A GENERIC TORELLI-TYPE THEOREM FOR SINGULAR ALGEBRAIC CURVES WITH AN INVOLUTION

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Abstract. We prove a generic Torelli-type theorem for a special class of singular algebraic curves with an involution. In order to obtain this result we introduce an appropriate mixed Hodge structure on the anti-invariant part of the first homology group, and study its properties.

Let \( X \) be an irreducible projective algebraic curve with an involution \( \sigma \). Suppose \( X \) has only ordinary singularities and let \( \Sigma \) be its singular locus. Let \( \pi: N \to X \) be the normalization of \( X \) and let \( \tau: N \to N \) be the involution induced by \( \sigma \). We suppose that the following condition is satisfied:

\[ (*) \]

The set of fixed points of \( \sigma \) coincides with \( \Sigma \) and the involution \( \tau \) is without fixed points.

Following J. Carlson [2] one can introduce a polarized mixed Hodge structure (PMHS) on the anti-invariant part of \( H_1(X, \mathbb{Z}) \) with respect to \( \sigma \), denoted by \( H_1^-(X, \mathbb{Z}) \):

\[ 0 \to H_1^-(N, \mathbb{Z}) \to H_1^-(X, \mathbb{Z}) \to A \to 0, \]

where \( H_1^-(N, \mathbb{Z}) \) is the anti-invariant part of \( H_1(N, \mathbb{Z}) \) with respect to \( \tau \) and has a polarized Hodge structure (PHS) of weight \(-1\); \( A \) has PHS of weight \(0\). It turns out that the latter is isomorphic to the lattice generated by a root-system of the type \( D_n \) when \( \#(\pi^{-1}(\Sigma)) > 2 \).

Using the generic Torelli theorem for the Prym map as proven by Friedman-Smith [4] and Kanev [5], the pair \((N, \tau)\) is uniquely determined by its Prym variety (equivalently by the PHS of \( H_1^-(N, \mathbb{Z}) \)) if the following condition is satisfied.

\( N/\tau \) is a sufficiently general curve of genus \( g \geq 7 \).

We prove the following result:

Theorem 7. Let \( X \) be the curve which satisfies \((*)\). Suppose that \( N/\tau \) is a general curve of genus \( g \geq 15 \). Then \( N, \tau \), and the set \( \pi^{-1}(\Sigma) \) are uniquely determined by the PMHS of \( H_1^-(X, \mathbb{Z}) \).
As a consequence we obtain

**Theorem 8.** If $X$ satisfies the conditions of Theorem 7 and has only one singular point then $X$ and $\sigma$ are uniquely determined by the PMHS of $H_1^-(X, \mathbb{Z})$.

1. Preliminaries

Let $X$ be an irreducible projective curve with ordinary singularities, and $n: N \to X$ be its normalization for which

(i) there exist involutions $\tau: N \to N$ and $\sigma: X \to X$, such that $\tau$ has no fixed points and the fixed points of $\sigma$ are the singularities of $X$;

(ii) $\pi \circ \tau = \sigma \circ \tau$.

Such curves can be constructed as follows:

(i) we fix $(N, \tau)$ to be a smooth irreducible projective algebraic curve with an involution without fixed points;

(ii) we choose a finite subset $\Omega \subset N$ such that $\Omega = \bigcup_{i=1}^k \Omega_i$, and for each $i$, $\tau(\Omega_i) = \Omega_i$ and $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$;

(iii) we define $X = N/\rho$, provided with the factor-topology and the induced involution $\sigma$, where by definition for each $a, b \in N$

\{apb\} iff \{either $a = b$ or $a, b \in \Omega_i$ for some $i$\}.

The condition that $X$ has only ordinary singularities uniquely determines $X$ as an algebraic curve.

Obviously $M = N/\tau$ is a smooth projective algebraic curve and the induced morphism $\psi: M \to Y = X/\sigma$ is a normalization. Furthermore $Y$ has at worst ordinary singularities.

Let $\phi: N \to M$ and $\lambda: X \to Y$ be the corresponding factor-morphisms.

2. Definition and properties of the main exact sequence

J. Carlson [2] constructed the following exact sequence for $\pi: N \to X$.

\[ 0 \to H_1(N, \mathbb{Z}) \xrightarrow{\pi^*} H_1(X, \mathbb{Z}) \xrightarrow{\partial} \zeta_x \to 0, \]

where $\partial$ is a boundary operator, $\zeta_x \subset \text{Div}^0 N$ is a finitely-generated group, which by means of the natural polarization on $\text{Div} N$ (for which the points of $N$ form an orthonormal base) has a representation as an orthogonal sum:

$\zeta_x = \bigoplus_{i=1}^k \zeta(y_i)$, where $y_i = \pi(\Omega_i)$. Furthermore $\zeta(y_i)$ is spanned by a root-system of type $A_{r(i)}$, where $r(i) = \#(\Omega_i) - 1$. The morphisms $\sigma$ and $\tau$ act on (1). Since $\pi \circ \tau = \sigma \circ \pi$ for the anti-invariant part of the corresponding members of (1) we have:

\[ H_1^-(N, \mathbb{Z}) \xrightarrow{\pi^{-\text{inv}}} H_1^-(X, \mathbb{Z}) \xrightarrow{\partial} \zeta_x^- \to 0. \]

Since $\tau(\Omega_i) = \Omega_i$ and $\tau$ has no fixed points we have $\zeta_x^- = \bigoplus_{i=1}^k \zeta^- (y_i) = \bigoplus_{i=1}^k \mathbb{Z}(x_i - \tau(x_i))$, which is an orthogonal sum. Here $2s = \#(\Omega)$ and $\Omega = \{x_1, \ldots, x_s, \tau(x_1), \ldots, \tau(x_s)\}$. Hence the nonzero elements of $\zeta_x^-$ with min-
The imal length are
\[ \{ \pm (x_i - \tau(x_i)) | x_i \in \Omega, \ i = 1, 2, \ldots, s \}. \]

Let \( A = \partial(H^-_1(X, Z)) \).

**Lemma 1.** (i) If \( \#(\Omega) = 2 \), then \( A \cong \mathbb{Z} \), a and \( (a, a) = 8 \);
(ii) If \( \#(\Omega) \geq 4 \) and \( R = \{ a \in \mathbb{R} \mid (a, a) = 4 \} \), then \( R \) is a root-system of type \( D_s \) \((2s = \#(\Omega))\) and \( R \) generates \( A \).

**Proof.** (a) We claim that there is no element \( b \in \mathbb{Z} \) for which \( (b, b) = 2 \)
and \( b \in A \). Indeed if there exists such an element then \( b = x - \tau(x) \in A \)
with \( x \in \Omega \), hence there exists \( c \in C_1(N) \) for which \( \partial(c) = x - \tau(x) \) and
\( \pi_*(c) \in H^-_1(X, Z) \). Furthermore \( c + \tau(c) \) is a cycle in \( H_1(N, Z) \), and \( \pi_*(c) \in H^-_1(X, Z) \) implies that \( \pi_*(c + \tau(c)) = 0 \) in \( H_1(X, Z) \). Since \( H_1(N, Z) \to H_1(X, Z) \) is injective, it follows that \( c + \tau(c) \) is homologous to \( 0 \) in \( H_1(N, Z) \).

Now consider \( \phi: N \to M \). Since \( M = N/\tau \), it follows that \( \phi_*(c + \tau(c)) = 2\phi_*(c) \) is homologous to \( 0 \) in \( H_1(M, Z) \). Since \( H_1(M, Z) \) is torsionfree, \( \phi_*(c) \) is also homologous to \( 0 \). Thus, as a loop in \( \pi_1(M) \), \( \phi_*(c) \) is in the commutator subgroup. Since \( \pi_1(N) \subset \pi_1(M) \) is a normal subgroup of index \( 2 \), \( \pi_1(N) \) contains the commutator subgroup. Thus \( \phi_*(c) \) lies in the image of \( \pi_1(N) \), which means that \( \phi_*(c) \) lifts to a closed loop in \( N \). This contradicts \( \partial(c) = x - \tau(x) \).

(b) We claim that if \( x, y \in \Omega \), then \( a = x - \tau(x) \pm (y - \tau(y)) \) is an element of \( A \). It is sufficient to prove the statement for \( a = x - \tau(x) + y - \tau(y) \) since \( x - \tau(x) - (y - \tau(y)) = x - \tau(x) + \tau(y) - \tau(y) \). Let \( c \in C_1(N) \) with \( \partial(c) = x - \tau(y) \). Then \( \partial(c - \tau(c)) = x - \tau(x) + y - \tau(y) \).

We complete the proof of Lemma 1 as follows:

**Case (i).** By (a) and (b) \( A \) is generated by \( a = 2(x - y), \) where \( \{x, y\} = \Omega \).

**Case (ii).** Put \( e_i = x_i - \tau(x_i) \), where \( \Omega = \{x_1, \ldots, x_s, \tau(x_1), \ldots, \tau(x_s)\} \).
Then \( R = \{e_i \pm e_j \mid i \neq j\} \), so \( R \) is a root-system of type \( D_s \). Since \( R = \{e_i \pm e_j\} \), it follows that \( R \) spans a sublattice \( R' \) of index \( 2 \) in \( \mathbb{Z}^{s} \) \( \mathbb{Z}e_i \).
Then (b) implies \( R' \subset A \), so that \( R' = A \) or \( A = \bigoplus_{i=1}^{s} \mathbb{Z}e_i \). By (a), the latter cannot happen, which proves that \( R \) generates \( A \). Q.E.D.

**Lemma 2.** The sequence
\[ 0 \to H_1^-(N, Z) \xrightarrow{\pi_*} H_1^-(X, Z) \xrightarrow{\partial} A \to 0 \]
is exact.

**Proof.** The exactness at the first member of (2) is derived from the exactness of the first member of (1); the exactness in the third member of (2) is derived from the definition of \( A \). It remains to prove that \( \text{Ker}\partial = \text{Im}\pi_* \). Obviously \( \text{Ker}\partial \supset \text{Im}\pi_* \). Let \( a \in H_1^-(X, Z) \) and \( \partial(a) = 0 \), hence there exists \( b \in H_1(N, Z) \) such that \( \pi_*(b) = a \), since (1) is exact. We have
\[ 0 = a\sigma_*(a) = \pi_*(b) + \sigma_*\pi_*(b) = \pi_*(b + \tau(b)), \]
hence \( b + \tau(b) = 0 \), since \( \pi_* \) is an injection. Hence \( b \in H^{-1}_1(N, \mathbb{Z}) \) and \( \pi_*(b) = a \), i.e., \( \text{Ker} \partial \subset \text{Im} \pi_* \). Q.E.D.

3. Construction of PMHS for \( H^{-1}_1(X, \mathbb{Z}) \)

From the Poincaré duality we get an exact sequence:

\[
0 \to \tilde{A} \to H^{-1}_1(X, \mathbb{Z}) \to H^{-1}_1(N, \mathbb{Z}) \to 0,
\]

where \( \tilde{A} = \text{Hom}_Z(A, \mathbb{Z}) \).

Following J. Carlson [2] we define a mixed Hodge structure on \( H^{-1}_1(X, \mathbb{Z}) \):

(a) Weight filtration on \( H^{-1}_1(X, \mathbb{Z}) \):

\[
W_{-1} = 0, \quad W_0 = \text{Im} \tilde{\partial}, \quad W_1 = H^{-1}_1(X, \mathbb{Z});
\]

define polarizations on \( W_0 \) and \( W_1/W_0 \) via the polarizations on \( A \) and on \( H^{-1}_1(N, \mathbb{Z}) \) introduced above.

(b) Hodge filtration on \( H^{-1}_1(X, \mathbb{C}) = H^{-1}_1(X, \mathbb{Z}) \otimes \mathbb{C} \):

\[
F^0 = H^{-1}_1(X, \mathbb{C});
\]

\[
F^1 = \left\{ \omega \in H^0(X - \Sigma, \Omega^1(X - \Sigma)) \mid \int_{X - \Sigma} \omega \wedge \overline{\omega} < \infty, \sigma^* \omega + \omega = 0 \right\},
\]

here \( \Sigma \) is the singular locus of \( X \).

\[
F^2 = 0.
\]

**Lemma 3.** The map \( \pi^* \) gives an isomorphism

\[
F^1 \cong H^0(N, \Omega^1_N).
\]

It follows from Lemma 3 that \( F^1 \cap W_0 = 0, W_1/W_0 \otimes \mathbb{C} \cong F^1 \oplus \overline{F^1} \) which means that we have MHS on \( H^{-1}_1(X, \mathbb{Z}) \), \( W_0 \) has PHS of pure weight 0, and \( W_1/W_0 \) has PHS of pure weight 1 (cf. [2]).

**Proof.** Let \( \alpha \in F^1 \). Then \( \pi^*(\alpha) \in H^0(N - \Omega, \Omega^1(N - \Omega)) \). Since

\[
\int_{N - \Omega} \pi^* \alpha \wedge \overline{\pi^* \alpha} = \int_{X - \Omega} \alpha \wedge \overline{\alpha} < \infty,
\]

then \( \pi^* \alpha \in H^0(N, \Omega^1_N) \). We have \( \pi^* \alpha + \tau^* \circ \pi^* \alpha = \pi^* (\alpha + \sigma^* \alpha) = 0 \) on \( N - \Omega \). The 1-form \( \pi^* \alpha \) is holomorphic, so \( \pi^* \alpha + \tau^* \circ \pi^* \alpha = 0 \) on \( N \). Thus we have a linear map \( \pi^*: F^1 \to H^0(N, \Omega^1_N) \). The inverse of this map is obviously defined since \( N \) and \( X \) are birationally isomorphic. Q.E.D.

**Proposition 4.** The group \( H^{-1}_1(X, \mathbb{Z}) \) has a polarized mixed Hodge structure for which:

(i) \( W_0 H^{-1}_1(X) = H^{-1}_1(X, \mathbb{Z}), W_{-1} H^{-1}_1(X) = \text{Im} \pi_* = H^{-1}_1(N, \mathbb{Z}), W_{-2} H^{-1}_1(X) = 0; \)
(ii) $F^1H^-(X) = 0$, $F^0H^-(X) = \text{ann}_R(F^1H^1_-(X)) \cong \overline{F^1H^1_-(X)}^* \otimes (A \otimes \mathbb{C})$, $F^{-1}H^-(X) = H^{-1}(X, \mathbb{C})$, here $F^1H^1_-(X)^* = \text{Hom}_R(F^1H^1_-(X), \mathbb{R})$ and the complex conjugation in $H^1_-(X, \mathbb{C})$ is induced by those in $\mathbb{C}$ through $H^1_-(X, \mathbb{C}) = H^1(X, \mathbb{Z}) \otimes \mathbb{C}$;

(iii) $\text{Gr}^{w}_{-1} = W_{-1}/W_{-2}$ has a pure weight $-1$, the polarization of $H^{-1}_1(N, \mathbb{Z})$ is transferred to $\text{Gr}^{w}_{-1}$ through $\pi_*$. $\text{Gr}^{w}_{0} = W_{0}/W_{-1}$ has a pure weight $0$ and the polarization of $A$ is transferred to $\text{Gr}^{w}_{0}$ through $\partial$.

Proof. This is an immediate consequence of Lemma 3 and the definition of MHS on a dual group (cf. [2]). Q.E.D.

It follows that the Hodge structure on $\text{Gr}^{w}_{-1} = W_{-1}H^-(X)$ is:

$$F^{-1}W_{-1}H^-(X) = F^{-1}H^-(X) \cap [W_{-1}H^1_-(X) \otimes \mathbb{C}] = F^1H^1_-(X)^* \oplus \overline{F^1H^1_-(X)}^* = W_{-1}H^-(X);$$

$$F^0W_{-1}H^-(X) = F^0H^-(X) \cap [W_{-1}H^1_-(X) \otimes \mathbb{C}] = \overline{F^1H^1_-(X)}^*;$$

$$F^1W_{-1}H^-(X) = 0.$$ 

4. Geometric description of the 1-motive map

By definition we have

$$L^0H^-(X) = [\text{Gr}^w_0H^-(X) \otimes \mathbb{C}] \cap [\text{Gr}^w_0H^-(X)]_\mathbb{Z},$$

$$J^0W_{-1}H^-(X) = [W_{-1}H^1_-(X) \otimes \mathbb{C}] / [W_{-1}H^1_-(X) + F^0W_{-1}H^-(X)].$$

It is clear from Proposition 4 and from the sequence (2) that $H^1_-(X, \mathbb{Z})$ is an extension of $\text{Gr}^w_0$ by $\text{Gr}^w_{-1}$. For extensions of this type J. Carlson [2] has constructed a map 1-motive:

$$u: L^0H^-(X) \rightarrow J^0W_{-1}H^-(X),$$

which depends only on the mixed Hodge structure of $H^1_-(X, \mathbb{Z})$ and is given as follows:

(i) Let $\{\omega_i\}$ be a basis of $\text{Gr}^w_0$ and $\{\omega^i\}$ be the dual basis in $(\text{Gr}^w_{-1})^*$;

(ii) Let $\{\Omega^i\} \subset H^1_-(X, \mathbb{Z})$, for which $\pi^*(\Omega^i) = \omega^i$ for each $i$;

(iii) If $\gamma \in \text{Gr}^w_0$ and $\Gamma \in H^1_-(X, \mathbb{Z})$ is such that $\partial(\Gamma) = \gamma$, then $\gamma \mapsto [\Sigma_i (\Omega^i, \Gamma)\omega_i]$, where $(\ , \ )$ is the canonical pairing between $H^1_-(X, \mathbb{Z})$ and $H^1_-(X, \mathbb{Z})$; and $[\alpha]$ is the class of $\alpha$ in $J^0W_{-1}H^1_-(X)$.  

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Let us recall that for \((N, \tau)\) a Prym variety \(P(N, \tau) = H^1,0_-(N)^*/H^1_-(N, \mathbb{Z})\) is defined and a map \(PA: \text{Div}^0(N) \to P(N, \tau)\), where
\[
PA(P - Q) = \left( \int_Q \phi_1, \ldots, \int_Q \phi_{g-1} \right) \quad (\text{mod} \ H^1_-(N, \mathbb{Z})).
\]
Here \(\{\phi_1, \ldots, \phi_{g-1}\}\) is a basis of \(H^1_-(N)\) and \(H^1_-(N, \mathbb{Z})\) is injected into \(H^1,0_-(N)^*\) by integration. \(PA\) is called the Abel–Prym's map.

**Proposition 5.** The following diagram is commutative
\[
\begin{array}{ccc}
A & & \to P(N, \tau) \\
\downarrow \cong & & \downarrow \cong \\
L^0H^1_-(X) & \overset{\mu}{\longrightarrow} & J^0W_{-1}H^1_-(X)
\end{array}
\]
where \(\mu\) is given by the identification of \(W_{-1}H^1_-(X)\) with \(H^1_-(N, \mathbb{Z})\) and by using the duality given by integration; \(\mu\) is the Abel–Prym's map, restricted on \(A \subset \text{Div}^0 N\).

**Proof.** Obviously \(L^0H^1_-(X) = G_{10}^0 H^1_-(X)\) which is isomorphic to \(A\) through \(\partial\) (cf. Proposition 4).

\[
J^0W_{-1}H^1_-(X) = \frac{W_{-1}H^1_-(X) \otimes \mathbb{C}}{W_{-1}H^1_-(X) + J^0W_{-1}H^1_-(X)} = \frac{F^1H^1_-(X)^* + F^1H^1_-(X)^*}{W_{-1}H^1_-(X) + F^1H^1_-(X)^*}
\]
\[
\cong \frac{F^1H^1_-(X)^*}{W_{-1}H^1_-(X)} \cong \frac{H^1,0_-(X)^*}{H^1_-(N, \mathbb{Z})} \cong P(N, \tau).
\]

To calculate \(\mu\) we introduce the well-known symplectic base of \(H_1(N, \mathbb{Z})\):
\[
\{a_1, b_2; a_{g+1}, b_{g+1}; \ldots ; a_{g-1}, b_{g-1}; a_{2g-1}, b_{2g-1}; a_g, b_g\}
\]
for which
\[
\tau_*(a_i) = a_{i+g}, \quad \tau_*(b_i) = b_{i+g} \quad \text{for} \quad i = 1, 2, \ldots, g - 1;
\]
\[
\tau_*(a_g) = a_g, \quad \tau_*(b_g) = b_g.
\]
Furthermore, choose a basis \(\{\omega^1, \ldots, \omega^{2g-1}\}\) of \(H^{1,0}_-(N)\) with \(\int_a \omega^j = \delta^i_j\) for \(i, j = 1, 2, \ldots, 2g - 1\).

Then \(H^{1,0}_-(N) = (\omega^1 - \omega^{g-1}) \cdot \mathbb{C} \oplus \cdots \oplus (\omega^{g-1} - \omega^{2g-1}) \cdot \mathbb{C}\), hence
\[
H^{1,0}_-(N)^* = \left( \int_{a_1 - a_{g+1}} \right) \cdot \mathbb{C} \oplus \cdots \oplus \left( \int_{a_{g-1} - a_{2g-1}} \right) \cdot \mathbb{C}.
\]
In this case \(\mu\) is given as follows:
\[
\gamma \to \left[ \sum_{i=1}^{g-1} \left( \int_{a_i - a_{i+g}} (\omega^1 - \omega^{g-1}) \right), \left( \int_{a_i - a_{i+g}} \right) \right],
\]
which is exactly the map \(\gamma \to PA(\gamma)\). Q.E.D.
In fact using the same basis of \( H^{1,0}(N) \) we have a map \( j: P(N, \tau) \to J(N) \), where \( J(N) \) is the Jacobi variety of \( N \) and

\[
j \circ \mu(\gamma) = \left[ \left( \int_{\Gamma} (\omega - \omega^{g+1}) \right), \ldots, \left( \int_{\Gamma} (\omega^{g-1} - \omega^{2g-1}) \right), 0, -\int_{\Gamma} (\omega^1 - \omega^{g+1}), \ldots, -\int_{\Gamma} (\omega^{g-1} - \omega^{2g-1}) \right].
\]

It is clear that \( j \circ \mu = (1 - \tau) \circ Ab \), where \( Ab: N \to J(N) \) is the Abel's map for \( N \).

5. Proof of Theorem 8

For the Abel's map \( Ab: N \to J(N) \) we have \((1 - \tau) \circ Ab(D) = Ab((1 - \tau)D)\) for each \( D \in \text{Div}^0 N \). Since each generator of \( A \) with minimal length has type \((1 - \tau)(P - Q)\) then

\[
j \circ \mu((1 - \tau)(P - Q)) = (1 - \tau) \circ Ab \circ (1 - \tau)(P - Q) = Ab \circ (1 - \tau)^2(P - Q)
\]

\[
= 2Ab \circ (1 - \tau)(P - Q) = 2PA(P - Q).
\]

It follows that we must consider the map

\[
\Phi: N \times N \to P(N, \tau) \subset J(N), \quad (P, Q) \to 2 \cdot PA(P - Q).
\]

Lemma 6. Let \( N \) be a smooth projective curve with an involution \( \tau \) without fixed points. If \( N \) is neither a 4-, nor 8-sheeted covering of \( \mathbb{P}^1 \), then \( \Phi(P_1, Q_1) = \Phi(P_2, Q_2) \) if and only if: either \((P_1, Q_1) = (P_2, Q_2)\) or \((P_1, Q_1) = (\tau(Q_2), \tau(P_2))\) or \((P_1, P_2) = (Q_1, Q_2)\).

Proof. \( \Phi(P_1, Q_1) = \Phi(P_2, Q_2) \) iff \( 2PA(P_1 + Q_2 - Q_1 - P_2) = 0 \) in \( J(N) \). We consider \( D = 2(1 - \tau)(P_1 + Q_2 - Q_1 - P_2) \) as an element of \( \text{Div}^0 N \). If \( D \neq 0 \) then \( D = D_+ - D_- \), \( D_+ > 0 \), \( \deg D_+ = \deg D_- = 4 \) or 8 and \( \text{supp} D_+ \cap \text{supp} D_- = \emptyset \). Since \( Ab(D) = 0 \) in \( J(N) \), then by Abel's theorem for \( N \) we conclude that there exists a map \( N \to \mathbb{P}^1 \) of degree 4 or 8 which is impossible by hypothesis. Thus \( D = 0 \), which is possible only in the cases listed in the lemma. Q.E.D.

Theorem 7. Let \( X \) be a curve of 1, for which \( N/\tau \) is a generic curve of genus \( g(N/\tau) \geq 15 \). Then using the mixed Hodge structure of \( H^1(X, \mathbb{Z}) \), constructed in 4, we can get \( N, \tau: N \to N \) and \( \Omega \).

Proof. The mixed Hodge structure gives us \( J^0W_{-1}H^1_-(X) \cong P(N, \tau) \). Using the generic Torelli theorem (V. Kanev [5], Friedman–Smith [4]) we obtain \( N \) and \( \phi: N \to M \), which gives us \( (N, \tau) \). Let \( R \) be the finite subset of \( L^0H^1_-(X) \) defined as follows. If there exist elements \( b \) of \( L^0H^1_-(X) \) with \((b, b) = 4\), then \( R = \{a|\langle a, a \rangle = 4\} \). Otherwise put \( R = \{a|\langle a, a \rangle = 8\} \) (see Lemma 1). Consider the 1-motive: \( u: L^0H^1_-(X) \to J^0W_{-1}H^1_-(X) \). Since \( g(N/\tau) \geq 15 \) then \( N/\tau \) is not a covering of \( \mathbb{P}^1 \) of degree 4 or 8 (see [1, p. 214]) implying \( N \)}
is not a covering of \( \mathbb{P}^1 \) of degree 4 or 8. Then by Proposition 5 and Lemma 6 the set \( u(R) \) uniquely determines the set \( \Omega \). Q.E.D.

**Theorem 8.** Let \( X \) satisfy the conditions of Theorem 7. Suppose \( X \) has only one singular point. Then \( X \) and \( \sigma \) are uniquely determined by the PMHS defined in 4.

**Proof.** By using Theorem 7 we reconstruct \( N, \tau, \) and \( \Omega \); since \( X \) has an ordinary singular point, which is obtained by identification of the points of \( \Omega \), we recover \( X \) and \( \sigma \). Q.E.D.

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