(H)$, there is a B in $B(H)$ such that

$$\langle B\varphi_j , \varphi_i \rangle = m_{ij} \langle A\varphi_j , \varphi_i \rangle ,$$

for any finite sequence of complex numbers

$$\{\xi_i\}_{i=1}^{N} , \qquad \sum_{ij} m_{ij} \xi_i \bar{\xi}_j \geq 0 .$$

We see that M defines a bounded positive linear operator from $B(H)$ into $B(H)$. It is known that M is a positive Herz-Schur multiplier if and only if there is a Hilbert space R and a sequence $\{x_i\} \in R$ such that, for some constant c, $\|x_i\| \leq c$ for all i and $m_{ij} = \langle x_i , x_j \rangle$ (see [2]), but we will not use this fact.

Proposition 2.4. *M is $p - C^*$ summing if and only if $\sum_i m_{ii}^p < \infty$; moreover, $c_p(M) = (\sum_i m_{ii}^p)^{1/p}$.*

Proof. Assuming that M is $p - C^*$ summing, we obtain

$$\sum_{i} m_{ii}^p = \sum_{i} \|M(\varphi_i \otimes \bar{\varphi}_i)\|^p \leq c_p^p(M) .$$

The converse is an immediate consequence of Corollary 1.4. □

Remark 2.5. It is easily seen, in view of Corollary 1.4, that a positive Herz-Schur multiplier is $p - C^*$ summing if and only if its image is contained in $C_p(H)$.

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Institute of Mathematics, University of Wrocław, 50-348 Wrocław, Poland

Current address: Department of Mathematics, Washington University in St. Louis, St. Louis, Missouri 63130