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COMPACTNESS IN L_1, DUNFORD-PETTIS OPERATORS, GEOMETRY OF BANACH SPACES

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ABSTRACT. A type of oscillation modeled on BMO is introduced to characterize norm compactness in L_1. This result is used to characterize the bounded linear operators from L_1 into a Banach space $\mathfrak{X}$ that map weakly convergent sequences onto norm convergent sequences (i.e., are Dunford-Pettis). This characterization is used to study the geometry of Banach spaces $\mathfrak{X}$ with the property that all bounded linear operators from L_1 into $\mathfrak{X}$ are Dunford-Pettis.

1. INTRODUCTION

The main result of this paper is a characterization of L_1-norm compactness that is based on a BMO-style oscillation. From this result we obtain a characterization of the bounded linear operators from L_1 into a Banach space $\mathfrak{X}$ that map weakly convergent sequences onto norm convergent sequences (i.e., are Dunford-Pettis). With such characterizations in hand, we study the geometry of Banach spaces $\mathfrak{X}$ with the property that all bounded linear operators from L_1 into $\mathfrak{X}$ are Dunford-Pettis.

One way to insure that a relatively weakly compact set K in L_1 is relatively norm compact is to be able to find, for each $\varepsilon > 0$, a finite measurable partition π of $[0, 1]$ such that for each f in K and each A in π,

$$\operatorname{osc} f\big|_A \equiv \operatorname*{ess\,sup}_{\omega\in A} f(\omega) - \operatorname*{ess\,inf}_{\omega\in A} f(\omega) < \varepsilon.$$

This condition is too strong to characterize L_1-norm compactness.

Bourgain showed that the Bourgain property, a weakening of the above condition, also guarantees relative norm compactness of relatively weakly compact subsets in L_1. A subset K of L_1 has the *Bourgain property* if *for each* $\varepsilon > 0$ *and subset* B *of* $[0, 1]$ *with positive measure there is a finite collection* $\mathcal{F}$ *of subsets of* B*, each with positive measure, such that for each* $f \in K$ *there is an* A *in* $\mathcal{F}$ *with* $\operatorname{osc} f\big|_A < \varepsilon$. However, this condition also is too strong to characterize L_1-norm compactness.

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We introduce an L_1-oscillation that is a weakening of the L_∞-oscillation used in the above conditions. The *Bocce oscillation* of an L_1-function f on a subset A is given by

$$\text{Bocce-osc } f|_A \equiv \frac{\int_A |f - \frac{\int_A f\,d\mu}{\mu(A)}|\,d\mu}{\mu(A)},$$

observing the convention that $0/0$ is 0. A subset K of L_1 satisfies the *Bocce criterion* if *for each* $\varepsilon > 0$ *and subset* B *of* $[0, 1]$ *with positive measure there is a finite collection* $\mathcal{F}$ *of subsets of* B, *each with positive measure, such that for each* $f \in K$ *there is an* A *in* $\mathcal{F}$ *with Bocce-osc* $f|_A < \varepsilon$. The Bocce criterion is a weakening of the Bourgain property. The main result of this paper is that a relatively weakly compact subset of L_1 is relatively L_1-norm compact if and only if it satisfies the Bocce criterion.

Throughout this paper, $\mathfrak{X}$ denotes an arbitrary Banach space, $\mathfrak{X}^*$ the dual space of $\mathfrak{X}$, $B(\mathfrak{X})$ the closed unit ball of $\mathfrak{X}$, and $S(\mathfrak{X})$ the unit sphere of $\mathfrak{X}$. The triple (Ω, Σ, μ) refers to the Lebesgue measure space on $[0, 1]$, Σ^+ to the sets in Σ with positive measure, L_1 to $L_1(\Omega, \Sigma, \mu)$. Recall that a subset of L_1 is relatively weakly compact if and only if it is bounded and uniformly integrable. All notation and terminology, not otherwise explained, are as in [DU].

2. Main result

Call a subset K of L_1 a *set of small Bocce oscillation* if *for each* $\varepsilon > 0$ *there is a finite positive measurable partition* π *of* Ω *such that for each* f *in* K

$$\sum_{A \in \pi} \mu(A) \textit{ Bocce-osc } f|_A < \varepsilon .$$

Bocce oscillation, Bocce criterion, and L_1-norm compactness are related by Theorem 2.1, the main theorem of this paper.

Theorem 2.1. *For a relatively weakly compact subset* K *of* L_1, *the following statements are equivalent.*

(1) K *is relatively norm compact.*
(2) K *is a set of small Bocce oscillation.*
(3) K *satisfies the Bocce criterion.*

Proof. It is well-known and easy to check that a bounded subset K of L_1 is relatively norm compact if and only if for each $\varepsilon > 0$ there is a finite measurable partition π of Ω such that $\| f - E_\pi(f) \|_{L_1} < \varepsilon$ for each f in K. Here, $E_\pi(f)$ denotes the conditional expectation of f relative to the sigma field generated by π. The equivalence of (1) and (2) follows directly from this observation, the computation below,

$$\begin{aligned}\| f - E_\pi(f) \|_{L_1} &= \int_\Omega \left| f - \sum_{B\in\pi} \frac{\int_B f\,d\mu}{\mu(B)} \chi_B \right| d\mu \\ &= \sum_{B\in\pi} \int_B \left| f - \frac{\int_B f\,d\mu}{\mu(B)} \right| d\mu \quad = \quad \sum_{B\in\pi} \mu(B) \text{ Bocce-osc } f|_B ,\end{aligned}$$

and the definition of a set of small Bocce oscillation.

Viewed as a function from Σ into the real numbers, Bocce-osc $f|_{(\cdot)}$ is not increasing (see Example 2.3); however, $\mu(\cdot)$ Bocce-osc $f|_{(\cdot)}$ is increasing (see Remark 2.4). With this in mind, we now show that (2) implies (3) in Theorem 2.1.

Let the subset K of L_1 be a set of small Bocce oscillation. Fix $\varepsilon > 0$ and B in Σ^+. Since K is a set of small Bocce oscillation, there is a positive measurable partition $\pi = \{A_1, A_2, \dots, A_n\}$ of Ω such that for each f in K

$$\sum_{i=1}^{n} \mu(A_i)(\text{ Bocce-osc } f|_{A_i}) < \varepsilon\mu(B).$$

Set $\mathscr{F} = \{A_i \cap B : A_i \in \pi \quad\text{and}\quad \mu(A_i \cap B) > 0\}$.

Fix f in K. Since the function $\mu(\cdot)$ Bocce-osc $f|_{(\cdot)}$ is increasing, if Bocce-osc $f|_{A_i\cap B} \geq \varepsilon$ for each set $A_i \cap B$ in $\mathscr{F}$ then we would have that

$$\begin{aligned}\varepsilon \cdot \mu(B) = \sum_{i=1}^{n} \varepsilon \cdot \mu(A_i \cap B) &\leq \sum_{i=1}^{n} \mu(A_i \cap B) \text{ Bocce-osc } f|_{A_i\cap B} \\ &\leq \sum_{i=1}^{n} \mu(A_i) \text{ Bocce-osc } f|_{A_i} < \varepsilon \cdot \mu(B).\end{aligned}$$

This cannot be, so there is a set $A_i \cap B$ in $\mathscr{F}$ such that the Bocce-osc $f|_{A_i\cap B} < \varepsilon$. Thus the set K satisfies the Bocce criterion.

We need the following lemma which we will prove after the proof of Theorem 2.1. This lemma is the key step in the proof that (3) implies (1).

Lemma 2.2. *Let the relatively weakly compact subset K of L_1 satisfy the Bocce criterion. If $\varepsilon > 0$ and $\{f_n\}$ is a sequence in K, then there are disjoint sets $B_1, \dots, B_p$ in Σ^+ and a subsequence $\{g_n\}$ of $\{f_n\}$ satisfying*

$$(**) \qquad \int_\Omega \left| g_n - \sum_{j=1}^{p} \frac{\int_{B_j} g_n\,d\mu}{\mu(B_j)} \chi_{B_j} \right| d\mu \leq 2\varepsilon.$$

We now proceed with showing that (3) implies (1) in Theorem 2.1. Let the relatively weakly compact subset K of L_1 satisfy the Bocce criterion. Choose a sequence $\{f_n\}$ in K and a sequence $\{\varepsilon_k\}$ of positive real numbers decreasing to zero.

It suffices to find a sequence $\{\pi_k\}$ of finite measurable partitions of Ω and a nested sequence $\{\{f_n^k\}_n\}_k$ of subsequences of $\{f_n\}$ such that for all positive integers n and k

$$(*) \qquad \| f_n^k - E_k(f_n^k) \|_{L_1} \leq 2\varepsilon_k ,$$

where $E_k(f)$ denotes the conditional expectation of f relative to the σ-field generated by π_k. For then the set $\{f_n^n\}$ is relatively L_1-norm compact since it can be uniformly approximated in the L_1-norm within $2\varepsilon_k$ by the relatively compact set $\{E_k(f_n^n)\}_{n\geq k} \cup \{f_n^n\}_{n<k}$; hence, there is a L_1-norm convergent subsequence of $\{f_n\}$.

Towards $(*)$, repeated applications of Lemma 2.2 yield for each positive integer k:

(1) disjoint subsets $B_1^k, \dots, B_{n_k}^k$ of Ω each with positive measure, and

(2) a subsequence $\{f_n^k\}_n$ of $\{f_n^{k-1}\}_n$ (where we write $\{f_n^0\}$ for $\{f_n\}$)

satisfying for each positive integer n

$$\int_\Omega \left| f_n^k - \sum_{j=1}^{n_k} \frac{\int_{B_j^k} f_n^k \, d\mu}{\mu(B_j^k)} \chi_{B_j^k} \right| d\mu \leq \varepsilon_k .$$

Set

$$B_0^k = \Omega \Big\backslash \bigcup_{j=1}^{n_k} B_j^k$$

and let π_k be the partition generated by $B_0^k, B_1^k, \dots, B_{n_k}^k$. Still observing the convention that $0/0$ is 0, two appeals to the above inequality yield for each positive integer n and k

$$\begin{aligned}
\| f_n^k - E_k(f_n^k) \|_{L_1} &= \int_\Omega \left| f_n^k - \sum_{j=0}^{n_k} \frac{\int_{B_j^k} f_n^k \, d\mu}{\mu(B_j^k)} \chi_{B_j^k} \right| d\mu \\
&\leq \int_\Omega \left| f_n^k - \sum_{j=1}^{n_k} \frac{\int_{B_j^k} f_n^k \, d\mu}{\mu(B_j^k)} \chi_{B_j^k} \right| d\mu + \int_\Omega \left| \frac{\int_{B_0^k} f_n^k \, d\mu}{\mu(B_0^k)} \chi_{B_0^k} \right| d\mu \\
&\leq \varepsilon_k + \int_{B_0^k} | f_n^k | \, d\mu \\
&= \varepsilon_k + \int_{B_0^k} \left| f_n^k - \sum_{j=1}^{n_k} \frac{\int_{B_j^k} f_n^k \, d\mu}{\mu(B_j^k)} \chi_{B_j^k} \right| d\mu \\
&\leq \varepsilon_k + \int_\Omega \left| f_n^k - \sum_{j=1}^{n_k} \frac{\int_{B_j^k} f_n^k \, d\mu}{\mu(B_j^k)} \chi_{B_j^k} \right| d\mu \\
&\leq 2\varepsilon_k .
\end{aligned}$$

Thus the proof of Theorem 2.1 is finished as soon as we verify Lemma 2.2.

Proof of Lemma 2.2. Let the relatively weakly compact subset K of L_1 satisfy the Bocce criterion. Fix $\varepsilon > 0$ and a sequence $\{f_n\}$ in K. The proof is an exhaustion-type argument.

Let $\mathscr{E}_1$ denote the collection of subsets B of Ω such that there are disjoint subsets $B_1, \ldots, B_p$ of B each with positive measure and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ satisfying

$$\int_B \left| f_{n_k} - \sum_{j=1}^{p} \frac{\int_{B_j} f_{n_k}\, d\mu}{\mu(B_j)} \chi_{B_j} \right| d\mu \leq \varepsilon\mu(B).$$

Since K satisfies the Bocce criterion, there is an A in Σ^+ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ satisfying

$$\int_A \left| f_{n_k} - \frac{\int_A f_{n_k}\, d\mu}{\mu(A)} \right| d\mu \leq \varepsilon\mu(A).$$

Thus the collection $\mathscr{E}_1$ is not empty. If there is a set B in $\mathscr{E}_1$ with corresponding subsets $B_1, \ldots, B_p$ and a subsequence $\{g_n\}$ of $\{f_n\}$ satisfying

$$\text{(1)} \qquad \int_\Omega \left| g_n - \sum_{j=1}^{p} \frac{\int_{B_j} g_n\, d\mu}{\mu(B_j)} \chi_{B_j} \right| d\mu \leq \varepsilon,$$

then we are finished since (1) implies (**).

Otherwise, let j_1 be the smallest positive integer for which there is a C_1 in $\mathscr{E}_1$ with $\frac{1}{j_1} \leq \mu(C_1)$. Accordingly, there is a finite sequence $\{C_j^1\}$ of disjoint subsets of C_1 each with positive measure and a subsequence $\{f_n^1\}$ of $\{f_n\}$ satisfying

$$\text{(2)} \qquad \int_{C_1} \left| f_n^1 - \sum_j \frac{\int_{C_j^1} f_n^1\, d\mu}{\mu(C_j^1)} \chi_{C_j^1} \right| d\mu \leq \varepsilon\mu(C_1).$$

Note that since condition (1) was not satisfied, $\mu(C_1) < \mu(\Omega) = 1$.

Let $\mathscr{E}_2$ denote the collection of subsets B of $\Omega \setminus C_1$ such that there are disjoint subsets $B_1, \ldots, B_p$ of B each with positive measure and a subsequence $\{f_{n_k}\}$ of $\{f_n^1\}$ satisfying

$$\int_B \left| f_{n_k} - \sum_{j=1}^{p} \frac{\int_{B_j} f_{n_k}\, d\mu}{\mu(B_j)} \chi_{B_j} \right| d\mu \leq \varepsilon\mu(B).$$

Since K satisfies the Bocce criterion and $\mu(\Omega \setminus C_1) > 0$, we see that $\mathscr{E}_2$ is not empty. If there is a set B in $\mathscr{E}_2$ with corresponding finite sequence $\{C_j^2\}$ of subsets of B and a subsequence $\{g_n\}$ of $\{f_n^1\}$ satisfying

$$\text{(3)} \qquad \int_{\Omega\setminus C_1} \left| g_n - \sum_j \frac{\int_{C_j^2} g_n\, d\mu}{\mu(C_j^2)} \chi_{C_j^2} \right| d\mu \leq \varepsilon\mu(\Omega \setminus C_1),$$

then we are finished. For in this case, $(**)$ holds since inequalities (2) and (3) insure that

$$\int_\Omega \left| g_n - \sum_{\substack{k=1,2\\ j}} \frac{\int_{C_j^k} g_n\, d\mu}{\mu(C_j^k)} \chi_{C_j^k} \right| d\mu \le \varepsilon\mu(C_1) + \varepsilon\mu(\Omega \setminus C_1) = \varepsilon .$$

Otherwise, let j_2 be the smallest positive integer for which there is a C_2 in $\mathscr{E}_2$ with $\frac{1}{j_2} \le \mu(C_2)$. Accordingly, there is a finite sequence $\{C_j^2\}$ of disjoint subsets of C_2 each with positive measure and a subsequence $\{f_n^2\}$ of $\{f_n^1\}$ satisfying

$$\int_{C_2} \left| f_n^2 - \sum_j \frac{\int_{C_j^2} f_n^2\, d\mu}{\mu(C_j^2)} \chi_{C_j^2} \right| d\mu \le \varepsilon\mu(C_2).$$

Note that since condition (3) was not satisfied, $\mu(C_2) < \mu(\Omega \setminus C_1)$.

Continue in this way. If the process stops in a finite number of steps then we are finished. If the process does not stop, then diagonalize the resultant sequence $\{\{f_n^k\}_{n=1}^\infty\}_{k=1}^\infty$ of sequences to obtain the sequence $\{f_n^n\}_{n=1}^\infty$ and set $C_\infty = \bigcup_{k=1}^\infty C_k$.

Note that $\mu(\Omega \setminus C_\infty) = 0$. For if $\mu(\Omega \setminus C_\infty) > 0$ then, since K satisfies the Bocce criterion, there is a subset B of $\Omega \setminus C_\infty$ with positive measure and a subsequence $\{h_n\}$ of $\{f_n^n\}$ satisfying

$$\int_B \left| h_n - \frac{\int_B h_n\, d\mu}{\mu(B)} \right| d\mu < \varepsilon\mu(B).$$

Since for each positive integer m

$$\sum_{k=1}^m \frac{1}{j_k} \le \mu\left(\bigcup_{k=1}^m C_k \right) \le \mu(\Omega)$$

and $j_m > 1$, we can choose an integer $m > 1$ such that

$$\frac{1}{j_m - 1} \le \mu(B).$$

But this implies that B is in $\mathscr{E}_m$ since

$$B \subset \Omega \setminus C_\infty \subset \Omega \setminus \bigcup_{k=1}^{m-1} C_k$$

and $\{h_n\}_{n=m-1}^\infty$ is a subsequence of $\{f_n^n\}_{n=m-1}^\infty$, which in turn is a subsequence of $\{f_n^{m-1}\}_{n=1}^\infty$. This contradicts the choice of j_m. Thus $\mu(\Omega \setminus C_\infty) = 0$.

Since K is relatively weakly compact, it is uniformly integrable. Thus, there exists $\delta > 0$ such that if $\mu(A) < \delta$ then $\int_A |f|\, d\mu < \varepsilon$ for each $f \in K$. Pick an integer m so that

$$\mu\left(\Omega \setminus \bigcup_{k=1}^m C_k \right) < \delta .$$

Since $\{f_n^n\}_{n=m}^{\infty}$ is a subsequence of $\{f_n^m\}_{n=1}^{\infty}$, for each f in $\{f_n^n\}_{n=m}^{\infty}$ we have

$$\begin{aligned}\int_{\Omega}\left|f-\sum_{\substack{1\le k\le m\\ j}}\frac{\int_{C_j^k} f\,d\mu}{\mu(C_j^k)}\chi_{C_j^k}\right|d\mu &= \sum_{k=1}^{m}\int_{C_k}\left|f-\sum_{j}\frac{\int_{C_j^k} f\,d\mu}{\mu(C_j^k)}\chi_{C_j^k}\right|d\mu+\int_{\Omega\setminus\bigcup_{k=1}^m C_k}|f|\,d\mu\\ &\le \sum_{k=1}^{m}\varepsilon\mu(C_k)+\varepsilon\le 2\varepsilon.\end{aligned}$$

So the disjoint subsets $\{C_j^k : j\ge 1 \text{ and } k=1,\dots,m\}$ along with the subsequence $\{f_n^n\}_{n=m}^{\infty}$ of $\{f_n\}$ satisfy the conditions of the lemma.

This completes the proof of Theorem 2.1.

We close this section with a few observations about Bocce oscillation.

Example 2.3. Let $A=[0,1/4]$ and $B=[0,1]$. Define the function f from [0,1] into the real numbers by $f(t)=\chi_C(t)$ where $C=[1/8,1]$. It is straightforward to verify that Bocce-osc $f|_A=1/2$ but Bocce-osc $f|_B=7/32$.

Remark 2.4. In the proof of Theorem 2.1, we used the fact that the function

$$\mu(\cdot)\text{ Bocce-osc } f|_{(\cdot)}\equiv\int_{(\cdot)}\left|f-\frac{\int_{(\cdot)} f\,d\mu}{\mu(\cdot)}\right|d\mu$$

is an increasing function from Σ into the real numbers. To see this, let A be a subset of B with A and B in Σ^+. Let

$$m_A=\frac{\int_A f\,d\mu}{\mu(A)}\quad\text{and}\quad m_B=\frac{\int_B f\,d\mu}{\mu(B)}.$$

Note that

$$\begin{aligned}\int_A|f-m_A|\,d\mu &\le \int_A|f-m_B|\,d\mu+\int_A|m_B-m_A|\,d\mu\\ &=\int_B|f-m_B|\,d\mu-\int_{B\setminus A}|f-m_B|\,d\mu+\int_A|m_B-m_A|\,d\mu,\end{aligned}$$

and

$$\begin{aligned}\int_A | m_B - m_A | \, d\mu &= \mu(A) \, | m_B - m_A | \\ &= \left| \frac{\mu(A)}{\mu(B)} \int_B f \, d\mu - \int_A f \, d\mu \right| \\ &= \left| \frac{\mu(A)}{\mu(B)} \int_B f \, d\mu - \left(\int_B f \, d\mu - \int_{B\setminus A} f \, d\mu \right) \right| \\ &= \left| \int_{B\setminus A} f \, d\mu - \frac{\mu(B) - \mu(A)}{\mu(B)} \int_B f \, d\mu \right| \\ &= \left| \int_{B\setminus A} (f - m_B) \, d\mu \right| .\end{aligned}$$

Thus, as needed,

$$\int_A | f - m_A | \, d\mu \leq \int_B | f - m_B | \, d\mu .$$

Remark 2.5. The Bocce oscillation has been implicitly studied. By definition, a function f in L_1 is of BMO (bounded mean oscillation) provided $\sup_I$ Bocce-osc $f|_I < \infty$ where the sup is over all intervals I contained in $[0, 1]$.

3. Applications

Fix a bounded linear operator T from L_1 into $\mathfrak{X}$ and consider the uniformly bounded subset $T^*(B(\mathfrak{X}^*)) = \{T^*(x^*) : \| x^* \| \leq 1\}$ of L_1. The oscillation behavior of elements in $T^*(B(\mathfrak{X}^*))$ provides information about T.

Ghoussoub, Godefroy, Maurey, and Schachermayer [GGMS] showed that the operator T is strongly regular if and only if the subset $T^*(B(\mathfrak{X}^*))$ of L_1 has the Bourgain property. Recall that an operator is called Dunford-Pettis if it maps weakly convergent sequences onto norm convergent sequences. With Theorem 2.1, we obtain an analogous characterization of Dunford-Pettis operators. From the oscillation characterizations, it is easy to see that a strongly regular operator is Dunford-Pettis.

Corollary 3.1. *A bounded linear operator T from L_1 into a Banach space $\mathfrak{X}$ is Dunford-Pettis if and only if the subset $T^*(B(\mathfrak{X}^*))$ of L_1 satisfies the Bocce criterion.*

Corollary 3.1 follows directly from Theorem 2.1 and the fact below.

Fact 3.2. The following statements are equivalent.

(1) T is a Dunford-Pettis operator.
(2) T maps weak compact sets to norm compact sets.
(3) $T(B(L_\infty))$ is a relatively norm compact subset of $\mathfrak{X}$.
(4) The adjoint of the restriction of T to L_∞ from $\mathfrak{X}^*$ into L_∞^* is a compact operator.
(5) As a subset of L_1, $T^*(B(\mathfrak{X}^*))$ is relatively norm compact.

The equivalence of (2) and (3) follows from the fact that the subsets of L_1 that are relatively weakly compact are precisely those subsets that are bounded and uniformly integrable, which in turn, are precisely those subsets that can be uniformly approximated in L_1-norm by uniformly bounded subsets. The other implications are easy to verify [cf. DU].

Ghoussoub, Godefroy, Maurey, and Schachermayer also obtained an internal geometric description of the class of Banach spaces with the property that all bounded linear operators from L_1 into $\mathfrak{X}$ are strongly regular. Using Corollary 3.1, we obtain internal geometric descriptions of the class of Banach spaces with the property that all bounded linear operators from L_1 into $\mathfrak{X}$ are Dunford-Pettis. To provide a flavor of the techniques involved in these descriptions, we present an outline of one of the arguments.

Theorem 3.3. *If each bounded subset of a Banach space $\mathfrak{X}$ is weak-norm-one dentable, then each bounded linear operator from L_1 into $\mathfrak{X}$ is Dunford-Pettis.*

Recall that subset D is weak-norm-one dentable if for each $\varepsilon > 0$ there is a finite subset F of D such that for each x^* in $S(\mathfrak{X}^*)$ there is x in F satisfying

$$x \notin \overline{\text{co}}\left(D \backslash V_{\varepsilon, x^*}(x)\right) \equiv \overline{\text{co}}\,\{z \in D : |x^*(z-x)| \geq \varepsilon\}.$$

Sketch of proof. Let all bounded subsets of $\mathfrak{X}$ be weak-norm-one dentable. Fix a bounded linear operator T from L_1 into $\mathfrak{X}$. By Corollary 3.1, it is enough to show that the subset $T^*(B(\mathfrak{X}^*))$ of L_1 satisfies the Bocce criterion.

To this end, fix $\varepsilon > 0$ and B in Σ^+. Let F denote the vector measure from Σ into $\mathfrak{X}$ given by $F(E) = T(\chi_E)$. Since the subset $\{\frac{F(E)}{\mu(E)} : E \subset B \text{ and } E \in \Sigma^+\}$ of $\mathfrak{X}$ is bounded, it is weak-norm-one dentable. Accordingly, there is a finite collection $\mathscr{F}$ of subsets of B each in Σ^+ such that for each x^* in the unit ball of $\mathfrak{X}^*$ there is a set A in $\mathscr{F}$ such that if

$$\frac{F(A)}{\mu(A)} = \frac{1}{2}\frac{F(E_1)}{\mu(E_1)} + \frac{1}{2}\frac{F(E_2)}{\mu(E_2)}$$

for some subsets E_i of B with $E_i \in \Sigma^+$, then

$$\frac{1}{2}\left|\frac{x^*F(E_1)}{\mu(E_1)} - \frac{x^*F(A)}{\mu(A)}\right| + \frac{1}{2}\left|\frac{x^*F(E_2)}{\mu(E_2)} - \frac{x^*F(A)}{\mu(A)}\right| < \varepsilon.$$

Fix $x^* \in B(\mathfrak{X}^*)$ and find the associated A in $\mathscr{F}$. By definition, the set $T^*(B(\mathfrak{X}^*))$ will satisfy the Bocce criterion if Bocce-osc $(T^*x^*)|_A \leq \varepsilon$.

For a finite positive measurable partition π of A, denote

$$f_\pi = \sum_{E \in \pi} \frac{F(E)}{\mu(E)} \chi_E.$$

Set

$$E_\pi^+ = \bigcup\left\{E \in \pi : \frac{x^*F(E)}{\mu(E)} \geq \frac{x^*F(A)}{\mu(A)}\right\},$$

and

$$E_\pi^- = \bigcup \left\{ E \in \pi : \frac{x^*F(E)}{\mu(E)} < \frac{x^*F(A)}{\mu(A)} \right\}.$$

We may assume that the L_1-function T^*x^* is not constant a.e. on A. Furthermore, we may find an increasing sequence $\{\pi_n\}$ of positive measurable partitions of A satisfying

$$\Omega\, \sigma(\pi_n) = \Sigma \cap A \qquad \text{and} \qquad \mu(E_{\pi_n}^+) = \frac{\mu(A)}{2} = \mu(E_{\pi_n}^-) \quad .$$

It is easy to verify that since $F(A)/\mu(A)$ has the proper form,

$$\begin{aligned} \int_A \left| x^* f_{\pi_n} - \frac{x^*F(A)}{\mu(A)} \right| d\mu \\ &= \mu(A) \left[\frac{1}{2} \left| \frac{x^*F(E_{\pi_n}^+)}{\mu(E_{\pi_n}^+)} - \frac{x^*F(A)}{\mu(A)} \right| + \frac{1}{2} \left| \frac{x^*F(E_{\pi_n}^-)}{\mu(E_{\pi_n}^-)} - \frac{x^*F(A)}{\mu(A)} \right| \right] \\ &< \mu(A)\varepsilon . \end{aligned}$$

Since $(x^* f_{\pi_n})|_A$ converges to $(T^*x^*)|_A$ in the L_1-norm,

$$\text{Bocce-osc } (T^*x^*)|_A \equiv \frac{\int_A \left| (T^*x^*) - \frac{\int_A (T^*x^*)\, d\mu}{\mu(A)} \right| d\mu}{\mu(A)} \leq \varepsilon .$$

Thus $T^*(B(\mathfrak{X}^*))$ satisfies the Bocce criterion, as needed. □

Using martingale techniques, one can show that the converse of Theorem 3.3 is true. For this argument and more results along these lines, we refer the reader to [G].

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