PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 111, Number 3, March 1991

Q-SETS, SIERPINSKI SETS, AND RAPID FILTERS

HAIM JUDAH AND SAHARON SHELAH

(Communicated by Andreas R. Blass)

ABSTRACT. In this work we will prove the following:

Theorem 1. cons(ZF) *implies* cons(ZFC + *there exists a Q-set of reals* + *there exists a set of reals of cardinality* $\aleph_1$ *which is not Lebesgue measurable*).

Theorem 2. cons(ZF) *implies* cons($ZFC + 2^{\aleph_0}$ *is arbitrarily larger than* $\aleph_2$+ *there exists a Sierpinski set of cardinality* $2^{\aleph_0}$ + *there are no rapid filters on* ω).

These theorems give answers to questions of Fleissner [Fl] and Judah [Ju].

0. INTRODUCTION

In this work we will solve two open problems about special sets of the reals. In order to state them we need some definitions.

0.1. **Definition.** A set of reals A is a Q-set iff every subset of A is a relative F_σ, i.e., it is a countable union of relatively closed subsets of A.

Q-sets are very strange: for example $2^{\aleph_0} < 2^{\aleph_1}$ implies that there are no Q-sets of cardinality $\aleph_1$. Also Q-sets have universal measure zero, but they do not necessarily have strong measure zero (see [Fl, JSh2, Mi2]).

In [Fl] it is asked if the existence of a Q-set of cardinality $\aleph_1$ implies that every $\aleph_1$-set of reals is of Lebesgue measure zero. Our first theorem answers this question negatively by showing

Theorem. cons(ZF) *implies* cons(ZFC + *there exists a Q-set of reals* + *there exists a set of reals of cardinality* $\aleph_1$ *which is not Lebesgue measurable*).

We show this theorem as follows. We begin by forcing a set A of reals of cardinality $\aleph_1$, and then we force, with a countable support iteration of length ω_2, making A a Q-set in the generic extension. We prove that this composition of forcing notions satisfies the Sacks property (studied in [Sh]) and, in the end

Received by the editors December 4, 1987 and, in revised form, April 5, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 03E35; Secondary 03E15.
The first author would like to thank the NSF for its partial support under grant DMS-8701828.
The second author thanks the United States–Israel Binational Foundation for partially supporting his research.

of the section, we prove that if a forcing notion has the Sacks property then in the generic extension the old reals have outer measure one. Clearly this implies, if we begin from L, that in our generic extension there exists an uncountable Q-set and a $\aleph_1$-set of reals which is not Lebesgue measurable.

0.2. **Definition.** (a) A set of reals A is a Sierpinski set iff for every measure zero set M, $A \cap M$ is countable.

(b) $[\omega]^\omega = \{x : x \subseteq \omega \wedge |x| = \aleph_0\}$; $[\omega]^{<\omega} = \{x : x \subseteq \omega \wedge |x| < \aleph_0\}$.

(c) A subset $F \subseteq [\omega]^\omega$ is a rapid filter iff

(i) $(\forall x, y \in F)(x \cap y \in [\omega]^\omega)$ and $(\forall x \forall y)(x \in F \wedge x \subseteq y \rightarrow y \in F)$,

(ii) $(\forall f \in \omega^\omega \exists x \in F)(\forall n \in \omega)(|f(n) \cap x| < n)$.

Clearly, if the Sierpinski set has the cardinality of the continuum then the real line cannot be the union of less than $2^{\aleph_0}$-many measure zero sets.

In [Ju] it was remarked that if the reals are not the union of less than $2^{\aleph_0}$-many meager sets then there exists a rapid filter on ω. Therefore it was asked: if the reals are not the union of less than $2^{\aleph_0}$-many measure zero sets then does there exist a rapid filter on ω? The next theorem will answer this question negatively.

Theorem. cons(ZF) *implies* cons$(ZFC + 2^{\aleph_0}$ *is arbitrarily larger than* $\aleph_2$ + *there exists a Sierpinski set of cardinality* $2^{\aleph_0}$ + *there are no rapid filters on* $\omega)$.

This theorem has some applications. For example, the existence of a Sierpinski set of cardinality $2^{\aleph_0}$ implies that every Δ^1_2-set of reals is measurable (see [JSh1]); also in this model $\omega_1^L = \omega_1$, and therefore, we get a model for "Every Δ^1_2-set of reals is Lebesgue measurable $+\omega_1^L = \omega_1+$ there is no rapid filter on ω." This says that it is impossible to improve the following result of Raisonnier [Ra]:

"If every Σ^1_2-set of reals is Lebesgue measurable and $\omega_1^L = \omega_1$ then there is a rapid filter on ω."

We prove this theorem in §2. The model is gotten by adding ω_2-many Mathias reals and afterward adding random reals. It was remarked by A. Miller in [Mi1] that in the model obtained by iterating ω_2-Mathias reals over L there is no rapid filter on ω.

We assume that the reader knows the material given in [Ba], about countable support iterated forcing and forcing notion satisfying the Axiom A (for the notation). The rest of the notation is standard.

1. Q-SETS

In this section we build a model of set theory where there exists a Q-set of reals and there exists an outer measure one set of reals of cardinality $\aleph_1$. This is the model given in 1.6. For the basic definitions the reader may consult the introduction (§0) and Fleissner [Fl]. We also need some definitions used in the construction.

1.1. **Definition.** $\overline{A} = \langle a_i, A_i : i < \omega_1 \rangle$ is a suitable sequence if and only if

(a) $A_i \in [\omega]^\omega$ for every $i \in \omega_1$;

(b) if $i < j < \omega_1$ then $A_i \subseteq^* A_j$ $(\exists n(A_i - n \subseteq A_j))$ and $A_j - A_i \in [\omega]^\omega$;

(c) $a_i \in [A_{i+1} - A_i]^\omega$ for every $i \in \omega_1$.

1.2. **Definition.** For $\overline{A} = \langle a_i, A_i : i < \omega_1 \rangle$ suitable, and $X \subseteq \omega_1$ we define the partially ordered set $P(\overline{A}, X)$ by stipulating that h belongs to $P(\overline{A}, X)$ if and only if

(i) h is a partial function from ω to $\{0, 1\}$;

(ii) there exists $i = i(h)$ such that

$$\operatorname{Dom} h \subseteq^* A_i \quad \text{(take such } i \text{ minimal)};$$

(iii) for every $j < i(h)$ we have

$$a_j \subseteq^* \operatorname{Dom}(h),$$
$$\text{if } j \in X \text{ then } a_j \subseteq^* h^{-1}(\{1\}),$$
$$\text{if } j \notin X \text{ then } a_j \subseteq^* h^{-1}(\{0\}).$$

For $h_1, h_2 \in P(\overline{A}, X)$ we set $h_1 \leq h_2$ if and only if $h_1 \subseteq h_2$.

1.3. **Lemma.** *If $\overline{A} = \langle a_i, A_i : i < \omega_1 \rangle$ and $X \subseteq \omega_1$, $P(\overline{A}, X)$ are as in* 1.2 *and $h \in P(\overline{A}, X)$, hence $i(h) = \alpha$ is well defined, $\alpha < \beta < \omega_1$, then there exists $h^* \in P(\overline{A}, X)$ such that*

$$h \subseteq h^* \quad \textit{and} \quad i(h^*) \geq \beta.$$

Proof. There exists $g : [\alpha, \beta) \to \omega$ such that

(a) $\alpha \leq \gamma < \beta$ implies $(\operatorname{Dom} h) \cap a_\gamma \subseteq g(\gamma) \supseteq a_\gamma - A_\beta$;

(b) $\alpha \leq \gamma < \delta < \beta$ implies $(a_\gamma - g(\gamma)) \cap (a_\delta - g(\delta)) = \varnothing$ (simply let $\langle \gamma_l : l < l^* \leq \omega \rangle = [\alpha, \beta)$ and construct $g(\gamma_l)$ by induction on l).

Now $\operatorname{Dom} h^* = (\operatorname{Dom} h) \cup \bigcup_{\gamma \in [\alpha, \beta)} (a_\gamma - g(\gamma))$ and

$$h^*(n) = \begin{cases} h(n) & \text{if } n \in \operatorname{Dom} h, \\ 0 & \text{if } n \in a_\gamma - g(\gamma) \text{ and } \gamma \notin X, \\ 1 & \text{if } n \in a_\gamma - g(\gamma) \text{ and } \gamma \in X. \end{cases} \quad \square$$

1.4. **Lemma.** *Let V be a model of ZFC satisfying*

(i) *$\overline{A} = \langle a_i, A_i : i < \omega_1 \rangle$ is suitable, $\overline{A} \in V$;*

(ii) *for every $X \subseteq \omega_1$ there exists $M \subseteq V$ such that $X \in M$, $\overline{A} \in M$, and therefore, $P(\overline{A}, X)$ is definable in M;*

(iii) *there exists $G \in V$ such that $G \subseteq P(\overline{A}, X) \cap M$ and G is generic over M.*

Then $B(\overline{A}) = \{f \in 2^\omega : (\exists i < \omega_1)(\operatorname{char}(a_i) = f)\}$ is a Q-set in V.

Proof. Use 1.3 and the hypothesis. □

1.5. **Definition.** Let $\overline{Q} = \langle P_i ; Q_j : i < \omega_2, j < \omega_2 \rangle$ be a countable support iterated forcing system satisfying

(a) $Q_0 = \langle \{ \langle a_i, A_i : i < \alpha \rangle : a < \omega_1$ and $\langle a_i, A_i : i < \alpha \rangle$ is an initial segment of a suitable sequence $\}, \subseteq \rangle$.

Let $\overline{\mathbf{A}}$ be the Q_0-name of the suitable sequence generated by the Q_0-generic object.

(b) Let $0 < i < \omega_2$; then there exists a P_i-name $\mathbf{X}$ such that

$$\Vdash_{P_i} \text{"}\mathbf{X} \subseteq \omega_1 \text{ and } \mathbf{Q}_i = P(\overline{\mathbf{A}}, \mathbf{X})\text{"}.$$

(c) If $i < \omega_2$ and $\mathbf{X}$ is a P_i-name such that

$$\Vdash_{P_i} \text{"}\mathbf{X} \subseteq \omega_1\text{"}$$

then there exists $j \geq i$ and $\mathbf{Y}$ a P_j-name satisfying

$$\Vdash_{P_j} \text{"}\mathbf{X}[\mathbf{G} \restriction i] = \mathbf{Y} \text{ and } Q_j = P(\overline{\mathbf{A}}, \mathbf{Y})\text{"}.$$

1.6. **Theorem.** *Let $P\omega_2$ be the directed limit of the iterated forcing system $\overline{Q}$ defined in* 1.5. *Let $G \subseteq P\omega_2$ be generic over $V \models$ "GCH". Then the following holds*:

(a) *For every $i < \omega_2$*

$$\Vdash_{P_i} \text{"}Q_i \textit{ satisfies } \aleph_2 - \text{c.c"}.$$

Therefore $P\omega_2$ satisfies $\aleph_2 -$ c.c.

(b) *$P\omega_2$ is a Proper Forcing notion, moreover $P\omega_2$ satisfies the Sacks property. Therefore $V[G] \models$ "$2^\omega \cap V$ has outer measure one*" (*see* 1.8).

(c) *If $V[G]$ we have*

$$B(\overline{\mathbf{A}}[G]) \textit{ is a } Q\textit{-set}.$$

Proof. (a) easy; (c) use 1.4. The proof of (b) is sharp:

(In this work we say that a forcing notion P satisfies the Sacks property iff $(\forall \mathbf{f} \in V^P \forall p \in P)$ (if $p \Vdash_P$ "$\mathbf{f} \in^\omega V$" then $(\exists q \geq p \exists g \in V \cap^\omega V)(q \Vdash$ "$\mathbf{f}(n) \in g(n)$") and $(\forall n \in \omega)(|g(n)| \leq 2^{n^2})$.)

Let χ be sufficiently large, and $p \in P\omega_2$. Let N be such that

$$N \prec \langle H(\chi), \varepsilon, \leq^* \rangle \quad (\leq^* \text{ is some fixed well order}),$$
$$p \in N, \qquad \overline{Q}, P\omega_2 \in N, \qquad \|N\| = \aleph_0.$$

Set $\delta = N \cap \omega_1$, and let $\langle w_n : n < \omega \rangle$ be such that $\bigcup \{ w_n : n < \omega \} = N \cap \omega_2 - \{0\}$

$$w_n \subsetneqq w_{n+1}, \qquad |w_n| = n.$$

Also let $\langle \tau_n : n < \omega \rangle$ be an enumeration of the $P\omega_2$-names of ordinal numbers that belong to N. Let $\langle \alpha_n : n < \omega \rangle$ be such that $\alpha_n < \alpha_{n+1}$ and $\sup_{n<\omega} \alpha_n = \delta$. And fix $\mathbf{h} \in N$ such that $\mathbf{h}$ is a $P\omega_2$-name of a function from ω to V.

$(*)$ We will choose, by induction on ω, conditions $p_n \in P\omega_2$ and finite sets u_n such that

(i) $p \leq p_n \leq p_{n+1} \in P\omega_2 \cap N$;
(ii) $p_{n+1} \Vdash$ "$\tau_n \in N \cap \mathrm{Ord}$";
(iii) there exists $b_n \in [V]^{2^{n^2}}$ such that

$$p_{n+1} \Vdash \text{“}\mathbf{h}(n) \in b_n\text{”};$$

(iv) $u_n \subseteq u_{n+1} \subseteq \omega$ and $\|u_n\| = n$;
(v) if $\beta \in w_{m+1} - w_m$ and $m \leq n$ then

$$p_n \lceil \beta \Vdash \text{“}\mathrm{Dom}(p_n(\beta)) \cap u_n \subseteq u_{m+1}\text{”};$$

(vi) $\alpha_n < lg(p_n(0))$ and $u_n \cap A_{\alpha_m} \subseteq u_{m+1}$, for $m < n$.

Claim. $(*)$ implies (b).

Proof of the claim. Let p_ω be defined by

$$p_\omega(0) = \langle a_j, A_j : i < \delta \rangle \cup \langle A_\delta \rangle$$

when for every n

$$p_n(0) = \langle a_j, A_j : i < lg(p_n(0)) \rangle \quad \text{and} \quad A_\delta = \omega - \bigcup_{n \in \omega} u_n .$$

If $\beta \in \bigcup_n \mathrm{Dom}(p_n) - \{0\}$ then, from the hypothesis, for $\langle p_n(\beta) : n < \omega \rangle$ there exists $p_\omega(\beta) = \bigcup_n p_n(\beta)$ such that for every $n \in \omega$ $p_n(\beta) \subseteq p_\omega(\beta)$. (This holds in V^{P_β} where $p_\omega \lceil \beta$ belongs to the generic set.) Then p_ω is $\langle N, P\omega_2 \rangle$-generic and $p_\omega \Vdash$ "$(\forall n)(\mathbf{h}(n) \in b_n)$". $\square$

Therefore the problem is to show that the inductive construction given in $(*)$ is realizable: suppose that p_n and u_n were given satisfying all conditions of $(*)$.

Let $w_{n+1} = \{\alpha_0 < \alpha_1 < \cdots < \alpha_n < \alpha_{n+1}\}$. We try to extend p_n to a condition satisfying (ii) and (iii):

Notation. If $r \in P(\overline{\mathbf{A}}, \mathbf{X})$ and $h : \omega \to \{0, 1\}$ is finite then $r^{[h]} \in P(\overline{\mathbf{A}}, \mathbf{X})$ when $r^{[h]}$ is such that

$$r^{[h]}(i) = \begin{cases} r(i) & \text{if } i \in \mathrm{Dom}(r) - \mathrm{Dom}(h), \\ h(i) & \text{if } i \in \mathrm{Dom}(h). \end{cases}$$

Clearly for every finite $h : \omega \to \{0, 1\}$ if $r \in P(\overline{\mathbf{A}}, \mathbf{X})$ then $r^{[h]} \in P(\overline{\mathbf{A}}, \mathbf{X})$.

Let $\langle \langle h_l^\alpha : \alpha \in w_{n+1} \rangle : l \leq l_0 \leq 2^{(n+1)^2} \rangle$ be a list of all $\langle h^\alpha : \alpha \in w_{n+1} \rangle$ such that for every $\alpha \in w_{m+1} - w_m$

$$h^\alpha : u_n - u_{m+1} \to \{0, 1\}.$$

Now we choose by induction on $l \leq l_0$, $p_{n,l}$ such that

(a) $p_{n,0} = p_n$.

(b) $p_{n,l} \leq p_{n,l+1} \in P\omega_2 \cap N$.
(c) For every $\alpha \in w_{m+1} - w_m$, $m < n$ we have

$$p_{n,l}\lceil\alpha \Vdash \text{“}\operatorname{Dom} p_{n,l}(\alpha) \cap u_n \subseteq u_{m+1}\text{”}.$$

(d) If $p'_{n,l+1}$ is such that

$$p'_{n,l+1}(\alpha) = \begin{cases} p_{n,l+1}(\alpha)^{[h_l^\alpha]} & \text{if } \alpha \in w_n, \\ p_{n,l+1}(\alpha) & \text{if } \alpha \notin w_n, \end{cases}$$

then $p'_{n,l+1}$ forces values for $\boldsymbol{\tau}_i$ $(i \leq n+1)$ and for $\mathbf{h}(i)$ $(i \leq n+1)$.

The induction.

for $l = 0 : p_{n,0} = p_n$,
for $l+1 : p_{n,l}$ was defined satisfying (a), (b), (c), (d), and let $p^*_{n,l}$ be such that

$$p^*_{n,l}(\alpha) = \begin{cases} p_{n,l}(\alpha)^{[h_l^\alpha]} & \text{if } \alpha \in w_n, \\ p_{n,l}(\alpha) & \text{if } \alpha \notin w_n. \end{cases}$$

There exists $q_{n,l} \geq p^*_{n,l}$ such that $q_{n,l}$ forces values for $\boldsymbol{\tau}_i$ $(i \leq n+1)$, $\mathbf{h}(i)$ $(i \leq n+1)$. Fix such $q_{n,l}$ in N, and then we define $p_{n,l+1}$ satisfying

$$p_{n,l+1}(\alpha) = \begin{cases} q_{n,l}(\alpha)\lceil(\omega - \operatorname{Dom} h_l^\alpha) & \text{if } \alpha \in w_n, \\ q_{n,l}(\alpha) & \text{if } \alpha \notin w_n. \end{cases}$$

Clearly $p_{n,l+1}$ satisfies (a), (b), (c), and (d). P_{n,l_0} is almost p_{n+1}, only we need to extend it in order to find the u_{n+1} required. Clearly p_{n,l_0} fix 2^{n^2}-many possible values to $\mathbf{h}(n)$. The next lemma is exactly what we need.

1.7. **Lemma.** *If $w \subseteq \omega_2$ is finite, and $p \in P\omega_2$ and $n < \omega$ then there exists $k \in \omega$ and $q \in P\omega_2$ such that*

(a) $n < k < \omega$;
(b) $p \leq q$;
(c) *for every* $\alpha \in w$

$$q\lceil\alpha \Vdash \text{“}\operatorname{Dom} q(\alpha) \cap [0, n) = \operatorname{Dom} p(\alpha) \cap [0, n)\text{”};$$

(d) *for every* $\alpha \in w$

$$q\lceil\alpha \Vdash \text{“}k \notin \operatorname{Dom} q(\alpha)\text{”}.$$

Proof. Let $w = \{\alpha_1 < \alpha_2 < \cdots < \alpha_m\}$. We choose by (decreasing) induction

$$q_m, q_{m-1}, \ldots, q_1, q_0$$

satisfying

(i) $q_m = p$,
(ii) $q_l\lceil\alpha_l \leq q_{l-1} \in P_{\alpha_l}$,

(iii) q_{l-1} forces less than $2^{n\times m}$-many possible values for the following names of ordinals (i.e., satisfies condition (c)):

(A) γ_l such that

$$\mathrm{Dom}(q_l(\alpha_l)) =^* [A_{\gamma_l}]^{q_{l-1}(0)}$$

(remember that Q_0 fixes A_{γ_l});

(B) k_l such that

$$1 + \max(\mathrm{Dom}\, q_l(\alpha_l) - [A_{\gamma_l}]^{q_{l-1}(0)}) < k_l$$

(use the above construction in order to get such q_{l-1}).

Therefore there exists k_l^i $(i = 1, \dots, t_l < \omega)$; γ_l^i $(i = 1, \dots, s_l < \omega)$ such that

$$q_l \Vdash_{P_{\alpha_l}} \text{“}(\exists i < t_l \exists j < s_l)(\gamma_l = \gamma_l^j \wedge k_l = k_l^i)\text{”}.$$

Now let

$$\overline{\gamma}_l = \max\{1, \gamma_l', \dots, \gamma_l^{sl}\},$$
$$\bar{k}_l = \max\{1, k_l', \dots, k_l^{t_l}\}$$

and

$$q = \bigcup_{l=0}^{m} q_l, \quad \text{i.e., if } \alpha_0 = -1 \text{ and } \alpha_{m+1} = \omega_2$$

then

$$q(\alpha) = q_l(\alpha) \quad \text{where } \alpha_{l-1} \le \alpha < \alpha_l.$$

Then we choose k such that

$$k > \bar{k}_l (l \le m), \qquad k \notin A_{\gamma 0} \cup A_{\gamma 1} \cup \dots \cup A_{\gamma m}.$$

This concludes the proof of part (b) of Theorem 1.6. $\square$

1.8. **Theorem.** *If V is a model for ZFC and $P \in V$ is a forcing notion satisfying the Sacks property then* $\Vdash_P$ “$\mu^*(2^\omega \cap V) = 1$”.

Proof. Suppose that there exists $p \in P$ such that $p \Vdash_P$ “$\mu(2^\omega \cap V) = 0$”. Then there exists $(\mathbf{I}_n : n < \omega)$ such that for every $n \in \omega$ $\mathbf{I}_n$ is a P-name of a rational interval, and

$$0 \Vdash \text{“}\sum \mu(\mathbf{I}_n) < \infty\text{”},$$
$$p \Vdash \text{“}2^\omega \cap V \subseteq \bigcap_n \bigcup_{m \ge n} \mathbf{I}_n\text{”}.$$

Then there exists $\mathbf{g}$ such that $\mathbf{g}$ is a P-name of a function from ω to ω and satisfying

$$0 \Vdash \text{“}\sum_{m > \mathbf{g}(n)} \mu(\mathbf{I}_n) < \frac{1}{2^{n^2+n}}\text{”}.$$

Fix $q \geq p$ and $g \in^{\omega} \omega$, given by the Sacks property such that

$$q \Vdash \text{“}(\forall n)(\mathbf{g}(n) < g(n))\text{”}.$$

Again using the Sacks property, let $r \geq q$ and $\langle\langle\langle I^k_{j,i} : g(i) \leq k < g(i+1)\rangle : j < 2^{i^2}\rangle : i < \omega\rangle$ satisfying for every $i < \omega$, there exists $j < 2^{i^2}$

$$r \Vdash \text{“}\forall k \in [g(i), g(i+1))(\mathbf{I}_k = I^k_{j,i})\text{”}$$

and under this notation we define

$$J_i = \bigcup_{j<2^{i^2}} \bigcup_{g(i)\leq k<g(i+1)} I^k_{j,i}.$$

Then for every i

$$\mu(J_i) \leq 2^{i^2} \cdot \frac{1}{2^{i^2+i}} = \frac{1}{2^i}.$$

Therefore $\sum_{i\in\omega} \mu(J_i) < \infty$ and this implies that there exists $x \in 2^\omega$ such that $x \notin \bigcap_i \bigcup_{j\geq i} J_j$, and by the hypothesis this implies that

$$r \Vdash \text{“}x \notin \bigcap_i \bigcup_{j\geq i} \mathbf{I}_j\text{”}$$

a contradiction. □

It may be possible that these ideas help to solve the question in [JSh]: remember that the Sacks property implies the Laver property.

2. Rapid filters and Sierpinski sets

2.0. **Theorem.** cons$(ZF) \Rightarrow$ cons$(ZFC$ + *there exists a Sierpinski set* + $2^{\aleph_0}$ *is a regular cardinal* + *there are no rapid filters on* $\omega)$.

Proof. We begin with $V = L$ and let P be the ω_2-iteration of Mathias reals with countable support and let $\mathbf{R}$ be a P-name of the product of $\aleph_\alpha$ random reals (i.e., the measure algebra). Then we will prove that if $G \subseteq P * \mathbf{R}$ then

$(*)$ $\quad V[G] \vDash$ “there are no rapid filters on ω”.

Clearly this is enough in order to obtain the inclusion of the theorem. The proof of $(*)$ will take the remainder of this section.

2.1. **Definition.** Let $\mathbf{a}$ be an R-name of a set of ordinals; then we define

$$\mathbf{a}(n) = \mu(\|n \in \mathbf{a}\|).$$

Then $\mathbf{a}(\,)$ is a function from ordinals to $[0, 1]$.

2.2. *Fact.* Let M be a model of ZFC and let R be any product of random reals, i.e., a measure algebra. Let $\langle n_i : i < \omega\rangle$ be in M an increasing sequence of natural numbers. Assume that $\mathbf{a}$ is in M^R and

$$M \vDash \text{“} \Vdash \text{“}(\forall k \in \omega)(|n_k \cap \mathbf{a}| < k)\text{” ”}.$$

Then for every $k \in \omega$

$$M \models \text{“} \sum_{m=0}^{n_k - 1} \mathbf{a}(m) \leq k \text{”}.$$

Proof. We will work in M. Let $\varphi_i(m)$ be the formula that says

“m is the ith member of a”.

Then $\mathbf{a}(m) = \sum_{i=0}^{m} \mu(\|\varphi_i(m)\|)$. Also if $m \neq n$ then $\mu(\|\varphi_i(m)\| \cdot \|\varphi_i(n)\|) = 0$. Assume that there exists $k \in \omega$ such that

$$\sum_{m=0}^{n_k - 1} \mathbf{a}(m) > k.$$

Then

$$\sum_{m=0}^{n_k - 1} \sum_{i=1}^{m} \mu(\|\varphi_i(m)\|) > k.$$

Therefore

$$\sum_{m=0}^{n_k - 1} \sum_{i=1}^{k} \mu(\|\varphi_i(m)\|) > k \quad \text{(because } m < n_k\text{)}$$

and hence

$$\sum_{i=1}^{k} \sum_{m=0}^{n_k - 1} \mu(\|\varphi_i(m)\|) > k.$$

And thus there exists i such that $\sum_{m=0}^{n_k - 1} \mu(\|\varphi_i(m)\|) > 1$ and this implies that there exists $n \neq m$ such that

$$\mu(\|\varphi_i(m)\| \cdot \|\varphi_i(n)\|) \neq \varnothing,$$

a contradiction. □

From now on we fix $M \models ZFC$. Let P in M be the ω_2-iteration of Mathias reals. Each Mathias real adds a sequence of natural numbers. $P(0)$ is the first coordinate of P. Let $\langle \mathbf{n}_i : i < \omega \rangle$ be a P-name for the sequence added by $P(0)$. Let $\mathbf{R}$ be a P-name for a product of random reals (i.e., a measure algebra). If $H \subseteq P$ is generic over M then $\mathbf{R}[H]$ is the realization of $\mathbf{R}$ in $M[H]$. Let $\mathbf{a} \in M[H]$ be an $\mathbf{R}[H]$-name for a sequence of natural numbers such that

$$M[H] \models \text{“} \Vdash_{\mathbf{R}[H]} \text{“} |\mathbf{n}_k[H] \cap \mathbf{a}| < k \text{” ”}.$$

Therefore, by 2.2,

$$M[H] \models \sum_{m=0}^{n_k[H] - 1} \mathbf{a}(m) \leq k.$$

The function $\mathbf{a}(\ ) : \omega \to [0, 1]$, defined by $\mathbf{a}$, lies in $M[H]$, so it has a P-name. Let $\mathbf{f}$ be such a name (we omit this relation with $\mathbf{a}$ because it is clear).

2.3. **Lemma.** *Let* $p \in P$, *and* $\varepsilon > 0$ *given; then there exists* $w_0 \subseteq w_1 \subseteq \cdots \subseteq w_n \subseteq \cdots$, p_1^n, p_2^n, $k_0 < k_1 < \cdots < k_n < \cdots$, $B_i^0 \subseteq \cdots \subseteq B_i^n \subseteq \cdots$, $i = 1, 2$, *satisfying*

(a) *for every* n, $w_n \subseteq \omega_2$; $k_n \in \omega$; $p_i^n \in P$, $i = 1, 2$; $B_i^n \in [\omega]^{<\omega}$, $i = 1, 2$; $B_1^n \cap B_2^n = B_1^0 \cap B_2^0$, $\bigcup B_1^n \cup \bigcup B_2^n = \omega$.

(b) $\bigcup_n \{\mathrm{Dom}(p_1^n) \cup \mathrm{Dom}(p_2^n)\} = \bigcup_n w_n$.

(c) *For every* n, $p_i^n \leq_{w_n}^n p_i^{n+1}$, $i = 1, 2$ (*see* [Ba, §7]).

(d) $p_i^0 = p$ *for* $i = 1, 2$.

(e) *If* m *is even, then for every* $k > k_m$ *there exists* $q_{w_{m+1}}^{m+1} \geq p_1^m$ *such that*

(i) $q \Vdash_P$ "$\sum_{l=k_m}^k \mathbf{f}(l) < \varepsilon/10^m$",

(ii) $q \Vdash_P$ "$\sum_{\substack{l \notin B_1^m \\ l < k_m}} \mathbf{f}(l) < \varepsilon\left(\frac{1}{10} + \frac{1}{10^2} + \cdots + \frac{1}{10^{m-1}}\right)$",

(iii) $p_2^m \Vdash_P$ "$\sum_{\substack{l \notin B_2^m \\ l < k_m}} \mathbf{f}(l) < \varepsilon\left(\frac{1}{10} + \cdots + \frac{1}{10^{m-1}}\right)$".

(f) *If* m *is odd, then for every* $k > k_m$ *there exists* $q_{w_{m+1}}^{m+1} \geq p_2^m$ *such that*

(i) $q \Vdash$ "$\sum_{l=k_m}^k \mathbf{f}(l) < \varepsilon/10^m$",

(ii) $q \Vdash$ "$\sum_{\substack{l \notin B_2^m \\ l < k_m}} \mathbf{f}(l) < \varepsilon\left(\frac{1}{10} + \cdots + \frac{1}{10^{m-1}}\right)$",

(iii) $p_1^m \Vdash \sum_{\substack{l \notin B_1^m \\ l < k_m}} \mathbf{f}(l) < \varepsilon\left(\frac{1}{10} + \cdots + \frac{1}{10^{m-1}}\right)$".

(g) $0 \in w_0$.

Proof. By induction on m.

$m = n + 1$: By the symmetry of (e) and (f) without loss of generality n is even. In this stage we need to take care of the relation $\leq_{w_n}^n$.

We define $w_{n+1} \supseteq w_n$ such that w_{n+1} contains the first n numbers of each $\sup(p_1^j) \cup \sup(p_2^j)$, for $j \leq n$, in some fixed enumeration of $\sup(p_1^j) \cup \sup(p_2^j)$.

For every natural number ζ, there exists $p_2^{n,\zeta}$, $\langle f^{\zeta,l} : l < 2^{|w_{n+1}| \times n+1} \rangle$ such that

(α) $p_2^n \leq_{w_{n+1}}^{n+1} p_2^{n,\zeta} \in P$,

(β) The $n+2$-th member of the infinite part of $p_2^{n,\zeta}(0)$ is larger than ζ,

(γ) $p_2^{n,\zeta} \Vdash$ "$\exists l < 2^{|w_{n+1}| \times n+1} \forall \xi < \zeta (|f^{\zeta,l}(\xi) - \mathbf{f}(\xi)| < 2^{-t(\zeta)})$" where $t(\zeta) = 2^{2^{2^{\zeta+n+1+|w_{n+1}|}}}$.

(Use [Ba, §9.5] and an approximation to $\mathbf{f}(\xi)$ with error $t(\zeta)$.)

Now let $\langle \zeta_i : i < \omega \rangle$ be such that for every $\xi \in \omega$, and for every $l < 2^{|w_{n+1}| \times n+1}$ the sequence $\langle f^{\zeta_i,l}(\xi) : i < \omega \rangle$ converges to a real number in $[0, 1]$. We call this limit $f^{*,l}$. (Use a diagonal argument and the compactness theorem.)

Claim. For every $k \in \omega$ $\sum_{k<\xi<\omega} f^{*,l}(\xi) \leq n + 2$.

Proof. If not, then there exists $\bar{k} > k$ such that

$$\sum_{k<\xi\leq\bar{k}} f^{*,l}(\xi) > n+2 .$$

Then there exists i such that $\zeta_i > \bar{k}$ and

$$\sum_{k<\xi\leq\bar{k}} f^{\zeta_i,l}(\xi) > n+2$$

and this is a contradiction to 2.2. □

Therefore there exists $k_{n+1} > k_n$ such that for each $l < 2^{|w_{n+1}|\times n+1}$ we have that

$$\sum_{k_{n+1}<\xi}^{\omega} f^{*,l}(\xi) < \varepsilon/10^{n+8} .$$

Then we define

$$\begin{aligned} B_1^{n+1} &= B_1^n \cup [k_n , k_{n+1}) , \\ B_2^{n+1} &= B_2^n , \\ p_2^{n+1} &= p_2^n . \end{aligned}$$

Now, by the induction hypothesis, p_1^n has an extension p_1^{n+1} such that

$$p_1^{n+1} \Vdash \sum_{\substack{l \notin B_1^{n+1} \\ l<k_{n+1}}} \mathbf{f}(l) < \varepsilon \left(\frac{1}{10} + \cdots + \frac{1}{10^{n-1}} + \frac{1}{10^n} \right) .$$

This completes the induction. The reader may check that this works.

Let $\mathbf{U}$ be a P-name of the $\mathbf{R}$-name for a rapid filter. Then using 2.3 we get the following

Conclusion 1. There exists p_1 , p_2 , B_1 , B_2 such that

$$(*) \qquad p_1 \Vdash_P \mu(\|B_2 \in \mathbf{U}\|) < \varepsilon ,$$

$$(**) \qquad p_2 \Vdash_P \mu(\|B_1 \in \mathbf{U}\|) < \varepsilon ,$$

and $B_1 \cap B_2$ is finite, $B_1 \cup B_2 = \omega$.

Using Conclusion 1, we get the following

Conclusion 2. There exists $\delta < \omega_2$, such that for every $\alpha < \delta(\mathrm{cof}(\delta) = \omega_1)$ and for every $\langle B_1 , B_2\rangle \in V[G\lceil\alpha]$ (where $G\lceil\alpha$ is $\mathbf{G}\lceil P_\alpha$) if there exists β such that in $V[G\lceil\beta]$ Conclusion 1 holds for B_1 and B_2, then there exists $\beta < \delta$ such that in $V[G\lceil\beta]$ Conclusion 1 holds, and $\mathrm{supp}(p_1) \cup \mathrm{supp}(p_2) \subset \delta$.

Now using Lemma 2.3, working in $V[G\lceil\delta]$, there exists B_1 , B_2 , p_1 , p_2 such that

$$V[G\lceil\delta] \models \text{“}(*) \wedge (**) \wedge B_1 \cup B_2 = \omega \wedge B_1 \cap B_2 \text{ is finite”}.$$

(Remember that $P \cong P_{\delta\omega_2}$.) Then, by the hypothesis on δ, and using the fact that P_δ is proper, we have that there exists $\beta < \delta$ and p_1', p_2' such that

$$B_i \in V[G\lceil\beta], \quad i = 1, 2,$$

and in $V[G\lceil\beta]$ we have (*), (**) for p_1', p_2' respectively.

Without loss of generality $p_1' \in G\lceil\delta$. Now $p_1' \cup p_2$ is a condition and w.l.o.g. $p_1' \cup p_2 \in G\lceil\omega_2$. Therefore in $V[G\lceil\omega_2]$ we have

(a) $\mu(\|B_1 \in \mathbf{U}[G\lceil\omega_2\|) < \varepsilon$,
(b) $\mu(\|B_2 \in \mathbf{U}[G\lceil\omega_2\|) < \varepsilon$.

If ε was chosen small, we can deduce that $\mu(\|B_1 \cap B_2 \in \mathbf{U}[G\lceil\omega_2]\|) \neq 0$, and this implies that

$$\mu(\|\mathbf{U}[G\lceil\omega_2] \text{ is not a filter}\|) > 0 \quad (\text{because } B_1 \cap B_2 \text{ is finite}),$$

a contradiction.

References

[Ba] J. Baumgartner, *Iterated forcing*, Surveys in Set Theory (A. R. D. Mathias, ed.), London Math Soc. Lecture Notes Series 87, Cambridge Univ. Press, 1983.

[Fl] W. Fleissner, *Current research on Q-sets*, Topology, vol. 1, Colloq. Math. Soc. Janos Bolyai, vol. 23, North-Holland, 1980.

[Ju] H. Judah, *Strong measure zero sets and rapid filters*, J. Symbolic Logic **53** (1988).

[JSh1] H. Judah and S. Shelah, *Δ^1_2-set of reals*, Ann. Pure Appl. Logic **42**.

[JSh2] ——, *Q-set do not necessarily have strong measure zero*, Proc. Amer. Math. Soc. **102** (1988).

[Ku] K. Kunen, *Set theory*, North-Holland, 1980.

[Mi1] A. Miller, *There are no Q-points in Laver's model for the Borel conjecture*, Proc. Amer. Math. Soc. **178** (1980).

[Mi2] ——, *Special subsets of the real line*, Handbook of Set-Theoretic Topology (K. Kunen and J. E. Vaughan, eds.), North-Holland, 1984.

[Ra] J. Raisonnier, *A mathematical proof of S. Shelah's theorem on the measure problem and related results*, Israel J. Math. **48** (1984).

[Sh] S. Shelah, *Proper forcing*, Lecture Notes in Math., vol. 940, Springer-Verlag, New York.

Department of Mathematics, Bar Ilan University, 52900 Ramat Gan, Israel

Institute of Mathematics, The Hebrew University of Jerusalem, Israel