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ALGEBRAS OF INVARIANT FUNCTIONS ON THE ŠILOV BOUNDARY OF GENERALIZED HALF-PLANES

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ABSTRACT. Let $\mathscr{N}$ be the nilpotent Lie group identified to the Šilov boundary of a symmetric generalized half-plane $\mathscr{D}$ and L a compact group acting on $\mathscr{N}$ by automorphisms, arising from the realization of $\mathscr{D}$ as hermitian symmetric space. Is then $(L \ltimes \mathscr{N}, L)$ a Gelfand pair? We study the problem and resolve it in the case of classical families.

0. INTRODUCTION

Let G be a locally compact group, L a compact group of automorphisms of G and $L \ltimes G$ the semidirect product of L and G; $(L \ltimes G, L)$ is a *Gelfand pair* if and only if the algebra of L-invariant integrable functions on G is commutative with respect to the convolution product.

Recently, some authors have been studying Gelfand pairs of the form $(L \ltimes N, L)$, where N is a nilpotent Lie group and L is a compact Lie group acting on N by automorphisms; for instance, Korányi (see [7]) considers the nilpotent Lie group N arising from an Iwasawa decomposition $G = KAN$ of a connected real semisimple Lie group of real rank one and the centralizer L of A in K (for other examples, see [5] and [6]). In [1], Benson, Jenkins and Ratcliff prove that there can be Gelfand pairs $(L \ltimes N, L)$ only if N is at most of step two and then develop a classification theory for such Gelfand pairs.

In this paper we are interested in finding such Gelfand pairs in the following setting: let $M = G^o/K$ be a hermitian symmetric space of noncompact type (see [4]), $\mathscr{D}$ the realization of M as generalized half-plane, $\check{S} = K/L$ the Šilov boundary of M in its compact dual M^* and $\mathscr{N}$ the Šilov boundary of $\mathscr{D}$ ($\mathscr{N}$ is in correspondence, by the Cayley transform, with a dense open subset of $\check{S}$) (see [8]); the compact group L acts on $\mathscr{N}$ by automorphisms. Trivially, $(L \ltimes \mathscr{N}, L)$ is a Gelfand pair when $\mathscr{D}$ is a tube domain (since in this case $\mathscr{N}$ is commutative) and the general problem can obviously be reduced to the irreducible case. In this paper we give a general result, independent of classification theory of symmetric generalized half-planes (Theorem 4) and the

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complete answer in the case of *classical* domains (Theorem 6) (the problem for the exceptional non-tube domain EIII remains open).

Our paper is organized as follows: in §1, we recall some results of the theory of representations of $\mathscr{N}$ (from [9] and [12]), describe the group L acting on $\mathscr{N}$ and restate our characterization of Gelfand pairs in the case of semidirect product (from [2]); in §2, we give general results concerning the stability subgroups and the intertwining representations, independently of classification theory; in §3, we solve the question in the classical cases: we always have Gelfand pairs except for $M = SU(p, p+q)/S(U(p) \times U(p+q))$ in the case $p > 2$ and $q \geq 2$.

1. Preliminaries

Siegel domain, Šilov boundary, and their representations. Our main references are [9] and [12].

Let $\mathscr{D}$ be a *Siegel domain of type* II (or *generalized half-plane*), i.e.,

$$\mathscr{D} = \mathscr{D}(\Omega, \mathscr{F}) = \{(Z, W) \in \mathscr{V} \times \mathscr{W} : \operatorname{Im}(Z) - \mathscr{F}(W, W) \in \Omega\}$$

where $\mathscr{V}$ and $\mathscr{W}$ are complex vector spaces of finite dimension, respectively n and m, $\mathscr{R}$ is a real form of $\mathscr{V}$, Ω is a regular cone in $\mathscr{R}$ and $\mathscr{F} : \mathscr{W} \times \mathscr{W} \longrightarrow \mathscr{V}$ is a Ω–positive hermitian map.

The real dual of $\mathscr{R}$ will be denoted with $\mathscr{R}'$ and the value of $\lambda \in \mathscr{R}'$ at $Z \in \mathscr{V}$ by $\langle Z, \lambda \rangle$.

We fix Haar measures dX and dW on the abelian groups $\mathscr{R}$ and $\mathscr{W}$; the Haar measure $d\lambda$ on $\mathscr{R}'$ is normalized so that the Fourier inversion formula holds:

$$f(x) = \int_{\mathscr{R}'} \exp(i\langle X, \lambda \rangle) \hat{f}(\lambda) d\lambda .$$

We fix a basis $\beta = \{\beta_1, \dots, \beta_m\}$ of $\mathscr{W}$ which is compatible with dW, i.e., $\int_{\mathscr{W}} g(W)dW = \int_{\mathbf{C}} \cdots \int_{\mathbf{C}} g(\sum_{j=1}^m w_j \beta_j) dw_1 \cdots dw_m$ for all $g \in L^1(\mathscr{W})$.

The Šilov boundary of the Siegel domain $\mathscr{D}$ (i.e., the minimal closed subset $\check{S}$ of $\partial\mathscr{D}$ such that $\max_{p \in \overline{\mathscr{D}}} | f(p) | = \max_{p \in \check{S}} | f(p) |$, for every function f holomorphic in $\mathscr{D}$ and reaching its maximum modulus in $\overline{\mathscr{D}}$) is

$$\check{S}_{\mathscr{D}} = \{(Z, W) \in \mathscr{V} \times \mathscr{W} : \operatorname{Im}(Z) = \mathscr{F}(W, W)\} .$$

The group of affine automorphisms of $\mathscr{D}$ of the form

$$(Z, W) \longrightarrow (X, \zeta) \cdot (Z, W) = (Z + X + 2i\mathscr{F}(W, \zeta) + i\mathscr{F}(\zeta, \zeta), W + \zeta)$$

($(Z, W) \in \mathscr{D}$, $(X, \zeta) \in \mathscr{R} \times \mathscr{W}$), acts simply transitively on $\check{S}$ and so $\check{S}$ can be identified with the nilpotent Lie group of step two $\mathscr{N} = \mathscr{R} \times \mathscr{W}$ with the product

$$(X_1, W_1)(X_2, W_2) = (X_1 + X_2 + 2 \operatorname{Im} \mathscr{F}(W_1, W_2), W_1 + W_2) .$$

The center of $\mathscr{N}$ is $\mathscr{R}$ and so $\mathscr{N}$ is commutative if and only if $\mathscr{D}$ is a *tube domain* (i.e., $\mathscr{W} = 0$); the product of the Haar measures dX and dW is a Haar measure on $\mathscr{N}$.

From now on, we suppose that $\mathscr{D}$ is an *irreducible, symmetric* generalized half-plane.

We must recall some results about the theory of the irreducible unitary representations of $\mathscr{N}$; we suppose that $\mathscr{D}$ is not a tube domain (otherwise, each element in $\hat{\mathscr{N}}$ is simply a character).

If π is an irreducible unitary representation of $\mathscr{N}$, then the restriction of π to $\mathscr{R}$ is a scalar multiple of the identity, i.e., there exists $\lambda \in \mathscr{R}'$ such that

$$\pi(X, 0) = \exp(i\langle X, \lambda\rangle)\,\mathrm{Id} \qquad \forall X \in \mathscr{R}. \tag{1}$$

From ([12], Lemma 2 and Theorem 1), we know that π is finite dimensional if and only if $\lambda = 0$. In this case, π is one dimensional.

The infinite dimensional elements of $\hat{\mathscr{N}}$ can be given in the Schrödinger or in the Bargmann–Fock form; for our purposes it is more convenient to work with the second one. We need some definitions: we put, for every $\lambda \in \mathscr{R}'$ and $W_i \in \mathscr{W}$,

$$\mathscr{F}_\lambda(W_1, W_2) := 4\langle \mathscr{F}(W_1, W_2), \lambda\rangle,$$
$$\mathscr{B}_\lambda(W_1, W_2) := \mathrm{Im}(\mathscr{F}_\lambda(W_1, W_2))$$

and define

$$\Lambda := \{\lambda \in \mathscr{R}' : \mathscr{F}_\lambda \text{ is nondegenerate}\}.$$

Λ is an open subset of $\mathscr{R}'$, with total Haar measure in $\mathscr{R}'$. From now on, we suppose that the linear form λ defined in (1) is in Λ.

Let $\mathscr{F}_\lambda(\zeta)$ be the matrix of the hermitian form $\mathscr{F}_\lambda$ w.r. to the basis $\zeta = \{\zeta_1, \dots, \zeta_m\}$ of $\mathscr{W}$; then $\rho(\lambda) :=|\det(\mathscr{F}_\lambda(\zeta))|$ is independent of the compatible basis ζ chosen. It is possible to consider a compatible basis $\epsilon = \{\epsilon_1, \dots, \epsilon_m\}$ s.t. $\mathscr{F}_\lambda(\epsilon) = \mathrm{diag}(\lambda_1, \dots, \lambda_m)$, with $\lambda_j \in R\backslash(0)$. Let J be the complex structure on $\mathscr{W}$, $\mathscr{W}^{\mathbf{R}}$ the underlying real space of $\mathscr{W}$ with basis $\{\epsilon_1, \dots, \epsilon_m, J\epsilon_1, \dots, J\epsilon_m\}$ and $J_\lambda : \mathscr{W}^{\mathbf{R}} \longrightarrow \mathscr{W}^{\mathbf{R}}$ defined w.r. to the ϵ-basis by

$$J_\lambda = \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix} \quad \text{where } \sigma = \mathrm{diag}(\mathrm{sign}\,\lambda_1, \dots, \mathrm{sign}\,\lambda_m).$$

$\mathscr{W}_\lambda$ will denote the space $\mathscr{W}^{\mathbf{R}}$ with the complex structure J_λ. For $W, W_1, W_2 \in \mathscr{W}$, we put

$$W_1 \cdot W_2 := \mathscr{B}_\lambda(J_\lambda W_1, \overline{W_2}) + i\mathscr{B}_\lambda(W_1, \overline{W_2}),$$
$$|W|^2 := W \cdot \overline{W}$$

($\overline{W}$ denotes the conjugate of W w.r. to J_λ).

Let $\Xi_\lambda(\mathscr{W})$ be the space of $\mathscr{C}^\infty$–functions F on $\mathscr{W}$ such that

(i) F is holomorphic w.r. to J_λ (i.e., $\langle dF(\zeta), J_\lambda W\rangle = i\langle dF(\zeta), W\rangle$ $\forall \zeta, W \in \mathscr{W}$),

(ii) if $G(W) := \exp(-\frac{|W|^2}{4})F(W)$ then $G \in L^2(\mathscr{W})$.

With the inner product

$$(F_1, F_2) := \int_{\mathscr{W}} F_1(W)\overline{F_2(W)} \exp(-\frac{|W|^2}{2}) dW,$$

$\Xi_\lambda(\mathscr{W})$ is a Hilbert space.
We have, from [11] and ([9], Prop.4.2, 5.2 and 6.2):

Theorem 1. *Let π be an irreducible unitary representation of $\mathscr{N}$ and λ the element of $\mathscr{R}'$ defined by the restriction of π to $\mathscr{R}$; we suppose $\lambda \in \Lambda$. Then π is equivalent to the Fock representation on $\Xi_\lambda(\mathscr{W})$ given by*

$$(2) \qquad (\pi^\lambda(X, W)F)(Z) := \exp\left\{-i\langle X, \lambda\rangle - \frac{|W|^2}{4} + \frac{Z \cdot \overline{W}}{2}\right\} F(Z - W).$$

If $\lambda, \mu \in \Lambda$ then π^λ is equivalent to π^μ if and only if $\lambda = \mu$.

The Plancherel measure for $\hat{\mathscr{N}}$ is concentrated on Λ and is given by $\rho(\lambda)\, d\lambda$ (therefore Λ is the *reduced dual* of $\mathscr{N}$).

A compact group of automorphisms of the Šilov boundary. We use notations, definitions and results of [8] (for a summary of the Korányi–Wolf theory, sufficient to our aim, see [3]).

We consider the symmetric generalized half-plane $\mathscr{D}$ as the realization of a hermitian symmetric space of noncompact type $M = G^o/K$ as generalized half-plane (therefore we have $\mathscr{V} = \underline{p}_1^-$, $\mathscr{W} = \underline{p}_2^-$, $\mathscr{R} = \underline{n}_1^-$ and so on). The Šilov boundary $\mathscr{N}$ of $\mathscr{D}$ can be identified, by the Cayley transform c, with a dense open subset of $\check{S} = K/L$, Šilov boundary of $M = G^o(\dot{x})$ in its compact dual $M^* = G^c/K^cP^+$ ($\dot{x}$ is the identity coset in M^* and L is the isotropy subgroup in K of the point $c(\dot{x})$). L is the product $L = L_1L_2$ where L_i is the analytic subgroup of K corresponding to the Lie algebra $\underline{l}_i$ (for the definition of $\underline{l}_1$ and $\underline{l}_2$, see ([8], 4.5). The compact group L normalizes the subgroup N^- of $\mathrm{Ad}(c)G^o$ ([8], 5.7); since N^- is isomorphic to the Šilov boundary $\mathscr{N}$, we can consider L as a group of automorphisms of $\mathscr{N}$; the action of an element $l = (l_1, l_2) \in L$ $(l_i \in L_i)$ on $\mathscr{N}$ is given by

$$(3) \qquad l(X, W) = (\mathrm{Ad}(l_1)X, \mathrm{Ad}(l)W)$$

where $(X, W) \in \mathscr{N} = \underline{n}_1^- \times \underline{p}_2^-$ and Ad is the adjoint representation on $\underline{p}^- = \underline{p}_1^- \times \underline{p}_2^-$; $\underline{p}_1^-$ and $\underline{p}_2^-$ are invariant under $\mathrm{Ad}(L)$, $\underline{n}_1^-$ under $\mathrm{Ad}(L_1)$ and $\mathrm{Ad}(L_2)$ is trivial on $\underline{p}_1^-$ ([8], 6.6).

In order to determine if the algebra of integrable L-invariant functions on $\mathscr{N}$ is commutative, i.e., if $(L \ltimes \mathscr{N}, L)$ is a Gelfand pair, we shall use our characterization.

Theorem 2 [2]. *Let G be a separable locally compact group, L a separable compact group of automorphisms of G; if $\pi \in \hat{G}$ we put*

$\pi_l(g) := \pi(l(g)) \quad \forall l \in L, \forall g \in G,$

$L_\pi := \{l \in L : \pi_l$ *is equivalent to* $\pi\}$ *"stability subgroup" of* π *in* L,
$U_\pi^\omega :=$ *projective- (* ω*-)representation of* L_π *which intertwines* π *and* π_l, *i.e.*

$$\pi_l(g) = [U_\pi^\omega(l)]^{-1}[\pi(g)][U_\pi^\omega(l)] \qquad \forall g \in G, \forall l \in L_\pi .$$

Then $(L \ltimes G, L)$ *is a Gelfand pair if and only if for all* $\pi \in \hat{G}$ *and for all T in* $\hat{L}_\pi^\omega$ *(the* ω*-dual of* L_π*) T occurs with multiplicity at most* 1 *in* U_π^ω (in the "if part" it is sufficient that the condition holds for all π in the reduced dual $\hat{G}_r$, chosen one from each L-orbit).

2. A GENERAL RESULT

Let $\mathscr{N}$ be the Šilov boundary of a symmetric irreducible non-tube generalized half-plane and L the compact group of automorphisms associated to $\mathscr{N}$ as in §1. We have the following results:

Lemma 3. *For every* $\lambda \in \Lambda$ *let* π^λ *be the Fock representation of* $\mathscr{N}$ *defined in* (2). *Then*

(i) *the stability subgroup of* π^λ *in L is*

$$L_{\pi^\lambda} = \{l_1 \in L_1 : \mathrm{Ad}(l_1^{-1})\lambda = \lambda\} \times L_2 ;$$

(ii) *for every* $\lambda \in L_{\pi^\lambda}$ *and* $F \in \Xi_\lambda(\mathscr{W})$ *we define*

$$[U_{\pi^\lambda}(l)F](W) := F(\mathrm{Ad}(l)W) \quad \forall W \in \mathscr{W} .$$

U_{π^λ} *is a (ordinary) representation of* L_{π^λ} *in* $\Xi_\lambda(\mathscr{W})$ *which intertwines* π^λ *and* π_l^λ, *i.e.*

$$\pi_l^\lambda(X, W) = \pi^\lambda(l(X, W)) = [U_{\pi^\lambda}(l)]^{-1}[\pi^\lambda(X, W)][U_{\pi^\lambda}(l)] \qquad \forall (X, W) \in \mathscr{N}, \ \forall l \in L_{\pi^\lambda} .$$

Theorem 4. $(L \ltimes \mathscr{N}, L)$ *is a Gelfand pair if and only if for every* λ *in* Λ, *each irreducible representation of* L_{π^λ} *occurs with multiplicity at most one in* U_{π^λ}.

Proof of Lemma 3. *Proof of* (i).

The real vector space $\mathscr{R} = \underline{n}_1^-$ is identified with its real dual $\mathscr{R}'$ by the real positive definite symmetric bilinear form B_ν ($B_\nu(U, V) := -B(U, \nu V)$ with $U, V \in \mathscr{W} = \underline{p}_2^-$, ν involution of $\underline{g}^c$ w.r.t. to $\underline{g}$ and $B(.,.)$ Killing form of $\underline{g}^c$ ([8], 6.1); we again denote with λ the element in $\mathscr{R}$ corresponding to the linear form $\lambda \in \Lambda \subset \mathscr{R}'$ and write $\langle X, \lambda\rangle := B_\nu(X, \lambda)$.

If $(\cdot)^*$ denotes the adjoint of a linear transformation $(\cdot)$ on $\underline{g}^c$ with respect to the hermitian form $B_\nu(.,.)$, then it is easy to see that $\mathrm{Ad}(l)^* = \mathrm{Ad}(l^{-1})$ for every $l \in L$. Since we have, for every $l_1 \in L_1$,

$$\begin{aligned}\pi_l^\lambda(X, 0) &= \pi^\lambda(\mathrm{Ad}(l_1)X, 0) = \exp\{-i\langle \mathrm{Ad}(l_1)X, \lambda\rangle\}\,\mathrm{Id} \\ &= \exp\{-i\langle X, (\mathrm{Ad}(l_1)^*)\lambda\rangle\}\,\mathrm{Id} = \exp\{-i\langle X, \mathrm{Ad}(l_1^{-1})\lambda\rangle\}\,\mathrm{Id},\end{aligned}$$

then, by Proposition 1, it follows that π_l^λ is equivalent to π^λ if and only if $\mathrm{Ad}(l_1^{-1})\lambda = \lambda$, and so (i) is proved.

Proof of (ii).

(a) For every $l \in L_{\pi^\lambda}$, the transformation $\mathrm{Ad}(l)$ of $\mathscr{W}$ commutes with the complex structure J_λ.

If $l \in L_{\pi^\lambda}$ then, for every $U, V \in \mathscr{W}$

$$\begin{aligned}\tfrac{1}{4}\mathscr{F}_\lambda(\mathrm{Ad}(l)U, \mathrm{Ad}(l)V) &= \langle \mathscr{F}(\mathrm{Ad}(l)U, \mathrm{Ad}(l)V), \lambda\rangle = \langle \mathrm{Ad}(l)\mathscr{F}(U, V), \lambda\rangle \\ &= \langle \mathrm{Ad}(l_1)\mathscr{F}(U, V), \lambda\rangle = \langle \mathscr{F}(U, V), \mathrm{Ad}(l_1^{-1})\lambda\rangle \\ &= \tfrac{1}{4}\mathscr{F}_\lambda(U, V)\end{aligned}$$

(the second equality is by [8], 6.4) and so $\mathrm{Ad}(l)$ commutes with the transformation of $\mathscr{W}$ defined w.r. to the ϵ-basis by the matrix $\mathscr{F}_\lambda(\epsilon) = \mathrm{diag}(\lambda_1, \dots, \lambda_m)$. Since the complex structure J_λ is defined w.r. to the ϵ-basis by the matrix $J_\lambda(\epsilon) = \mathrm{diag}(i\,\mathrm{sign}\,\lambda_1, \dots, i\,\mathrm{sign}\,\lambda_m)$, point (a) follows.

(b) $U \cdot \overline{V} = \mathrm{Ad}(l)U \cdot \overline{\mathrm{Ad}(l)V} \qquad \forall U, V \in \mathscr{W} \quad \forall F \in \Xi_\lambda(\mathscr{W})$.

Indeed, from (a)

$$\begin{aligned}(\mathrm{Ad}(l)U)\cdot\overline{(\mathrm{Ad}(l)V)} &= \mathrm{Im}\,\mathscr{F}_\lambda(J_\lambda \mathrm{Ad}(l)U, \mathrm{Ad}(l)V) + i\,\mathrm{Im}\,\mathscr{F}_\lambda(\mathrm{Ad}(l)U, \mathrm{Ad}(l)V) \\ &= \mathrm{Im}\,\mathscr{F}_\lambda(\mathrm{Ad}(l)J_\lambda U, \mathrm{Ad}(l)V) + i\,\mathrm{Im}\,\mathscr{F}_\lambda(\mathrm{Ad}(l)U, \mathrm{Ad}(l)V) \\ &= \mathrm{Im}\,\mathscr{F}_\lambda(J_\lambda U, V) + i\,\mathrm{Im}\,\mathscr{F}_\lambda(U, V) = U\cdot\overline{V}.\end{aligned}$$

(c) For every $l \in L_{\pi^\lambda}$ and $F \in \Xi_\lambda(\mathscr{W})$, $U_{\pi^\lambda}(l)F$ is again a holomorphic function on $\mathscr{W}_\lambda$ (by (a)). From (b) it follows that $U_{\pi^\lambda}(l)$ is an isometry onto $\Xi_\lambda(\mathscr{W})$; a trivial computation shows that U_{π^λ} is a representation of L_{π^λ} which intertwines π^λ and π_l^λ and so (ii) is proved.

Remark. We observe that the simple form obtained for the intertwining representation depends on the fact that we have chosen the "Fock form" of the representation.

Proof of Theorem 4. The proof follows by Theorem 1, Theorem 2 and Lemma 3.

3. The classical cases

As it is known (see [10]), all the symmetric irreducible generalized half-planes fall into four large classical classes but two exceptional domains. The non-tube cases are included in two classical families

$$(AIII : M = SU(p, p+q)/S(U(p)\times U(p+q))$$

and

$$DIII : M = SO^*(2n)/U(n)),$$

besides one of the exceptional domains ([4], p. 528). In this section we shall determine if $(L \ltimes \mathscr{N}, L)$ is a Gelfand pair (or not) when $\mathscr{N}$ is the Šilov boundary of a classical non-tube domain. The explicit realizations as Siegel domains, the formulas for the Cayley transform, the various involved spaces and the groups L which act on the Šilov boundaries, that we use below, can be obtained using the Korányi-Wolf theory, as shown in [3]. We have, from [3], Theorem 2, Theorem 4 and Remarks

Proposition 5.

(i) *Let* $M = SU(p, p+q)/S(U(p) \times U(p+q))$ $(p > 0, q \geq 0)$. *The realization of* M *as generalized half-plane is*

$$\mathscr{D}_{p,q} = \{(Z, W) \in M_p(\mathbf{C}) \times M_{q,p}(\mathbf{C}) : \operatorname{Im} Z - \tfrac{1}{2}W^*W \in H_p^+\}$$

where $M_{q,p}(\mathbf{C})$ *is the vector space of all* $q \times p$ *complex matrices;* $M_p(\mathbf{C}) = M_{p,p}(\mathbf{C})$; $H_p := \{Z \in M_p(\mathbf{C}) : Z = Z^*\}$ (*real form of* $M_p(\mathbf{C})$) *and* $H_p^+ := \{Z \in H_p : Z$ *is positive definite*$\}$.

The group L *is isomorphic to* $SU(p) \times U(q)$ *and its action on the Šilov boundary* $\mathscr{N}_{p,q}$ *of* $\mathscr{D}_{p,q}$ *is given by* $l(X, W) = (gXg^{-1}, hWg^{-1})$ *where* $l = (g, h) \in SU(p) \times U(q)$ *and* $(X, W) \in \mathscr{N}_{p,q} = H_p \times M_{q,p}(\mathbf{C})$.

(ii) *Let* $M = SO^*(2n)/U(n)$ $(n \geq 2)$.

If $n = 2s+1$ $(s > 1)$, *then the realization of* M *as generalized half-plane is*

$$\mathscr{D}_{2s+1} = \{(Z, W) \in M'_{2s}(\mathbf{C}) \times M_{2s,1}(\mathbf{C}) : \operatorname{Im} Z - \mathscr{F}(W, W) \in (H'_{2s})^+\}$$

where

$$M'_{2s}(\mathbf{C}) := \left\{Z \in M_{2s}(\mathbf{C}) : Z^tJ = JZ, \quad J = \begin{pmatrix} 0 & I_s \\ -I_s & 0 \end{pmatrix}\right\},$$

$$H'_{2s} := \{Z \in M'_{2s}(\mathbf{C}) : Z = Z^*\} \text{ real form of } M'_{2s}(\mathbf{C}),$$

$$(H'_{2s})^+ := \{Z \in H'_{2s} : Z \text{ is positive definite}\},$$

$$\mathscr{F} : M_{2s,1}(\mathbf{C}) \times M_{2s,1}(\mathbf{C}) \to M'_{2s}(\mathbf{C}), \qquad \mathscr{F}(W_1, W_2) = \tfrac{1}{2}(W_1W_2^* - J\overline{W_2}W_1^tJ).$$

The group L is isomorphic to $\mathrm{Sp}(s) \times \mathbf{T}$ *and its action on the Šilov boundary* $\mathscr{N}_{2s+1}$ *of* $\mathscr{D}_{2s+1}$ *is given by* $l(X, W) = (uXu^{-1}, e^{i\theta}uW)$ *where* $l = (u, e^{i\theta}) \in \mathrm{Sp}(s) \times \mathbf{T}$ $(\mathrm{Sp}(s) = \{u \in U(2s) : u^tJu = J\})$ *and* $(X, W) \in \mathscr{N}_{2s+1} = H'_{2s} \times M_{2s,1}(\mathbf{C})$.

If $n = 3$ *then* $\mathscr{D}_3 \simeq \mathscr{D}_{1,2}$.
If $n = 2s$ $(s > 1)$ *then* $\mathscr{D}_{2s} = \{Z \in M'_{2s}(\mathbf{C}) : \operatorname{Im} Z \in (H'_{2s})^+\}$.
If $n = 2$ *then* $\mathscr{D}_2 \simeq \mathscr{D}_{1,0}$.

Therefore, we have to consider the domains $\mathscr{D}_{p,q}$ $(q > 0)$ and $\mathscr{D}_{2s+1}$ $(s > 0)$. Our result is

Theorem 6.

(i) *Let* $\mathscr{N}_{p,q} = H_p \times M_{q,p}(\mathbf{C})$ *and* $L = SU(p) \times U(q)$. *Then* $(L \ltimes \mathscr{N}, L)$ *is a Gelfand pair if and only if* $q \leq 1$ *or* $p \leq 2$.

(ii) *Let* $\mathscr{N}_{2s+1} = H'_{2s} \times M_{2s,1}(\mathbf{C})$ *and* $L = \mathrm{Sp}(s) \times \mathbf{T}$. *Then* $(L \ltimes \mathscr{N}, L)$ *is a Gelfand pair for every* $s \geq 1$.

Proof of (i). We proved (i) if $p \leq q$ in [2]. The proof in the general case $p > 0$, $q \geq 0$ is similarly carried out and we omit it.

Remark. If $p = 1$, then we have the *Heisenberg group* $\mathbf{H}_q = \mathbf{R} \times \mathbf{C}^q$ and the commutative algebra of *radial* functions on $\mathbf{H}_q$; therefore, we recover the results of ([5] and [7], case $G = SU(1, n+1)$; see the introduction).

Proof of (ii). (We can suppose $s > 1$.)

For every $\lambda \in H'_{2s}$ we have

$$\begin{aligned}\mathscr{F}_\lambda(W_1, W_2) &= 4\langle \mathscr{F}(W_1, W_2), \lambda\rangle = 4B_\nu(\mathscr{F}(W_1, W_2), \lambda)\\ &= 4(2n-2)\operatorname{Tr}(\mathscr{F}(W_1, W_2)\lambda^*)\\ &= (4n-4)\operatorname{Tr}[(W_1W_2^* - J\overline{W_2}W_1^tJ)\lambda] = 8(n-1)\operatorname{Tr}(W_1W_2^*\lambda)\end{aligned}$$

since

$$\begin{aligned}\operatorname{Tr}(J\overline{W_2}W_1^tJ\lambda) &= \operatorname{Tr}(\overline{W_2}W_1^tJ\lambda J) = -\operatorname{Tr}(\overline{W_2}W_1^t\lambda^t)\\ &= -\operatorname{Tr}(\lambda W_1W_2^*) = -\operatorname{Tr}(W_1W_2^*\lambda)\end{aligned}$$

and therefore

$$\Lambda = \{\lambda \in H'_{2s} : \det(\lambda) \neq 0\}. \tag{4}$$

We need the following:

Lemma 7. *If the intertwining representation* U_{π^λ} *is decomposed without multiplicities greater than* 1 *for every*

$$\lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_s, \lambda_1, \dots, \lambda_s) \in \Lambda, \qquad \lambda_i \neq \lambda_j \text{ if } i \neq j \tag{5}$$

then $(L \ltimes \mathscr{N}_{2s+1}, L)$ *is a Gelfand pair.*

Proof of the lemma. Let $\lambda \in \Lambda$; if $\underline{v} := (x_1, \dots, x_s, y_1, \dots, y_s)^t$ is an eigenvector of λ, then $\underline{\tilde{v}} := (-\overline{y_1}, \dots, -\overline{y_s}, \overline{x_1}, \dots, \overline{x_s})^t$ is also an eigenvector of the same eigenvalue, orthogonal to $\underline{v}$. Let $\lambda_1, \dots, \lambda_s, \lambda_1, \dots, \lambda_s$ be the $2s$ eigenvalues of λ and $\underline{v}_k$ ($1 \leq k \leq s$) eigenvector of λ_k, with $\underline{v}_k$ orthogonal to $\underline{v}_j$. The matrix u, whose columns are $\underline{v}_1, \dots, \underline{v}_s, \underline{\tilde{v}}_1, \dots, \underline{\tilde{v}}_s$, is an element of $\operatorname{Sp}(s)$ s.t. $\mu = u^{-1}\lambda u = \operatorname{diag}(\lambda_1, \dots, \lambda_s, \lambda_1, \dots, \lambda_s)$ and so μ and λ are in the same L-orbit.

The proof of the lemma follows by the sufficient condition of the Theorem 2, taking into account that, leaving out a set of Plancherel measure zero, we can suppose $\lambda_i \neq \lambda_j$ if $i \neq j$.

Coming to the proof of (ii), let λ be as in (5) and π^λ the Fock representation of $\mathscr{N}_{2s+1}$ associated to λ. From Lemma 3 (i) we have

$$L_{\pi^\lambda} = \{u \in \operatorname{Sp}(s) : u^{-1}\lambda u = \lambda\} \times \mathbf{T}.$$

We prove that $u \in \mathrm{Sp}(s)$ commutes with λ if and only if u is of the form

$$(6) \qquad u = \begin{pmatrix} \alpha_1 & & & \beta_1 & & \\ & \ddots & & & \ddots & \\ & & \alpha_s & & & \beta_s \\ -\overline{\beta}_1 & & & \overline{\alpha}_1 & & \\ & \ddots & & & \ddots & \\ & & -\overline{\beta}_s & & & \overline{\alpha}_s \end{pmatrix}$$

where

(7) α_j and β_j are complex numbers such that $|\alpha_j|^2 + |\beta_j|^2 = 1$.

The "if part" is trivial. The "only if " part: if $u\lambda = \lambda u$ then u preserves the eigenspace of the eigenvalue λ_j (for every j) and so all the elements of u must be zero except those shown in (6); the condition $u \in \mathrm{Sp}(s)$ implies (7).

Therefore

$$L_{\pi^\lambda} \simeq (SU(2))^s \times \mathbf{T}$$

where $(SU(2))^s$ denotes the direct product of s copies of $SU(2)$.

The Fock representation π^λ is realized on the space $\Xi_\lambda(M_{2s,1}(\mathbf{C})) = \Xi_\lambda(\mathbf{C}^{2s})$ of the $\mathscr{C}^\infty$- functions F of $2s$ complex variables $\xi_1, \dots, \xi_s, \eta_1, \dots, \eta_s$ which are holomorphic (resp. antiholomorphic) w.r. to ξ_j and η_j if $\lambda_j > 0$ (resp. $\lambda_j < 0$) and square integrable with respect to the weight

$$\exp\left\{-2\sum_{j=1}^{s} |\lambda_j| (|\xi_j|^2 + |\eta_j|^2)\right\}.$$

The intertwining representation U_{π^λ} of $L_{\pi^\lambda} = (SU(2))^s \times \mathbf{T}$ on $\Xi_\lambda(\mathbf{C}^{2s})$ is, by Lemma 3 (ii):

$$(8) \qquad [U_{\pi^\lambda}(u, e^{i\theta})F](W) = F(e^{i\theta}uW)$$

($(u, e^{i\theta}) \in (SU(2))^s \times \mathbf{T}$, $F \in \Xi_\lambda(\mathbf{C}^{2s})$ *and* $W = (\xi_1, \dots, \xi_s, \eta_1, \dots, \eta_s)^t \in \mathbf{C}^{2s}$). We can suppose $\lambda_k > 0$ if $k \le r$ and $\lambda_k < 0$ if $k > r$.

For each multiindex $\underline{m} = (m_1, \dots, m_s)$, m_j nonnegative integers, let $\mathscr{P}_{\underline{m}}$ be the space of all polynomials in $\xi_1, \dots, \xi_r, \overline{\xi}_{r+1}, \dots, \overline{\xi}_s, \eta_1, \dots, \eta_r, \overline{\eta}_{r+1}, \dots, \overline{\eta}_s$ which are homogeneous of degree m_k w.r. to ξ_k, η_k for $k \le r$ and w.r. to $\overline{\xi}_k$, $\overline{\eta}_k$ for $k > r$; every $\mathscr{P}_{\underline{m}}$ is U_{π^λ}-invariant. In order to compute the character $\chi_{\underline{m}}$ of the representation U_{π^λ} on $\mathscr{P}_{\underline{m}}$, we consider the orthogonal basis for $\mathscr{P}_{\underline{m}}$ given by the elements

$$(9)\ P_{\underline{j}} = P_{(j_1, \dots, j_s)}(W) = \xi_1^{j_1} \cdots \xi_r^{j_r} \overline{\xi}_{r+1}^{j_{r+1}} \cdots \overline{\xi}_s^{j_s} \eta_1^{m_1 - j_1} \cdots \eta_r^{m_r - j_r} \overline{\eta}_{r+1}^{m_{r+1} - j_{r+1}} \cdots \overline{\eta}_s^{m_s - j_s}$$

$$(0 \le j_h \le m_h \quad 1 \le h \le s).$$

If $u = \mathrm{diag}(e^{i\phi_1}, \dots, e^{i\phi_s}, e^{-i\phi_1}, \dots, e^{-i\phi_s})$ then, by (8) and (9)

$$(10) \qquad [U_{\pi^\lambda}(u, e^{i\theta})P_{\underline{j}}](W) = \exp\left\{ i\theta(m_1 + \dots + m_r - m_{r+1} - \dots - m_s) + i\sum_{h=1}^{r}[\phi_h(2j_h - m_h)]i\sum_{h=r+1}^{s}[\phi_h(2j_h - m_h)] \right\} P_{\underline{j}}(W).$$

We set $\tilde{m} = m_1 + \cdots + m_r - m_{r+1} - \cdots - m_s$; by (10)

$$\begin{aligned}
\chi_{\underline{m}}(u, e^{i\theta}) &= \mathrm{Tr}[U_{\pi^\lambda}(u, e^{i\theta})] \\
&= e^{i\tilde{m}\theta} \sum_{\underline{j}\,:\,0\le j_h\le m_h} \exp\left\{ i\sum_{h=1}^{r}[\phi_h(2j_h - m_h)] - i\sum_{h=r+1}^{s}[\phi_h(2j_h - m_h)] \right\} \\
&= e^{i\tilde{m}\theta} \prod_{h=1}^{r}\left[\sum_{j_h=0}^{m_h} \exp(i\phi_h(2j_h - m_h))\right] \prod_{h=r+1}^{s}\left[\sum_{j_h=0}^{m_h} \exp(i\phi_h(m_h - 2j_h))\right] \\
&= e^{i\tilde{m}\theta} \prod_{h=1}^{s} \chi_{m_h}(\phi_h)
\end{aligned}$$

where $\chi_n(\phi) = \sum_{k=0}^{n} e^{i\phi(2k-n)}$ is the character of the irreducible $(n+1)$-dimensional representation $d(n)$ of $SU(2)$. Therefore, the representation U_{π^λ} of $(SU(2))^s \times \mathbf{T}$ on $\mathscr{P}_{\underline{m}}$ is equivalent to the outer Kronecker product

$$d(m_1) \otimes \cdots \otimes d(m_s) \otimes e^{i\tilde{m}(\cdot)}$$

and so, if $\underline{m} \neq \underline{m}'$, U_{π^λ} on $\mathscr{P}_{\underline{m}}$ and U_{π^λ} on $\mathscr{P}_{\underline{m}'}$ are not equivalent; since $\bigoplus \mathscr{P}_{\underline{m}}$ is dense in $\Xi_\lambda(\mathbf{C}^{2s})$, we have proved that the representation U_{π^λ} of L_{π^λ} is decomposed without multiplicities greater than one; this concludes the proof, by Lemma 7.

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