

ON THE COHOMOLOGICAL EQUIVALENCE
OF A CLASS OF FUNCTIONS
UNDER AN IRRATIONAL ROTATION OF BOUNDED TYPE

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ABSTRACT. We prove that a linear combination of positive real powers of x , with integral 0 and equal values at 0 and 1, is a coboundary for any irrational rotation of bounded type. We apply this result to establish the ergodicity of related compact and noncompact skew products.

1. INTRODUCTION

We shall study an equivalence relation defined on real-valued functions on \mathbb{R}/\mathbb{Z} . Two functions, v_1 and v_2 , are *cohomologous* if there exists a Lebesgue measurable, real-valued function w such that

$$(1) \quad v_1(x) - v_2(x) = w(x) - w(x + \theta)$$

for almost all x , where θ is a fixed irrational.

This equivalence relation arises in the representation theory of nontype I groups [3, 8] and groups with ordered duals [6]. It is also related to the ergodicity of skew products of the form

$$T_v : \mathbb{R}/\mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} \times \mathbb{R} \\ T_v(x, y) = (x + \theta, y + v(x)).$$

If v is cohomologous to a constant c , then $f(x, y) = e^{2\pi i(w(x)+y)/c}$ is an invariant function for T_v , thus showing that T_v is not ergodic. If v is cohomologous to a nonconstant function w , then the ergodicity of T_w implies the ergodicity of T_v [9].

We will focus our attention on the cohomology relations among functions of the form $x^\alpha - 1/(\alpha + 1)$, for $\alpha > 0$, and on the ergodicity of the corresponding skew products $T_{x^\alpha - 1/(\alpha + 1)}$. (The constant $1/(\alpha + 1)$ is required since cohomological equivalence requires equal integrals and ergodicity requires an

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integral of 0.) Our results extend easily to linear combinations of functions of this type. We also obtain information about the multiplicative cohomology relations among the exponentials of these functions, and about the ergodicity of the compact skew products these exponentials define.

Previous results concerning cohomology of continuous functions required an L^2 derivative [2], and thus provided information only for $\alpha > 1/2$. Previous ergodicity results required the even more restrictive condition of a continuous derivative [5], so that $\alpha \geq 1$. In this paper, we establish the cohomological equivalence of the $x^\alpha - 1/(\alpha + 1)$ for all $\alpha > 0$, and we obtain as a corollary the ergodicity of $T_{x^\alpha - 1/(\alpha + 1)}$ for all $\alpha > 0$. However, we need to restrict the fixed irrational θ to be of bounded type. This means that there exists a $\delta > 0$ such that $|1 - e^{2\pi i n \theta}| \geq \delta/n$ for all $n > 0$. Some restriction has been shown to be essential for cohomology results; the only pairs of continuous functions that are cohomologous for all θ are those that differ by trigonometric polynomials [1]. Whether the restriction is necessary for the ergodicity result below is still unknown.

2.

We will establish cohomological equivalence, as in [2], by using Fourier techniques to find solutions to the functional equation (1). We write

$$v_1(x) - v_2(x) = \sum c_n(v_1 - v_2)e^{2\pi i n x},$$

and look for a solution $w(x) = \sum c_n(w)e^{2\pi i n x}$. Suppose that v_1 and v_2 are absolutely continuous, with $c_0(v_1 - v_2) = 0$, and that $v_1(0) - v_2(0) = v_1(1) - v_2(1)$. Then from (1), we see that we should look for a w with $c_n(w) = c_n(v_1 - v_2)/(1 - e^{2\pi i n \theta})$. The obvious difficulty lies in showing the convergence of $\sum c_n(w)e^{2\pi i n x}$ for this w . If $v_1 - v_2$ has an L^2 derivative, we write, for $n \neq 0$, $c_n(w) = c_n(v_1' - v_2')\sigma_n$, where $\sigma_n = 1/2\pi i n(1 - e^{2\pi i n \theta})$, and note that $\{\sigma_n\}$ is a bounded sequence for θ of bounded type. To extend this technique to functions with L^p derivatives, $1 < p < 2$, we use the following classical result.

Theorem (Marcinkiewicz). *Let $\{m_n\}$ be a bounded sequence on \mathbb{Z} . Then $\{m_n\}$ is a bounded multiplier on L^p , $p > 1$, if there exists a lacunary sequence $\{N_k\}$ and a constant M such that $\sum_{N_k \leq |j| < N_{k+1}} |m_j - m_{j+1}| \leq M$ for all k .*

Proof. See [4].

We will take $\{N_k\}$ to be the sequence of denominators of the convergents of the continued fraction expansion for θ . The convergents give the best rational approximations to θ (best relative to size of denominator). The theory of continued fractions gives estimates on how close these approximations are, and ties the property of being of bounded type to a growth condition for the $\{N_k\}$. In particular, we will need the following facts.

Lemma 1. For θ of bounded type, there exist positive constants δ (as in the definition of bounded type), δ' and δ'' such that:

(1) For all $j \leq N_k$,

$$\delta/N_k \leq \min_{\substack{0 \leq h \leq N_k \\ h \neq j}} |e^{2\pi i j \theta} - e^{2\pi i h \theta}| \leq \delta'/N_k.$$

(2) $\delta'' N_{k-1} \geq N_k \geq N_{k-1} + N_{k-2}$.

Proof. See [7].

Note that by (2), $N_k/N_{k-1} \geq 1 + 1/\delta'' > 1$, which shows that the $\{N_k\}$ forms a lacunary sequence.

The following lemma uses these properties of the convergents to give an estimate we will need in applying the Marcinkiewicz Theorem to show cohomological equivalence.

Lemma 2. There exist constants M_1 and M_2 such that

$$M_1 \ln(N_k) \leq \sum_{N_k \leq |j| < N_{k+1}} |\sigma_j - \sigma_{j+1}| \leq M_2 \ln(N_k).$$

Proof. By symmetry, we need only consider positive j . Now,

$$\begin{aligned} |\sigma_j - \sigma_{j+1}| &= \frac{1}{2\pi} \left| \frac{1}{j(j+1)(1 - e^{2\pi i j \theta})} - \frac{e^{2\pi i j \theta} (e^{2\pi i \theta} - 1)}{(j+1)(1 - e^{2\pi i j \theta})(1 - e^{2\pi i (j+1)\theta})} \right| \\ &\geq \left| \frac{\sigma_{j+1} (e^{2\pi i \theta} - 1)}{1 - e^{2\pi i j \theta}} \right| - \frac{1}{2\pi \delta (j+1)}. \end{aligned}$$

Thus, it will suffice, by (2) of Lemma 1, to prove the assertion with $\sigma_j - \sigma_{j+1}$ replaced by σ_{j+1} .

We define the sequence $\{j_n\}$ to be a reordering of the integers $N_k, N_k + 1, \dots, N_{k+1} - 1$, so that $j_1 \theta < j_2 \theta \dots < j_r \theta < 1/2 < j_{r+1} \theta < \dots < j_{(N_{k+1} - N_k)} \theta \pmod{1}$. Then

$$\begin{aligned} |1 - e^{2\pi i j_n \theta}| &\geq \frac{2}{\pi} (|1 - e^{2\pi i j_1 \theta}| + |e^{2\pi i j_1 \theta} - e^{2\pi i j_2 \theta}| + \dots + |e^{2\pi i j_{n-1} \theta} - e^{2\pi i j_n \theta}|) \\ &\geq 2n\delta/\pi N_k, \quad \text{for } 1 \leq n \leq r, \end{aligned}$$

and

$$\begin{aligned} |1 - e^{2\pi i j_n \theta}| &\geq \frac{2}{\pi} (|1 - e^{2\pi i j_{(N_{k+1} - N_k)} \theta}| + |e^{2\pi i j_{(N_{k+1} - N_k)} \theta} - e^{2\pi i j_{(N_{k+1} - N_k - 1)} \theta}| \\ &\quad + \dots + |e^{2\pi i j_{n+1} \theta} - e^{2\pi i j_n \theta}|) \\ &\geq 2(N_{k+1} - N_k - n)\delta/\pi N_k, \quad \text{for } r < n \leq N_{k+1} - N_k. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{N_k \leq |j| < N_{k+1}} |\sigma_j| &= \sum_{n=1}^{N_{k+1}-N_k} \frac{1}{2\pi j_n |1 - e^{2\pi i j_n \theta}|} \\ &\leq 2 \sum_{n=1}^{N_{k+1}-N_k} \frac{N_k}{4\delta n j_n} \\ &\leq 2 \sum_{n=1}^{N_{k+1}-N_k} \frac{1}{4\delta n} \\ &\leq M_2 \ln(N_k) \end{aligned}$$

for some constant M_2 .

Now for the lower bound, we note similarly that $|1 - e^{2\pi i j_n \theta}| \leq n\delta'/N_k$ for $1 \leq n \leq N_{k+1} - N_k$, so that

$$\begin{aligned} \sum_{n=1}^{N_{k+1}-N_k} \frac{1}{2\pi j_n |1 - e^{2\pi i j_n \theta}|} &\geq \sum_{n=1}^{N_{k+1}-N_k} \frac{N_k}{2\pi \delta' n j_n} \\ &\geq \sum_{n=1}^{N_{k+1}-N_k} \frac{1}{2\pi \delta'' \delta' n} \\ &\geq M_1 \ln(N_k) \end{aligned}$$

for some constant M_1 .

We see by Lemma 2 that $\{\sigma_n\}$ does not satisfy the hypothesis of the Marcinkiewicz Theorem. In the following theorem, we apply Marcinkiewicz using an appropriately modified sequence.

Theorem 1. *Suppose v is absolutely continuous, $v(0) = v(1)$, and $\{\ln(|n|)c_n(v')\}$ is the sequence of Fourier coefficients of an L^p function, $p > 1$. Then v is cohomologous to the constant $c_0(v)$.*

Proof. We have that $\{2\pi i n \ln(|n|)c_n(v)\}$ is the sequence of Fourier coefficients of an L^p function. Also, as in Lemma 2, we see that $\{\sigma_n / \ln(|n|)\}$ satisfies the hypotheses of the Marcinkiewicz Theorem. Thus $\{c_n(v)/(1 - e^{2\pi i n \theta})\}$ is the sequence of Fourier coefficients of an L^p function w . Then by substitution we see that v , $c_0(v)$, and w satisfy (1).

We use Theorem 1 to show the cohomological equivalence of functions of the form $x^\alpha - 1/(\alpha + 1)$.

Theorem 2. *The functions $x^\alpha - 1/(\alpha + 1)$ and $x^\beta - 1/(\beta + 1)$ are cohomologous, for any $\alpha, \beta > 0$.*

Proof. Let $v_\alpha(x) = x^\alpha - x$. By Theorem 1, it will suffice to show that $\{\ln(|n|) \times c_n(v'_\alpha)\}$ is the sequence of Fourier coefficients of an L^p function, $p > 1$. By previous results [2], we need only consider $0 < \alpha \leq 1/2$.

We use a change of variables to write, for $n \neq 0$, $c_n(v'_\alpha) = \alpha|n|^{-\alpha}I_{n,\alpha}$, where $I_{n,\alpha} = \int_0^{|n|} e^{-2\pi i x(\text{sgn}(n))} x^{\alpha-1} dx$. Choose an ε , $0 < \varepsilon < \alpha$, and note that $v'_{\alpha-\varepsilon}$ is in L^p for some $p > 1$. Now we have that

$$\ln(|n|)c_n(v'_\alpha) = \frac{\alpha \ln(|n|)I_{n,\alpha}}{(\alpha - \varepsilon)|n|^\varepsilon I_{n,\alpha-\varepsilon}} c_n(v'_{\alpha-\varepsilon}).$$

It will suffice to show that each of the three sequences $\{\ln(|n|)|n|^{-\varepsilon}\}$, $\{I_{n,\alpha}\}$, and $\{1/I_{n,\alpha-\varepsilon}\}$ satisfies the hypotheses of the Marcinkiewicz Theorem. We will use the fact that if a sequence $\{m_n\}$ is eventually monotone, then for large k , the Marcinkiewicz sum telescopes, so that

$$\sum_{N_k \leq |n| < N_{k+1}} |m_n - m_{n+1}| = |m_{N_k} - m_{N_{k+1}}| + |m_{(-N_k)} - m_{(-N_{k+1})}|.$$

Thus, a bounded, eventually monotone sequence gives a bounded multiplier.

This comment is enough to show that the first of our sequences is a bounded multiplier: for n large in absolute value, $\ln(|n|)|n|^{-\varepsilon}$ decreases monotonically as $|n|$ increases. Now we consider the sequence $\{I_{n,\alpha}\}$. We write

$$\begin{aligned} I_{n,\alpha} &= \int_0^{|n|} \frac{\cos 2\pi x}{x^{1-\alpha}} dx \pm i \int_0^{|n|} \frac{\sin 2\pi x}{x^{1-\alpha}} dx \\ &= a_n(\alpha) \pm ib_n(\alpha). \end{aligned}$$

Clearly, $\{b_n(\alpha)\}$ is increasing as $|n|$ increases. Integration by parts shows that $\{a_n(\alpha)\}$ is also increasing, and that both $\{a_n(\alpha)\}$ and $\{b_n(\alpha)\}$ are bounded. Thus by the remark above, the second sequence is also a bounded multiplier.

Finally, we consider the sequence $\{1/I_{n,\alpha-\varepsilon}\} = \{\overline{I_{n,\alpha-\varepsilon}}/|I_{n,\alpha-\varepsilon}|^2\}$. Now, $\{\overline{I_{n,\alpha-\varepsilon}}\}$ is a bounded multiplier by the same argument as above. Also, $|I_{n,\alpha-\varepsilon}|^2 = (a_n(\alpha-\varepsilon))^2 + (b_n(\alpha-\varepsilon))^2$, so that $\{1/|I_{n,\alpha-\varepsilon}|^2\}$ is eventually decreasing, and is thus a bounded multiplier.

Corollary 1. *If $v(x) = \sum_{j=1}^n \beta_j x^{\alpha_j}$, where $\beta_j \in \mathbb{R}$, $\alpha_j \geq 0$, and if $v(0) = v(1)$, then v is cohomologous to its integral $c_0(v)$.*

Proof. By Theorem 2, we have that for each j , with $\alpha_j \neq 0$, $\beta_j x^{\alpha_j} - \beta_j/(\alpha_j + 1)$ is cohomologous to $\beta_j(x - 1/2)$. Thus $v - c_0(v)$ is cohomologous to $(\sum_{\alpha_j \neq 0} \beta_j) \times (x - 1/2)$, which is 0 since $v(0) = v(1)$.

Corollary 2. *Let $v(x) = \sum_{j=1}^n \beta_j x^{\alpha_j}$, where $\beta_j \in \mathbb{R}$, $\alpha_j \geq 0$. If $v(0) \neq v(1)$ and if v has integral 0, then the skew product T_v is ergodic.*

Proof. As in Corollary 1, we see that v is cohomologous to $c(x - 1/2)$, where c is nonzero. The corollary then follows from the ergodicity of $T_{x-1/2}$, which is established in [5].

We now apply our cohomology results to compact skew products. We will

need the following definitions:

Two circle-valued functions f_1 and f_2 defined on \mathbb{R}/\mathbb{Z} are *multiplicatively cohomologous* if there exists a Lebesgue measurable circle-valued function g such that

$$(2) \quad f_1(x)/f_2(x) = g(x)/g(x + \theta)$$

for almost all x . A simple substitution shows that $e^{2\pi i v_1}$ is multiplicatively cohomologous to $e^{2\pi i v_2}$ if v_1 is cohomologous to v_2 .

A circle-valued function f on \mathbb{R}/\mathbb{Z} defines a compact skew product

$$S_f: \mathbb{R}/\mathbb{Z} \times \mathbb{T} \rightarrow \mathbb{R}/\mathbb{Z} \times \mathbb{T}, \\ S_f(x, \beta) = (x + \theta, \beta f(x)).$$

Just as with noncompact skew products, f_1 cohomologous to f_2 implies that the corresponding skew products S_{f_1} and S_{f_2} share either ergodicity or lack of ergodicity. The compact case is simpler than the noncompact in that a Fourier series argument can be used to show that S_f is ergodic if and only if f^k is not cohomologous to 1 for any nonzero integer k .

Applying Corollary 1 now gives us the following corollary.

Corollary 3. *If $v(x) = \sum_{j=1}^n \beta_j x^{\alpha_j}$, where $\beta_j \in \mathbb{R}$, $\alpha_j \geq 0$, then the compact skew product $S_{e^{2\pi i v}}$ is ergodic if and only if either*

$$(1) \quad v(0) \neq v(1)$$

or

$$(2) \quad v(0) = v(1) \text{ and } \int v \neq p + q\theta \text{ for any } p, q \in \mathbb{Q}.$$

Proof. If $v(0) \neq v(1)$, define a nonconstant linear function u by $u(x) = xv(1) + (1-x)v(0)$. Then by Corollary 1, $v - u$ is cohomologous to a constant c , so that $e^{2\pi i v}$ is multiplicatively cohomologous to $e^{2\pi i(u+c)}$. By [3], $e^{2\pi i k(u+c)}$ cannot be cohomologous to 1 for nonconstant u and $k \neq 0$.

If $v(0) = v(1)$ then by Corollary 1, v is cohomologous to its integral $c_0(v)$. Thus $e^{2\pi i v}$ is multiplicatively cohomologous to $e^{2\pi i c_0(v)}$, so that $S_{e^{2\pi i v}}$ is ergodic if and only if $S_{e^{2\pi i c_0(v)}}$ is ergodic. The latter is ergodic if and only if $e^{2\pi i k c_0(v)}$ is not cohomologous to 1 for any integer $k \neq 0$. This is true if and only if $e^{2\pi i k c_0(v)}$ is not an eigenvalue for rotation by θ , so not of the form $e^{2\pi i j \theta}$ for any integers j and $k \neq 0$.

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