PARTITIONING PAIRS OF COUNTABLE SETS

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Abstract. We make the translations of our partitions from [3] in the context of all countable subsets of a fixed uncountable set. A different translation was obtained recently by Velleman [4].

The purpose of this paper is to define a two-cardinal version of one of our partitions from [3].

Theorem. For every uncountable set $A$ there is a $c : [\mathcal{A}]^{\aleph_0} \to A$ such that, for every cofinal $U \subseteq [\mathcal{A}]^{\aleph_0}$ and $\alpha$ in $A$, there exist $x \subset y$ in $U$ such that $c(x, y) = \alpha$.

The proof will use straightforward generalization of one of the partitions from [3]. We shall assume that $A$ is equal to some initial ordinal $\theta$, and we shall fix an $r : [\theta]^{\aleph_0} \to \{0, 1\}^\omega$ such that $r_x \neq r_y$ for $x \subset y$. [Identifying $\omega_1$ with a subset of $\{0, 1\}^\omega$, let $r_x$ (including finite $x$) be the standard code of $(\text{tp} x, q_x)$, where $q_x$ is defined recursively on $\sup x$ as follows assuming that, for each ordinal $\alpha$ of cofinality $\omega_1$, we have a fixed increasing sequence $\{\alpha_i\}$ converging to $\alpha$: If $x$ has a maximal element $\xi$ set $q_x(0) = 1$ and $q_x(i + 1) = r_y(i)$, where $y = x \cap \xi$. If $x = \sup x$ is a limit ordinal, let $q_x(0) = 0$ and $q_x(2^i + 2j + 1) = r_x(i)$, where $x_i = x \cap \alpha_i$.] Moreover, we shall fix a one-to-one $e_x : x \to \omega$ for each $x$ in $[\theta]^{\aleph_0}$. For an integer $n$ and $x$ in $[\theta]^{\aleph_0}$, we set $x(n) = \{\xi \in x : e_x(\xi) \leq n\}$.

For $x \subset y$ in $[\theta]^{\aleph_0}$, let

$$\Delta(x, y) = \Delta(r_x, r_y),$$

i.e., the minimal place where the reals $r_x$ and $r_y$ disagree. Finally, for $x \subset y$ in $[\theta]^{\aleph_0}$ and an ordinal $\lambda \leq \theta$, we set $c_x \Delta(x, y) = \min(y(\Delta(x, y)) \setminus \sup(x \cap \lambda)) \setminus \sup(x \cap \lambda))$, 

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i.e., $c_\lambda(x, y)$ is the minimal ordinal of $\nu(\Delta(x, y))$ which is bigger than or equal to the supremum of $x \cap \lambda$; if this set is empty, set $c_\lambda(x, y) = 0$.

To state the basic property of our partitions $c_\lambda$, let $\lim(\omega)$ denote the class of all ordinals of cofinality $\omega$.

**Lemma.** Suppose $\kappa \neq \lambda$ are regular uncountable cardinals $\leq \theta$ and that $S$ and $T$ are stationary subsets of $\lim(\omega) \cap \kappa$ and $\lim(\omega) \cap \lambda$, respectively. Then, for every cofinal $U \subseteq [\theta]^{\omega_1}$, there exist $x \subset y$ in $U$ such that $c_\kappa(x, y) \in S$ and $c_\lambda(x, y) \in T$.

**Proof.** Choose a countable elementary submodel $M$ of $H_{(2^\kappa)^+}$ containing all relevant objects such that $\gamma = \sup(M \cap \kappa)$ is in $S$ and $\delta = \sup(M \cap \lambda)$ is in $T$. [If $\kappa < \lambda$, first pick submodel $N$ such that $N \cap \lambda$ is in $T$, then pick a countable submodel $M$ containing $N$ as an element such that $\sup(M \cap \kappa)$ is in $S$ and set $M = M \cap N$.] Choose a $y$ in $U$ containing $(M \cap 6) \cup \{y, \delta\}$ and an integer $n$ such that both $\gamma$ and $\delta$ are in $(y(n))$. For $s$ in $\{0, 1\}^{<\omega}$, let $U_s$ be the set of all $z$ in $U$ such that $r_s$ extends $s$. Let $\Sigma$ be the set of all $s$ in $\{0, 1\}^{<\omega}$ for which $U_s$ is cofinal. Note that, by elementarity of $M$, all restrictions of $r$ are in $\Sigma$ which splits from $r_y$ at some place $m \geq 1$. Pick $\alpha$ in $M \cap \kappa$ above every element of $y(m) \cap \gamma$, and also pick $\beta$ in $M \cap \lambda$ above every element of $y(m) \cap \delta$. Since $U_s$ is cofinal and since it is an element of $M$, we can find an $x$ in $U_s \cap M$ containing $\alpha$ and $\beta$. Then $\Delta(x, y) = m$, and $\gamma$ and $\delta$ are minimal ordinals of $y(m)$ above $\sup(x \cap \kappa)$ and $\sup(x \cap \lambda)$, respectively. This finishes the proof. \(\Box\)

Now the theorem follows easily: If $\theta$ is a regular cardinal we let $c$ be the composition of $c_\theta$ with a splitting of $\lim(\omega) \cap \theta$ into $\theta$ disjoint stationary sets. If $\theta$ is singular, let $\kappa$ be the maximum of $\{\omega_1, cf\theta\}$ and let $S_\xi$, $\xi < cf\theta$, be a partition of $\lim(\omega) \cap \kappa$ into disjoint stationary sets. Let $\lambda_\xi$, $\xi < cf\theta$, be an increasing sequence of regular cardinals converging to $\theta$ with $\lambda_0 > \kappa$. For each $\xi$ fix a partition $T_\xi^n$, $\eta < \lambda_\xi$, of $\lim(\omega) \cap \lambda_\xi$ into disjoint stationary sets. Finally, for $x \subset y$ in $[\theta]^{\omega_1}$, we let $c(x, y)$ be equal to $\langle \xi, \eta \rangle$ if $c_\kappa(x, y) \in S_\xi$ and $c_\lambda(x, y) \in T_\xi^n$. By the lemma, every $\langle \xi, \eta \rangle$ is realized in every cofinal $U \subseteq [\theta]^{\omega_1}$.

By looking at some other partitions of $[\theta]$ it is natural to ask whether we can define $c(x, y)$ to be an element of $[\theta]^{\omega_1}$ rather than an ordinal from $\theta$ with the hope that the set of values of $c$ on the square of every cofinal $U \subseteq [\theta]^{\omega_1}$ contains a closed and unbounded set. Unfortunately, such a hope cannot be realized since there might be an unbounded subset of $[\theta]^{\omega_1}$ of smaller size than any closed and unbounded set in $[\theta]^{\omega_1}$. In principle, it is still possible to have a stationary set $S \subset [\theta]^{\omega_1}$ such that every cofinal $U \subset [\theta]^{\omega_1}$ realizes every color from a closed and unbounded set restricted to $S$. This is the approach taken by Velleman [4]. Unfortunately, [4] assumes a too strong fact about
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[0], the existence of a stationary set of size $\theta$, which rules out many $\theta$'s, in particular, all $\theta$'s of cofinality $\omega$. A more relaxed approach goes as follows. Let stationary cofinality of $[A]^\kappa$ be the minimal cardinality of a stationary subset of $[A]^\kappa$. (We do not know whether the stationary cofinality of $[A]^\kappa$ is, in fact, equal to the cofinality of $[A]^\kappa$.) A Kurepa family in $[A]^\kappa$ is a family $K$ of countable subsets of $A$ such that the union of every uncountable subfamily of $K$ is uncountable. Let $K$ be a Kurepa family in $[A]^\kappa$, and let $A^*$ be the set of all finite sequences of elements of $A$. For each $k$ in $K$ fix an onto map $f_k: \omega \to k$, and let $k^*$ be the set of all proper initial subsequences of $f_k$. Then $K^* = \{k^*: k \in K\}$ is a Kurepa family on $A^*$ in the usual sense [2], i.e., for every $a$ in $[A]^\kappa$, there exist only countably many sets of the form $k^* \cap a$ for $k^*$ in $K^*$.

Theorem. Assume $S$ is stationary in $[A]^\kappa$ and that there is a Kurepa family in $[A]^\kappa$ of size $S$. Then there is a $d: [([A]^\kappa]^2) \to [A]^\kappa$ such that, for every cofinal $U \subset [A]^\kappa$, there is a closed and unbounded set of elements of $S$ of the form $d(x, y)$ for some $x \subset y$ in $U$.

Fix one-to-one $i: S \to [A]^\kappa$ such that $i''S$ is a Kurepa family in $[A]^\kappa$. For each $y$ in $[A]^\kappa$, we let

$$S_y = \{x \in S : x \cup i(x) \subseteq y\}$$

and fix $e_y: S_y \to \omega$ one-to-one. For an integer $n$ and $y$ in $[A]^\kappa$, set

$$y(n) = \{x \in S_y : e_y(x) \leq n\}.$$ 

Finally, for $x \subset y$, $d(x, y)$ is the $\subseteq$-minimal element of $y(\Delta(x, y))$ containing $x$, if such an element exists; otherwise $d(x, y) = y$. It is now quite easy to check that $d$ works.

Since the family of singletons of elements of $A$ is clearly a Kurepa family in $[A]^\kappa$, the hypothesis of the second theorem is satisfied when the stationary cofinality of $[A]^\kappa$ is equal to the size of $A$. It is well known that, besides the infinite products of infinite sets and their finite successors, this is true for $A$ of size $< \aleph_\omega$ (see [1]), or if the size of $A$ has uncountable cofinality provided $0^+$ does not exist. Thus, left open is the case $A = \theta$ and $cf \theta = \omega$. In [2, §8] (see also [3, §1]), we have produced a Kurepa family in $[\theta]^\kappa$ of size $\theta^+$ assuming $\square^\theta_\theta$. Thus, if, moreover, the stationary cofinality of $[\theta]^\kappa$ is equal to $\theta^+$, the hypothesis of the second theorem is satisfied. All this shows that, for example, the constructible universe is the natural model which satisfies this hypothesis for every set $A$. Note that some such hypothesis is needed here since, for example, the Chang's Conjecture for $(\kappa_{\omega+1}, \kappa_\omega)$ easily implies that every subset of $[\kappa_\omega]^\kappa$ of size $> \aleph_\omega$ contains an uncountable subset with countable union.

In the case $A = \theta$ and $cf \theta = \omega$, the number of colors given by the partition $d$ is larger than $\theta$. This follows from the fact that if $K$ is a Kurepa family in
Then every stationary set $S$ in $[A]^{\aleph_0}$ splits into $K$ disjoint stationary sets. To see this, for every $x$ in $[A]^{\aleph_0}$, fix one-to-one

$$f_x: \{k \in K : k \subseteq x\} \rightarrow \omega,$$

and, for $n$ in $\omega$ and $k$ in $K$, define

$$U^n_k = \{x \in [A]^{\aleph_0} : k \subseteq x \text{ and } f_x(k) = n\}.$$

Then for a fixed $k$, $U^n_k$ cover the supersets of $k$ while, for a fixed $n$, $U^n_k$ are disjoint. Thus we have an analogue of the Ulam matrix and we can proceed as in the classical case. (If the size of $K$ has countable cofinality, we first split $S$ into countably many disjoint stationary sets $S_i$ and then split each $S_i$ into a sufficient number of disjoint stationary sets.)

Note that if $K$ is a Kurepa family in $[A]^{\aleph_0}$ containing all singletons, then

$$F(x) = \{k \in K : k \subseteq x\}$$

is an isomorphical embedding of $[A]^{\aleph_0}$ into a cofinal subset of $[K]^{\aleph_0}$. Conversely, if $[\theta]^{\aleph_0}$ and $[\lambda]^{\aleph_0}$ (where $\theta \leq \lambda$) are cofinally similar (see [1]), then there is a Kurepa family in $[\theta]^{\aleph_0}$ of size $\lambda$. To see this pick a Tukey function $H : [\lambda]^{\aleph_0} \rightarrow [\theta]^{\aleph_0}$, i.e., a function mapping unbounded sets into unbounded sets. Then $\{H(\{\xi\}) : \xi < \lambda\}$ is a Kurepa family in $[\theta]^{\aleph_0}$ of size $\lambda$.

The basic result of this paper is true when $[A]^{\aleph_0}$ is replaced by $[A]^\kappa$ for every regular $\kappa$. This is proved using an analogue of the partition $a$ of [3, §5]: Given $x \subset y$ in $[A]^\kappa$ with $\text{sup}(x)$ in $y$, one first walks inside $y$ (along a fixed $c$-sequence on $\kappa^+$ and modulo the isomorphism of $y$ and its order type) from $\text{sup}(y)$ to $\text{sup}(x)$ and then picks the first place where the corresponding closed and unbounded sets have made a non-trivial oscillation provided the maximum of their intersection is in $x$.

**References**


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