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PARTITIONING PAIRS OF COUNTABLE SETS

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ABSTRACT. We make the translations of our partitions from [3] in the context of all countable subsets of a fixed uncountable set. A different translation was obtained recently by Velleman [4].

The purpose of this paper is to define a two-cardinal version of one of our partitions from [3].

Theorem. *For every uncountable set* A *there is a* $c : [[A]^{\aleph_0}]^2 \to A$ *such that, for every cofinal* $U \subseteq [A]^{\aleph_0}$ *and* α *in* A, *there exist* $x \subset y$ *in* U *such that* $c(x, y) = \alpha$.

The proof will use straightforward generalization of one of the partitions from [3]. We shall assume that A is equal to some initial ordinal θ, and we shall fix an

$$r : [\theta]^{\aleph_0} \to \{0, 1\}^{\omega}$$

such that $r_x \neq r_y$ for $x \subset y$. [Identifying ω_1 with a subset of $\{0, 1\}^{\omega}$, let r_x (including finite x) be the standard code of $(\operatorname{tp} x, q_x)$, where q_x is defined recursively on $\sup x$ as follows assuming that, for each ordinal α of cofinality ω, we have a fixed increasing sequence $\{\alpha_i\}$ converging to α: If x has a maximal element ξ set $q_x(0) = 1$ and $q_x(i+1) = r_y(i)$, where $y = x \cap \xi$. If $\alpha = \sup x$ is a limit ordinal, let $q_x(0) = 0$ and $q_x(2^i(2j+1)) = r_{x_i}(j)$, where $x_i = x \cap \alpha_i$.] Moreover, we shall fix a one-to-one $e_x : x \to \omega$ for each x in $[\theta]^{\aleph_0}$. For an integer n and x in $[\theta]^{\aleph_0}$, we set

$$x(n) = \{\xi \in x : e_x(\xi) \leq n\}.$$

For $x \subset y$ in $[\theta]^{\aleph_0}$, let

$$\Delta(x, y) = \Delta(r_x, r_y),$$

i.e., the minimal place where the reals r_x and r_y disagree. Finally, for $x \subset y$ in $[\theta]^{\aleph_0}$ and an ordinal $\lambda \leq \theta$, we set

$$c_\lambda(x, y) = \min(y(\Delta(x, y)) \setminus \sup(x \cap \lambda)),$$

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i.e., $c_\lambda(x, y)$ is the minimal ordinal of $y(\Delta(x, y))$ which is bigger than or equal to the supremum of $x \cap \lambda$; if this set is empty, set $c_\lambda(x, y) = 0$.

To state the basic property of our partitions c_λ, let $\lim(\omega)$ denote the class of all ordinals of cofinality ω.

Lemma. *Suppose $\kappa \neq \lambda$ are regular uncountable cardinals $\leq \theta$ and that S and T are stationary subsets of $\lim(\omega) \cap \kappa$ and $\lim(\omega) \cap \lambda$, respectively. Then, for every cofinal $U \subset [\theta]^{\aleph_0}$, there exist $x \subset y$ in U such that $c_\kappa(x, y) \in S$ and $c_\lambda(x, y) \in T$.*

Proof. Choose a countable elementary submodel M of $H_{(2^\theta)^+}$ containing all relevant objects such that $\gamma = \sup(M \cap \kappa)$ is in S and $\delta = \sup(M \cap \lambda)$ is in T. [If $\kappa < \lambda$, first pick submodel N such that $N \cap \lambda$ is in T, then pick a countable submodel $\bar{M}$ containing N as an element such that $\sup(\bar{M} \cap \kappa)$ is in S and set $M = \bar{M} \cap N$.] Choose a y in U containing $(M \cap \theta) \cup \{\gamma, \delta\}$ and an integer n such that both γ and δ are in $y(n)$. For s in $\{0, 1\}^{<\omega}$, let U_s be the set of all z in U such that r_z extends s. Let Σ be the set of all s in $\{0, 1\}^{<\omega}$ for which U_s is cofinal. Note that, by elementarity of M, all restrictions of r_y are in Σ. Since r_y is not in M there must be an s in Σ which splits from r_y at some place $m \geq n$. Pick α in $M \cap \kappa$ above every element of $y(m) \cap \gamma$, and also pick β in $M \cap \lambda$ above every element of $y(m) \cap \delta$. Since U_s is cofinal and since it is an element of M, we can find an x in $U_s \cap M$ containing α and β. Then $\Delta(x, y) = m$, and γ and δ are minimal ordinals of $y(m)$ above $\sup(x \cap \kappa)$ and $\sup(x \cap \lambda)$, respectively. This finishes the proof. □

Now the theorem follows easily: If θ is a regular cardinal we let c be the composition of c_θ with a splitting of $\lim(\omega) \cap \theta$ into θ disjoint stationary sets. If θ is singular, let κ be the maximum of $\{\omega_1, \operatorname{cf}\theta\}$ and let S_ξ, $\xi < \operatorname{cf}\theta$, be a partition of $\lim(\omega) \cap \kappa$ into disjoint stationary sets. Let λ_ξ, $\xi < \operatorname{cf}\theta$, be an increasing sequence of regular cardinals converging to θ with $\lambda_0 > \kappa$. For each ξ fix a partition T_ξ^η, $\eta < \lambda_\xi$, of $\lim(\omega) \cap \lambda_\xi$ into disjoint stationary sets. Finally, for $x \subset y$ in $[\theta]^{\aleph_0}$, we let $c(x, y)$ be equal to $\langle \xi, \eta \rangle$ if $c_\kappa(x, y) \in S_\xi$ and $c_{\lambda_\xi}(x, y) \in T_\xi^\eta$. By the lemma, every $\langle \xi, \eta \rangle$ is realized in every cofinal $U \subseteq [\theta]^{\aleph_0}$.

By looking at some other partitions of [3] it is natural to ask whether we can define $c(x, y)$ to be an element of $[\theta]^{\aleph_0}$ rather than an ordinal from θ with the hope that the set of values of c on the square of every cofinal $U \subseteq [\theta]^{\aleph_0}$ contains a closed and unbounded set. Unfortunately, such a hope cannot be realized since there might be an unbounded subset of $[\theta]^{\aleph_0}$ of smaller size than any closed and unbounded set in $[\theta]^{\aleph_0}$. In principle, it is still possible to have a stationary set $S \subset [\theta]^{\aleph_0}$ such that every cofinal $U \subset [\theta]^{\aleph_0}$ realizes every color from a closed and unbounded set restricted to S. This is the approach taken by Velleman [4]. Unfortunately, [4] assumes a too strong fact about

$[\theta]^{\aleph_0}$, the existence of a stationary set of size θ, which rules out many θ's, in particular, all θ's of cofinality ω. A more relaxed approach goes as follows. Let *stationary cofinality* of $[A]^{\aleph_0}$ be the minimal cardinality of a stationary subset of $[A]^{\aleph_0}$. (We do not know whether the stationary cofinality of $[A]^{\aleph_0}$ is, in fact, equal to the cofinality of $[A]^{\aleph_0}$.) A *Kurepa family* in $[A]^{\aleph_0}$ is a family K of countable subsets of A such that the union of every uncountable subfamily of K is uncountable. Let K be a Kurepa family in $[A]^{\aleph_0}$, and let A^* be the set of all finite sequences of elements of A. For each k in K fix an onto map $f_k : \omega \to k$, and let k^* be the set of all proper initial subsequences of f_k. Then $K^* = \{k^* : k \in K\}$ is a Kurepa family on A^* in the usual sense [2], i.e., for every a in $[A^*]^{\aleph_0}$, there exist only countably many sets of the form $k^* \cap a$ for k^* in K^*.

Theorem. *Assume S is stationary in $[A]^{\aleph_0}$ and that there is a Kurepa family in $[A]^{\aleph_0}$ of size S. Then there is a $d : [[A]^{\aleph_0}]^2 \to [A]^{\aleph_0}$ such that, for every cofinal $U \subset [A]^{\aleph_0}$, there is a closed and unbounded set of elements of S of the form $d(x, y)$ for some $x \subset y$ in U.*

Fix one-to-one $i : S \to [A]^{\aleph_0}$ such that $i''S$ is a Kurepa family in $[A]^{\aleph_0}$. For each y in $[A]^{\aleph_0}$, we let

$$S_y = \{x \in S : x \cup i(x) \subseteq y\}$$

and fix $e_y : S_y \to \omega$ one-to-one. For an integer n and y in $[A]^{\aleph_0}$, set

$$y(n) = \{x \in S_y : e_y(x) \leq n\}.$$

Finally, for $x \subset y$, $d(x, y)$ is the $\subseteq$-minimal element of $y(\Delta(x, y))$ containing x, if such an element exists; otherwise $d(x, y) = y$. It is now quite easy to check that d works.

Since the family of singletons of elements of A is clearly a Kurepa family in $[A]^{\aleph_0}$, the hypothesis of the second theorem is satisfied when the stationary cofinality of $[A]^{\aleph_0}$ is equal to the size of A. It is well known that, besides the infinite products of infinite sets and their finite successors, this is true for A of size $< \aleph_\omega$ (see [1]), or if the size of A has uncountable cofinality provided 0# does not exist. Thus, left open is the case $A = \theta$ and $\mathrm{cf}\,\theta = \omega$. In [2, §8] (see also [3, §1]), we have produced a Kurepa family in $[\theta]^{\aleph_0}$ of size θ^+ assuming $\square_\theta$. Thus, if, moreover, the stationary cofinality of $[\theta]^{\aleph_0}$ is equal to θ^+, the hypothesis of the second theorem is satisfied. All this shows that, for example, the constructible universe is the natural model which satisfies this hypothesis for *every* set A. Note that some such hypothesis is needed here since, for example, the Chang's Conjecture for $(\aleph_{\omega+1}, \aleph_\omega)$ easily implies that every subset of $[\aleph_\omega]^{\aleph_0}$ of size $> \aleph_\omega$ contains an uncountable subset with countable union.

In the case $A = \theta$ and $\mathrm{cf}\,\theta = \omega$, the number of colors given by the partition d is larger than θ. This follows from the fact that if K is a Kurepa family in

$[A]^{\aleph_0}$, then every stationary set S in $[A]^{\aleph_0}$ splits into K disjoint stationary sets. To see this, for every x in $[A]^{\aleph_0}$, fix one-to-one

$$f_x : \{k \in K : k \subseteq x\} \to \omega,$$

and, for n in ω and k in K, define

$$U_k^n = \{x \in [A]^{\aleph_0} : k \subseteq x \text{ and } f_x(k) = n\}.$$

Then for a fixed k, U_k^n cover the supersets of k while, for a fixed n, U_k^n are disjoint. Thus we have an analogue of the Ulam matrix and we can proceed as in the classical case. (If the size of K has countable cofinality, we first split S into countably many disjoint stationary sets S_i and then split each S_i into a sufficient number of disjoint stationary sets.)

Note that if K is a Kurepa family in $[A]^{\aleph_0}$ containing all singletons, then

$$F(x) = \{k \in K : k \subseteq x\}$$

is an isomorphical embedding of $[A]^{\aleph_0}$ into a cofinal subset of $[K]^{\aleph_0}$. Conversely, if $[\theta]^{\aleph_0}$ and $[\lambda]^{\aleph_0}$ (where $\theta \leq \lambda$) are cofinally similar (see [1]), then there is a Kurepa family in $[\theta]^{\aleph_0}$ of size λ. To see this pick a Tukey function $H : [\lambda]^{\aleph_0} \to [\theta]^{\aleph_0}$, i.e., a function mapping unbounded sets into unbounded sets. Then $\{H(\{\xi\}) : \xi < \lambda\}$ is a Kurepa family in $[\theta]^{\aleph_0}$ of size λ.

The basic result of this paper is true when $[A]^{\aleph_0}$ is replaced by $[A]^{\kappa}$ for every regular κ. This is proved using an analogue of the partition a of [3, §5]: Given $x \subset y$ in $[A]^{\kappa}$ with $\sup(x)$ in y, one first walks inside y (along a fixed c-sequence on κ^+ and modulo the isomorphism of y and its order type) from $\sup(y)$ to $\sup(x)$ and then picks the first place where the corresponding closed and unbounded sets have made a non-trivial oscillation *provided* the maximum of their intersection is in x.

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