ON \( p - C^* \) SUMMING OPERATORS

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Abstract. We prove that every bounded linear operator \( T : \mathcal{A} \to C_p(H) \) such that \( i \circ T : \mathcal{A} \to B(H) \) is positive (where \( \mathcal{A} \) is a unital \( C^* \)-algebra, \( C_p(H) \) a Schatten class, \( i \) the identity map from \( C_p(H) \) into \( B(H) \)) is \( p - C^* \) summing. This permits us to characterize \( p - C^* \) summing operators in some classes of multipliers.

Introduction

Gilles Pisier introduced the notion of \( p - C^* \) summing operator in order to prove Grothendieck's inequality for noncommutative \( C^* \)-algebras (see [5]). In fact he used, in his proof, only 4 and \( 2 - C^* \) summing operators. In this paper we prove that every bounded operator \( T : \mathcal{A} \to C_p(H) \) such that \( i \circ T \) is positive (where \( \mathcal{A} \) is a unital \( C^* \)-algebra, \( C_p(H) \) a Schatten class, \( i \) the canonical embedding of \( C_p(H) \) in \( B(H) \)) is \( p - C^* \) summing. We remark that, for \( 1 \leq p < 2 \), the assumption "\( i \circ T \) is positive" cannot be omitted. Using this result, we give the characterisation of \( p - C^* \) summing operators in the class of multiplier operators on \( B(H) \) and positive Herz-Schur multipliers.

1. On positive \( p - C^* \) summing operators

A linear map \( T \) from a \( C^* \)-algebra \( \mathcal{A} \) into a Banach space \( X \) is \( p - C^* \) summing (we assume \( p \geq 1 \)) if there is a constant \( c \) such that, for any finite sequence

\[ \{x_i\}_{i=1}^N \subset \mathcal{A}^h = \{ x \in \mathcal{A} : x^* = x \}, \]

the following condition holds:

\[ \left( \sum_{i=1}^N \|Tx_i\|^p \right)^{1/p} \leq c \left( \sum_{i=1}^N |x_i|^p \right)^{1/p}, \]

where

\[ |x| = (x^*x)^{1/2}. \]
The least constant \( c \) for which this condition is satisfied is denoted by \( c_p(T) \).

It is shown in [5] that \( T \) is \( p - C^* \) summing if and only if there is a constant \( c \) and a state \( \varphi \) on \( \mathcal{A} \) such that, for all \( x \) in \( \mathcal{A}^h \),

\[
\|Tx\| \leq c \varphi(|x|^p)^{1/p}.
\]

The least of those constants is equal to \( c_p(T) \).

Let \( \mathcal{A} \) and \( \mathcal{B} \) be two \( C^* \)-algebras and \( T : \mathcal{A} \to \mathcal{B} \) a linear map. \( T \) is
called positive if \( Tx \) is positive in \( \mathcal{B} \) for all positive \( x \) in \( \mathcal{A} \) and completely
positive if \( T \otimes i_n : \mathcal{A} \otimes M_n \to \mathcal{B} \otimes M_n \) is positive for all natural \( n \).

We will use the following notation:

- \( B(H) \)—algebra of all bounded linear operators on the Hilbert space \( H \) equipped with operator norm,
- \( C(H) \)—the ideal of compact operators on \( H \),
- \( C_p(H) \)—Schatten class, i.e., operators in \( C(H) \) of the form

\[
\sum_i \lambda_i \varphi_i \otimes \psi_i = \sum_i \lambda_i (\cdot, \psi_i) \varphi_i,
\]

where \( \{\varphi_i\}, \{\psi_i\} \) are orthonormal sets in \( H \) and \( \sum_i |\lambda_i|^p < \infty \) with
the norm \( \| \sum_i \lambda_i \varphi_i \otimes \psi_i \|_p = (\sum_i |\lambda_i|^p)^{1/p} \).

We will need

**Lemma 1.1.** [7, p. 95]. Let \( H \) be a Hilbert space, and let \( A, B \) be positive
operators in \( B(H) \). If \( p \geq 2 \) and \( B \in C_p(H) \) then \( \|AB\|_p \leq \|A^{p/2}B^{p/2}\|_2^p \).

**Theorem 1.2.** Let \( \mathcal{A} \) be a unital \( C^* \)-algebra and \( H \) be a Hilbert space. If
\( T : \mathcal{A} \to C_p(H) \) \( (p \geq 1) \) is a bounded linear map such that \( i \circ T : \mathcal{A} \to B(H) \)
is positive (where \( i : C_p(H) \to B(H) \) denotes the identity map), then \( T \) is \( p - C^* \)
summing and \( c_p(T) \leq \|T\| \).

**Proof.** Let \( x \) be a hermitian element of \( \mathcal{A} \), and let \( \mathcal{B} \) be a unital \( C^* \)-algebra
generated by \( x \). Since \( \mathcal{B} \) is commutative, \( T : \mathcal{B} \to B(H) \) is completely
positive and, by Stinespring’s theorem, may be represented in the form \( Ty = V^* \pi(y)V \) for all \( y \in \mathcal{B} \), where \( \pi : \mathcal{B} \to B(R) \) is a unital \( \ast \)-representation on
a Hilbert space \( R \) and \( V : H \to R \) is a bounded linear operator (see [1]).

Setting \( V_1 = \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix} \), an element of \( B(R \oplus H) \), and \( \pi_1(y) = \begin{bmatrix} \pi(y) & 0 \\ 0 & 0 \end{bmatrix} \), a
\( \ast \)-representation of \( \mathcal{B} \) in \( R \oplus H \), we have \( \pi_1(|x|) = |\pi_1(x)| = U^* \pi_1(x) \) for a
unitary \( U \in B(R \oplus H) \) which commutes with \( \pi_1(x) \), and we may write

\[
\|Tx\|_p = \|V^* \pi(x)V\|_p = \|V_1^* \pi_1(x) V_1\|_p
\]

\[
= \|V_1^* U \pi_1(|x|^{1/2}) \pi_1(|x|^{1/2}) V_1\|_p
\]

\[
\leq \|V_1^* U \pi_1(|x|^{1/2})\|_{2p} \|\pi_1(|x|^{1/2}) V_1\|_{2p}
\]

\[
= \|V_1^* \pi_1(|x|^{1/2}) U\|_{2p} \|\pi_1(|x|^{1/2}) V_1\|_{2p}
\]

\[
\leq \|\pi_1(|x|^{1/2}) V_1\|_{2p}^2 \leq \|\pi_1(|x|^{1/2}) V_1^*\|_{2p}^2.
\]
An application of Lemma 1.1 gives
\[ \|Tx\|_p \leq \|\pi_1(|x|^{p/2})V_1^{*}\|_2^{2/p} \]
\[ = (\text{tr}|V_1^{*}|^p \pi_1(|x|^p)V_1^{*})^{1/p} \]
\[ = (\text{tr}(V_1V_1^*)\pi_1(|x|^p))^{1/p}. \]

Since \((V_1V_1^*)^m/n = V_1(V_1^*V_1)^{m-1}V_1^*\) for all natural \(m, n\) such that \(m \geq n\) (to see this take the \(n\)th power of both sides), we have, by continuity argument, 
\( (V_1V_1^*)^p = V_1(V_1^*V_1)^{p-1}V_1^* \), and it follows that
\[ \|Tx\|_p \leq (\text{tr}V_1(V_1^*V_1)^{p-1}V_1^*\pi_1(|x|^{p}))^{1/p} \]
\[ = (\text{tr}(V_1^*V_1)^{p-1}V_1^*\pi_1(|x|^{p}))^{1/p} \]
\[ = (\text{tr}(V_1^*V_1)^{p-1}V_1^*\pi_1(|x|^{p}))^{1/p} \]
\[ = (\text{tr}(T^e)^{p-1}T(|x|^p))^{1/p}. \]

It is easily seen that the functional \( x \mapsto \text{tr}(T^e)^{p-1}T(x) \) is positive on \( \mathcal{A} \) and its norm equals \( \|T^e\|_p \), so the proof is complete. 

Remark 1.3. Proposition 2.3 shows that, for \( 1 < p < 2 \), the assumption \( i \circ T \) is positive is essential.

Corollary 1.4. Let \( \mathcal{A} \) be a \( C^* \)-algebra with the unit \( e \), and let \( H \) be a Hilbert space. If \( T : \mathcal{A} \rightarrow B(H) \) is a positive linear map and \( T e \in C_p(H) \) (\( p \geq 1 \)), then \( T \) is \( p - C^* \) summing, \( c_p(T) \leq \|T e\|_p \), \( T(\mathcal{A}) \subset C_p(H) \) and \( T \) is bounded as the map from \( \mathcal{A} \) into \( C_p(H) \) with the norm \( \|T e\|_p \).

Proof. Taking unitary element \( u \) instead of hermitian \( x \) in the first part of the proof of Theorem 1.2, we get \( \|Te\|_p = \|V^*V\|_p = \|V_1^*V_1\|_p \), so \( V_1 \in C_{2p}(R \oplus H) \) and \( \|V_1\|_{2p}^2 = \|Te\|_p \), \( \|Tu\|_p = \|V_1^*\pi_1(u)V_1\|_p = \|\pi_1(u)||\|V_1\|_{2p}^2 \leq \|Te\|_p \). Since \( \sup\{\|Tx\|_p : \|x\| \leq 1\} = \sup\{\|Tu\|_p : u \text{-unitary}\} \) (see [3]), we know that \( T \) is bounded as the map from \( \mathcal{A} \) into \( C_p(H) \) and its norm equals \( \|Te\|_p \). The rest immediately follows from Theorem 1.2. 

We will demonstrate that, in some cases, the assumption \( Te \in C_p(H) \) of Corollary 1.4 is not only sufficient but also necessary.

2. APPLICATION TO MULTIPLIER OPERATORS

Let \( B \in B(H) \) and \( T_B \) be a mapping from \( B(H) \) into \( B(H) \) defined by \( T_B(A) = BAB^* \).

Proposition 2.1. Let \( p \geq 1 \). Then \( T_B \) is a \( p - C^* \) summing operator if and only if \( B^*B \) belongs to \( C_p(H) \); moreover, \( c_p(T) = \|B^*B\|_p \).
Proof. Let us assume that $T_B$ is a $p - C^*$ summing operator and $\{\varphi_i\}$ an arbitrary orthonormal basis in $H$. We may write

$$\sum_{i=1}^{n} \langle B^* B \varphi_i, \varphi_i \rangle^p = \sum_{i=1}^{n} \| (\varphi_i \otimes \varphi_i) B^* B (\varphi_i \otimes \varphi_i) \|^p$$

$$= \sum_{i=1}^{n} \| B (\varphi_i \otimes \varphi_i) B^* \|^p \leq c_p^p(T_B).$$

Let $T = B^* B = \int_0^\|T\| \lambda dE(\lambda)$ be a spectral decomposition of $T$. We have

$$T \geq \int_\epsilon^\|T\| \lambda dE(\lambda) \geq \epsilon E((\epsilon, \|T\|)),$$

and, since

$$\sum_{i} \langle T \varphi_i, \varphi_i \rangle^p \leq c_p^p(T_B),$$

for any orthonormal basis $\{\varphi_i\}$, we get that the operator $E((\epsilon, \|T\|))$ is of finite rank. Hence $T$ is compact and may be represented in the following form: $T = \sum_i \lambda_i \psi_i \otimes \psi_i$, $\{\psi_i\}$ is an orthonormal basis in $H$. Since

$$\sum_{i} \lambda_i^p = \sum_{i} \langle T \psi_i, \psi_i \rangle^p \leq c_p^p(T_B),$$

we infer that $B^* B$ belongs to $C_p(H)$, $\|B^* B\|_p \leq c_p(T_B)$.

The converse is an immediate consequence of Corollary 1.4. \(\square\)

Now we consider the left regular representation on $B(H)$. For $B \in B(H)$, we define $L_B : B(H) \to B(H)$ by the formula $L_B(A) = BA$.

**Proposition 2.2.** Let $p \geq 2$. Then $L_B$ is $p - C^*$ summing if and only if $B$ belongs to $C_p(H)$; moreover, $c_p(L_B) = \|B\|_p$.

**Proof.** Let us assume that $L_B$ is $p - C^*$ summing and that $\{\varphi_i\}$ is an arbitrary orthonormal basis in $H$. Then

$$\sum_{i=1}^{n} \| (\varphi_i \otimes \varphi_i) B^* B (\varphi_i \otimes \varphi_i) \|^p/2$$

$$= \sum_{i=1}^{n} \| B (\varphi_i \otimes \varphi_i) \|^p \leq c_p^p(L_B).$$

Following the reasoning from Proposition 2.1, we state that $B^* B \in C_{p/2}(H)$ and $\|B^* B\|_{p/2} \leq c_p^2(L_B)$, hence $\|B\|_p \leq c_p(L_B)$. If $B \in C_p(H)$ then $\|BA\| = \|BA^2 B^*\|^{1/2} \leq c_p^{1/2}(T_B) \varphi(A)^{1/2}$, where $\varphi$ is a state on $B(H)$. We have $c_p(L_B) \leq c_p^{1/2}(T_B) = \|B^* B\|_p = \|B\|_p$. \(\square\)

**Proposition 2.3.** If $1 \leq p < 2$, $B \in B(H)$ and $B \neq 0$, then $L_B$ is not $p - C^*$ summing.

**Proof.** To see this, let us show first that, for any $\varphi \in H$, $\|\varphi\| = 1$, $L_{\varphi \otimes \varphi}$ is not $p - C^*$ summing. For all natural $n$ we can find orthonormal vectors $\{\varphi_i\}_{i=1}^{n}$
such that

\[ \varphi = \sum_{i=1}^{n} \frac{1}{\sqrt{n}} \varphi_i. \]

and

\[ \sum_{i=1}^{n} \| (\varphi \otimes \varphi) (\varphi_i \otimes \varphi_i) \|^p = \sum_{i=1}^{n} (\varphi, \varphi_i)^p = n^{(2-p)/2}, \]

so the operator \( L_{\varphi \otimes \varphi} \) is not \( p - C^* \) summing.

Let us assume that \( B \in B(H) \), \( B \neq 0 \) and \( L_B \) is \( p - C^* \) summing. We can find \( \xi \in H \), \( \| \xi \| = 1 \), such that

\[ (\xi \otimes \bar{\xi}) B = \xi \otimes \bar{B^*} \xi \neq 0. \]

We see that \( L_{\xi \otimes \bar{B^*} \xi} \) is \( p - C^* \) summing, so

\[ L_{B^* \xi \otimes \bar{\xi}} \circ L_{B \xi} = L_{(\xi \otimes \bar{B^*} \xi)^*} \circ L_{(\xi \otimes \bar{B^*} \xi)} \]

is non-zero and \( p - C^* \) summing, and we are done. \( \Box \)

Let \( H \) be a Hilbert space and \( \{\varphi_i\} \) an orthonormal basis of \( H \). Let \( M = (m_{ij}) \) be a positive Herz-Schur multiplier, i.e., for every \( A \) in \( B(H) \), there is a \( B \) in \( B(H) \) such that

\[ \langle B \varphi_j, \varphi_i \rangle = m_{ij} \langle A \varphi_j, \varphi_i \rangle, \]

for any finite sequence of complex numbers

\[ \{\xi_i\}_{i=1}^{N}, \quad \sum_{ij} m_{ij} \xi_i \bar{\xi_j} \geq 0. \]

We see that \( M \) defines a bounded positive linear operator from \( B(H) \) into \( B(H) \). It is known that \( M \) is a positive Herz-Schur multiplier if and only if there is a Hilbert space \( R \) and a sequence \( \{x_i\} \in R \) such that, for some constant \( c \), \( \|x_i\| \leq c \) for all \( i \) and \( m_{ij} = \langle x_i, x_j \rangle \) (see [2]), but we will not use this fact.

**Proposition 2.4.** \( M \) is \( p - C^* \) summing if and only if \( \sum_i m_{ii}^p < \infty \); moreover, \( c_p(M) = (\sum_i m_{ii}^p)^{1/p} \).

**Proof.** Assuming that \( M \) is \( p - C^* \) summing, we obtain

\[ \sum_i m_{ii}^p = \sum_i \| M(\varphi_i \otimes \varphi_i) \|^p \leq c_p(M)^p. \]

The converse is an immediate consequence of Corollary 1.4. \( \Box \)

**Remark 2.5.** It is easily seen, in view of Corollary 1.4, that a positive Herz-Schur multiplier is \( p - C^* \) summing if and only if its image is contained in \( C_p(H) \).
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References


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