

***Q*-SETS, SIERPINSKI SETS, AND RAPID FILTERS**

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(Communicated by Andreas R. Blass)

ABSTRACT. In this work we will prove the following:

Theorem 1. *cons(ZF) implies cons(ZFC + there exists a Q -set of reals + there exists a set of reals of cardinality \aleph_1 which is not Lebesgue measurable).*

Theorem 2. *cons(ZF) implies cons(ZFC + 2^{\aleph_0} is arbitrarily larger than \aleph_2 + there exists a Sierpinski set of cardinality 2^{\aleph_0} + there are no rapid filters on ω).*

These theorems give answers to questions of Fleissner [Fl] and Judah [Ju].

0. INTRODUCTION

In this work we will solve two open problems about special sets of the reals. In order to state them we need some definitions.

0.1. Definition. A set of reals A is a Q -set iff every subset of A is a relative F_σ , i.e., it is a countable union of relatively closed subsets of A .

Q -sets are very strange: for example $2^{\aleph_0} < 2^{\aleph_1}$ implies that there are no Q -sets of cardinality \aleph_1 . Also Q -sets have universal measure zero, but they do not necessarily have strong measure zero (see [Fl, JSh2, Mi2]).

In [Fl] it is asked if the existence of a Q -set of cardinality \aleph_1 implies that every \aleph_1 -set of reals is of Lebesgue measure zero. Our first theorem answers this question negatively by showing

Theorem. *cons(ZF) implies cons(ZFC + there exists a Q -set of reals + there exists a set of reals of cardinality \aleph_1 which is not Lebesgue measurable).*

We show this theorem as follows. We begin by forcing a set A of reals of cardinality \aleph_1 , and then we force, with a countable support iteration of length ω_2 , making A a Q -set in the generic extension. We prove that this composition of forcing notions satisfies the Sacks property (studied in [Sh]) and, in the end

Received by the editors December 4, 1987 and, in revised form, April 5, 1990.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 03E35; Secondary 03E15.

The first author would like to thank the NSF for its partial support under grant DMS-8701828.

The second author thanks the United States–Israel Binational Foundation for partially supporting his research.

of the section, we prove that if a forcing notion has the Sacks property then in the generic extension the old reals have outer measure one. Clearly this implies, if we begin from L , that in our generic extension there exists an uncountable Q -set and a \aleph_1 -set of reals which is not Lebesgue measurable.

0.2. Definition. (a) A set of reals A is a Sierpinski set iff for every measure zero set M , $A \cap M$ is countable.

(b) $[\omega]^\omega = \{x : x \subseteq \omega \wedge |x| = \aleph_0\}$; $[\omega]^{<\omega} = \{x : x \subseteq \omega \wedge |x| < \aleph_0\}$.

(c) A subset $F \subseteq [\omega]^\omega$ is a rapid filter iff

(i) $(\forall x, y \in F)(x \cap y \in [\omega]^\omega)$ and $(\forall x \forall y)(x \in F \wedge x \subseteq y \rightarrow y \in F)$,

(ii) $(\forall f \in \omega^\omega \exists x \in F)(\forall n \in \omega)(|f(n) \cap x| < n)$.

Clearly, if the Sierpinski set has the cardinality of the continuum then the real line cannot be the union of less than 2^{\aleph_0} -many measure zero sets.

In [Ju] it was remarked that if the reals are not the union of less than 2^{\aleph_0} -many meager sets then there exists a rapid filter on ω . Therefore it was asked: if the reals are not the union of less than 2^{\aleph_0} -many measure zero sets then does there exist a rapid filter on ω ? The next theorem will answer this question negatively.

Theorem. *cons(ZF) implies cons(ZFC + 2^{\aleph_0} is arbitrarily larger than \aleph_2 + there exists a Sierpinski set of cardinality 2^{\aleph_0} + there are no rapid filters on ω).*

This theorem has some applications. For example, the existence of a Sierpinski set of cardinality 2^{\aleph_0} implies that every Δ_2^1 -set of reals is measurable (see [JSh1]); also in this model $\omega_1^L = \omega_1$, and therefore, we get a model for "Every Δ_2^1 -set of reals is Lebesgue measurable + $\omega_1^L = \omega_1$ + there is no rapid filter on ω ." This says that it is impossible to improve the following result of Raisonnier [Ra]:

"If every Σ_2^1 -set of reals is Lebesgue measurable and $\omega_1^L = \omega_1$ then there is a rapid filter on ω ."

We prove this theorem in §2. The model is gotten by adding ω_2 -many Mathias reals and afterward adding random reals. It was remarked by A. Miller in [Mil] that in the model obtained by iterating ω_2 -Mathias reals over L there is no rapid filter on ω .

We assume that the reader knows the material given in [Ba], about countable support iterated forcing and forcing notion satisfying the Axiom A (for the notation). The rest of the notation is standard.

1. Q -SETS

In this section we build a model of set theory where there exists a Q -set of reals and there exists an outer measure one set of reals of cardinality \aleph_1 . This is the model given in 1.6. For the basic definitions the reader may consult the

introduction (§0) and Fleissner [Fl]. We also need some definitions used in the construction.

1.1. **Definition.** $\bar{A} = \langle a_i, A_i : i < \omega_1 \rangle$ is a suitable sequence if and only if

- (a) $A_i \in [\omega]^\omega$ for every $i \in \omega_1$;
- (b) if $i < j < \omega_1$ then $A_i \subseteq^* A_j$ ($\exists n(A_i - n \subseteq A_j)$) and $A_j - A_i \in [\omega]^\omega$;
- (c) $a_i \in [A_{i+1} - A_i]^\omega$ for every $i \in \omega_1$.

1.2. **Definition.** For $\bar{A} = \langle a_i, A_i : i < \omega_1 \rangle$ suitable, and $X \subseteq \omega_1$ we define the partially ordered set $P(\bar{A}, X)$ by stipulating that h belongs to $P(\bar{A}, X)$ if and only if

- (i) h is a partial function from ω to $\{0, 1\}$;
- (ii) there exists $i = i(h)$ such that

$$\text{Dom } h \subseteq^* A_i \quad (\text{take such } i \text{ minimal});$$

- (iii) for every $j < i(h)$ we have

$$\begin{aligned} a_j &\subseteq^* \text{Dom}(h), \\ \text{if } j \in X &\text{ then } a_j \subseteq^* h^{-1}(\{1\}), \\ \text{if } j \notin X &\text{ then } a_j \subseteq^* h^{-1}(\{0\}). \end{aligned}$$

For $h_1, h_2 \in P(\bar{A}, X)$ we set $h_1 \leq h_2$ if and only if $h_1 \subseteq h_2$.

1.3. **Lemma.** If $\bar{A} = \langle a_i, A_i : i < \omega_1 \rangle$ and $X \subseteq \omega_1$, $P(\bar{A}, X)$ are as in 1.2 and $h \in P(\bar{A}, X)$, hence $i(h) = \alpha$ is well defined, $\alpha < \beta < \omega_1$, then there exists $h^* \in P(\bar{A}, X)$ such that

$$h \subseteq h^* \quad \text{and} \quad i(h^*) \geq \beta.$$

Proof. There exists $g : [\alpha, \beta) \rightarrow \omega$ such that

- (a) $\alpha \leq \gamma < \beta$ implies $(\text{Dom } h) \cap a_\gamma \subseteq g(\gamma) \supseteq a_\gamma - A_\beta$;
- (b) $\alpha \leq \gamma < \delta < \beta$ implies $(a_\gamma - g(\gamma)) \cap (a_\delta - g(\delta)) = \emptyset$ (simply let $\langle \gamma_l : l < l^* \leq \omega \rangle = [\alpha, \beta)$ and construct $g(\gamma_l)$ by induction on l).

Now $\text{Dom } h^* = (\text{Dom } h) \cup \bigcup_{\gamma \in [\alpha, \beta)} (a_\gamma - g(\gamma))$ and

$$h^*(n) = \begin{cases} h(n) & \text{if } n \in \text{Dom } h, \\ 0 & \text{if } n \in a_\gamma - g(\gamma) \text{ and } \gamma \notin X, \\ 1 & \text{if } n \in a_\gamma - g(\gamma) \text{ and } \gamma \in X. \quad \square \end{cases}$$

1.4. **Lemma.** Let V be a model of ZFC satisfying

- (i) $\bar{A} = \langle a_i, A_i : i < \omega_1 \rangle$ is suitable, $\bar{A} \in V$;
- (ii) for every $X \subseteq \omega_1$ there exists $M \subseteq V$ such that $X \in M$, $\bar{A} \in M$, and therefore, $P(\bar{A}, X)$ is definable in M ;
- (iii) there exists $G \in V$ such that $G \subseteq P(\bar{A}, X) \cap M$ and G is generic over M .

Then $B(\bar{A}) = \{f \in 2^\omega : (\exists i < \omega_1)(\text{char}(a_i) = f)\}$ is a Q-set in V .

Proof. Use 1.3 and the hypothesis. \square

1.5. **Definition.** Let $\bar{Q} = \langle P_i; Q_j : i < \omega_2, j < \omega_2 \rangle$ be a countable support iterated forcing system satisfying

- (a) $Q_0 = \langle \{ \langle a_i, A_i : i < \alpha \rangle : a < \omega_1 \text{ and } \langle a_i, A_i : i < \alpha \rangle \text{ is an initial segment of a suitable sequence } \}, \subseteq \rangle$.

Let \bar{A} be the Q_0 -name of the suitable sequence generated by the Q_0 -generic object.

- (b) Let $0 < i < \omega_2$; then there exists a P_i -name \mathbf{X} such that

$$\Vdash_{P_i} \text{“} \mathbf{X} \subseteq \omega_1 \text{ and } Q_i = P(\bar{A}, \mathbf{X}) \text{”}.$$

- (c) If $i < \omega_2$ and \mathbf{X} is a P_i -name such that

$$\Vdash_{P_i} \text{“} \mathbf{X} \subseteq \omega_1 \text{”}$$

then there exists $j \geq i$ and \mathbf{Y} a P_j -name satisfying

$$\Vdash_{P_j} \text{“} \mathbf{X}[G \upharpoonright i] = \mathbf{Y} \text{ and } Q_j = P(\bar{A}, \mathbf{Y}) \text{”}.$$

1.6. **Theorem.** Let $P\omega_2$ be the directed limit of the iterated forcing system \bar{Q} defined in 1.5. Let $G \subseteq P\omega_2$ be generic over $V \models \text{“}GCH\text{”}$. Then the following holds:

- (a) For every $i < \omega_2$

$$\Vdash_{P_i} \text{“} Q_i \text{ satisfies } \aleph_2 - \text{c.c.} \text{”}.$$

Therefore $P\omega_2$ satisfies $\aleph_2 - \text{c.c.}$

- (b) $P\omega_2$ is a Proper Forcing notion, moreover $P\omega_2$ satisfies the Sacks property. Therefore $V[G] \models \text{“}2^\omega \cap V \text{ has outer measure one”}$ (see 1.8).

- (c) If $V[G]$ we have

$$B(\bar{A}[G]) \text{ is a } Q\text{-set.}$$

Proof. (a) easy; (c) use 1.4. The proof of (b) is sharp:

(In this work we say that a forcing notion P satisfies the Sacks property iff $(\forall f \in V^P \forall p \in P)$ (if $p \Vdash_P \text{“} f \in {}^\omega V \text{”}$ then $(\exists q \geq p \exists g \in V \cap {}^\omega V)(q \Vdash \text{“} f(n) \in g(n) \text{”})$ and $(\forall n \in \omega)(|g(n)| \leq 2^{n^2})$.)

Let χ be sufficiently large, and $p \in P\omega_2$. Let N be such that

$$N < \langle H(\chi), \varepsilon, \leq^* \rangle \quad (\leq^* \text{ is some fixed well order}), \\ p \in N, \quad \bar{Q}, P\omega_2 \in N, \quad \|N\| = \aleph_0.$$

Set $\delta = N \cap \omega_1$, and let $\langle w_n : n < \omega \rangle$ be such that $\bigcup \{w_n : n < \omega\} = N \cap \omega_2 - \{0\}$

$$w_n \subsetneq w_{n+1}, \quad |w_n| = n.$$

Also let $\langle \tau_n : n < \omega \rangle$ be an enumeration of the $P\omega_2$ -names of ordinal numbers that belong to N . Let $\langle \alpha_n : n < \omega \rangle$ be such that $\alpha_n < \alpha_{n+1}$ and $\sup_{n < \omega} \alpha_n = \delta$. And fix $\mathbf{h} \in N$ such that \mathbf{h} is a $P\omega_2$ -name of a function from ω to V .

(*) We will choose, by induction on ω , conditions $p_n \in P\omega_2$ and finite sets u_n such that

- (i) $p \leq p_n \leq p_{n+1} \in P\omega_2 \cap N$;
- (ii) $p_{n+1} \Vdash \text{“}\tau_n \in N \cap \text{Ord”}$;
- (iii) there exists $b_n \in [V]^{2^{n^2}}$ such that

$$p_{n+1} \Vdash \text{“}\mathbf{h}(n) \in b_n\text{”};$$

- (iv) $u_n \subseteq u_{n+1} \subseteq \omega$ and $\|u_n\| = n$;
- (v) if $\beta \in w_{m+1} - w_m$ and $m \leq n$ then

$$p_n \upharpoonright \beta \Vdash \text{“}\text{Dom}(p_n(\beta)) \cap u_n \subseteq u_{m+1}\text{”};$$
- (vi) $\alpha_n < lg(p_n(0))$ and $u_n \cap A_{\alpha_m} \subseteq u_{m+1}$, for $m < n$.

Claim. (*) implies (b).

Proof of the claim. Let p_ω be defined by

$$p_\omega(0) = \langle a_j, A_j : i < \delta \rangle \cup \langle A_\delta \rangle$$

when for every n

$$p_n(0) = \langle a_j, A_j : i < lg(p_n(0)) \rangle \quad \text{and} \quad A_\delta = \omega - \bigcup_{n \in \omega} u_n.$$

If $\beta \in \bigcup_n \text{Dom}(p_n) - \{0\}$ then, from the hypothesis, for $\langle p_n(\beta) : n < \omega \rangle$ there exists $p_\omega(\beta) = \bigcup_n p_n(\beta)$ such that for every $n \in \omega$ $p_n(\beta) \subseteq p_\omega(\beta)$. (This holds in V^{P_β} where $p_\omega \upharpoonright \beta$ belongs to the generic set.) Then p_ω is $\langle N, P\omega_2 \rangle$ -generic and $p_\omega \Vdash \text{“}(\forall n)(\mathbf{h}(n) \in b_n)\text{”}$. \square

Therefore the problem is to show that the inductive construction given in (*) is realizable: suppose that p_n and u_n were given satisfying all conditions of (*).

Let $w_{n+1} = \{\alpha_0 < \alpha_1 < \dots < \alpha_n < \alpha_{n+1}\}$. We try to extend p_n to a condition satisfying (ii) and (iii):

Notation. If $r \in P(\bar{A}, \mathbf{X})$ and $h : \omega \rightarrow \{0, 1\}$ is finite then $r^{[h]} \in P(\bar{A}, \mathbf{X})$ when $r^{[h]}$ is such that

$$r^{[h]}(i) = \begin{cases} r(i) & \text{if } i \in \text{Dom}(r) - \text{Dom}(h), \\ h(i) & \text{if } i \in \text{Dom}(h). \end{cases}$$

Clearly for every finite $h : \omega \rightarrow \{0, 1\}$ if $r \in P(\bar{A}, \mathbf{X})$ then $r^{[h]} \in P(\bar{A}, \mathbf{X})$.

Let $\langle \langle h_l^\alpha : \alpha \in w_{n+1} \rangle : l \leq l_0 \leq 2^{(n+1)^2} \rangle$ be a list of all $\langle h^\alpha : \alpha \in w_{n+1} \rangle$ such that for every $\alpha \in w_{m+1} - w_m$

$$h^\alpha : u_n - u_{m+1} \rightarrow \{0, 1\}.$$

Now we choose by induction on $l \leq l_0$, $p_{n,l}$ such that

- (a) $p_{n,0} = p_n$.

- (b) $p_{n,l} \leq p_{n,l+1} \in P\omega_2 \cap N$.
- (c) For every $\alpha \in w_{m+1} - w_m$, $m < n$ we have

$$p_{n,l} \Vdash \alpha \Vdash \text{“Dom } p_{n,l}(\alpha) \cap u_n \subseteq u_{m+1}\text{”}.$$

- (d) If $p'_{n,l+1}$ is such that

$$p'_{n,l+1}(\alpha) = \begin{cases} p_{n,l+1}(\alpha)^{[h_i^\alpha]} & \text{if } \alpha \in w_n, \\ p_{n,l+1}(\alpha) & \text{if } \alpha \notin w_n, \end{cases}$$

then $p'_{n,l+1}$ forces values for τ_i ($i \leq n+1$) and for $h(i)$ ($i \leq n+1$).

The induction.

for $l = 0 : p_{n,0} = p_n$,

for $l + 1 : p_{n,l}$ was defined satisfying (a), (b), (c), (d), and let $p^*_{n,l}$ be such that

$$p^*_{n,l}(\alpha) = \begin{cases} p_{n,l}(\alpha)^{[h_i^\alpha]} & \text{if } \alpha \in w_n, \\ p_{n,l}(\alpha) & \text{if } \alpha \notin w_n. \end{cases}$$

There exists $q_{n,l} \geq p^*_{n,l}$ such that $q_{n,l}$ forces values for τ_i ($i \leq n+1$), $h(i)$ ($i \leq n+1$). Fix such $q_{n,l}$ in N , and then we define $p_{n,l+1}$ satisfying

$$p_{n,l+1}(\alpha) = \begin{cases} q_{n,l}(\alpha) \upharpoonright (\omega - \text{Dom } h_i^\alpha) & \text{if } \alpha \in w_n, \\ q_{n,l}(\alpha) & \text{if } \alpha \notin w_n. \end{cases}$$

Clearly $p_{n,l+1}$ satisfies (a), (b), (c), and (d). P_{n,l_0} is almost p_{n+1} , only we need to extend it in order to find the u_{n+1} required. Clearly p_{n,l_0} fix 2^{n^2} -many possible values to $h(n)$. The next lemma is exactly what we need.

1.7. Lemma. *If $w \subseteq \omega_2$ is finite, and $p \in P\omega_2$ and $n < \omega$ then there exists $k \in \omega$ and $q \in P\omega_2$ such that*

- (a) $n < k < \omega$;
- (b) $p \leq q$;
- (c) for every $\alpha \in w$

$$q \Vdash \alpha \Vdash \text{“Dom } q(\alpha) \cap [0, n] = \text{Dom } p(\alpha) \cap [0, n]\text{”};$$

- (d) for every $\alpha \in w$

$$q \Vdash \alpha \Vdash \text{“}k \notin \text{Dom } q(\alpha)\text{”}.$$

Proof. Let $w = \{\alpha_1 < \alpha_2 < \dots < \alpha_m\}$. We choose by (decreasing) induction

$$q_m, q_{m-1}, \dots, q_1, q_0$$

satisfying

- (i) $q_m = p$,
- (ii) $q_l \Vdash \alpha_l \leq q_{l-1} \in P_{\alpha_l}$,

(iii) q_{l-1} forces less than $2^{n \times m}$ -many possible values for the following names of ordinals (i.e., satisfies condition (c)):

(A) γ_l such that

$$\text{Dom}(q_l(\alpha_l)) =^* [A_{\gamma_l}]^{q_{l-1}(0)}$$

(remember that Q_0 fixes A_{γ_l});

(B) k_l such that

$$1 + \max(\text{Dom } q_l(\alpha_l) - [A_{\gamma_l}]^{q_{l-1}(0)}) < k_l$$

(use the above construction in order to get such q_{l-1}).

Therefore there exists k_l^i ($i = 1, \dots, t_l < \omega$); γ_l^i ($i = 1, \dots, s_l < \omega$) such that

$$q_l \Vdash_{P_{\alpha_l}} \text{“}(\exists i < t_l \exists j < s_l)(\gamma_l = \gamma_l^j \wedge k_l = k_l^i)\text{”}.$$

Now let

$$\bar{\gamma}_l = \max\{1, \gamma_l^1, \dots, \gamma_l^{s_l}\},$$

$$\bar{k}_l = \max\{1, k_l^1, \dots, k_l^{t_l}\}$$

and

$$q = \bigcup_{l=0}^m q_l, \quad \text{i.e., if } \alpha_0 = -1 \text{ and } \alpha_{m+1} = \omega_2$$

then

$$q(\alpha) = q_l(\alpha) \quad \text{where } \alpha_{l-1} \leq \alpha < \alpha_l.$$

Then we choose k such that

$$k > \bar{k}_l (l \leq m), \quad k \notin A_{\gamma_0} \cup A_{\gamma_1} \cup \dots \cup A_{\gamma_m}.$$

This concludes the proof of part (b) of Theorem 1.6. \square

1.8. Theorem. *If V is a model for ZFC and $P \in V$ is a forcing notion satisfying the Sacks property then $\Vdash_P \text{“}\mu^*(2^\omega \cap V) = 1\text{”}$.*

Proof. Suppose that there exists $p \in P$ such that $p \Vdash_P \text{“}\mu(2^\omega \cap V) = 0\text{”}$. Then there exists $(\mathbf{I}_n : n < \omega)$ such that for every $n \in \omega$ \mathbf{I}_n is a P -name of a rational interval, and

$$0 \Vdash \text{“}\sum \mu(\mathbf{I}_n) < \infty\text{”},$$

$$p \Vdash \text{“}2^\omega \cap V \subseteq \bigcap_n \bigcup_{m \geq n} \mathbf{I}_m\text{”}.$$

Then there exists \mathbf{g} such that \mathbf{g} is a P -name of a function from ω to ω and satisfying

$$0 \Vdash \text{“}\sum_{m > \mathbf{g}(n)} \mu(\mathbf{I}_n) < \frac{1}{2^{n^2+n}}\text{”}.$$

Fix $q \geq p$ and $g \in {}^\omega \omega$, given by the Sacks property such that

$$q \Vdash "(\forall n)(g(n) < g(n))".$$

Again using the Sacks property, let $r \geq q$ and $\langle \langle I_{j,i}^k : g(i) \leq k < g(i+1) \rangle : j < 2^{i^2} \rangle : i < \omega \rangle$ satisfying for every $i < \omega$, there exists $j < 2^{i^2}$

$$r \Vdash " \forall k \in [g(i), g(i+1)) (I_k = I_{j,i}^k) "$$

and under this notation we define

$$J_i = \bigcup_{j < 2^{i^2}} \bigcup_{g(i) \leq k < g(i+1)} I_{j,i}^k.$$

Then for every i

$$\mu(J_i) \leq 2^{i^2} \cdot \frac{1}{2^{i^2+i}} = \frac{1}{2^i}.$$

Therefore $\sum_{i \in \omega} \mu(J_i) < \infty$ and this implies that there exists $x \in 2^\omega$ such that $x \notin \bigcap_i \bigcup_{j \geq i} J_j$, and by the hypothesis this implies that

$$r \Vdash "x \notin \bigcap_i \bigcup_{j \geq i} I_j"$$

a contradiction. \square

It may be possible that these ideas help to solve the question in [JSh]: remember that the Sacks property implies the Laver property.

2. RAPID FILTERS AND SIERPINSKI SETS

2.0. Theorem. $\text{cons}(ZF) \Rightarrow \text{cons}(ZFC + \text{there exists a Sierpinski set} + 2^{\aleph_0}$ is a regular cardinal + there are no rapid filters on ω).

Proof. We begin with $V = L$ and let P be the ω_2 -iteration of Mathias reals with countable support and let \mathbf{R} be a P -name of the product of \aleph_α random reals (i.e., the measure algebra). Then we will prove that if $G \subseteq P * \mathbf{R}$ then

$$(*) \quad V[G] \models \text{"there are no rapid filters on } \omega \text{"}.$$

Clearly this is enough in order to obtain the inclusion of the theorem. The proof of (*) will take the remainder of this section.

2.1. Definition. Let \mathbf{a} be an R -name of a set of ordinals; then we define

$$\mathbf{a}(n) = \mu(\|n \in \mathbf{a}\|).$$

Then $\mathbf{a}(\cdot)$ is a function from ordinals to $[0, 1]$.

2.2. Fact. Let M be a model of ZFC and let R be any product of random reals, i.e., a measure algebra. Let $\langle n_i : i < \omega \rangle$ be in M an increasing sequence of natural numbers. Assume that \mathbf{a} is in M^R and

$$M \models \Vdash "(\forall k \in \omega)(|n_k \cap \mathbf{a}| < k)"$$

Then for every $k \in \omega$

$$M \models \text{“} \sum_{m=0}^{n_k-1} \mathbf{a}(m) \leq k \text{”}.$$

Proof. We will work in M . Let $\varphi_i(m)$ be the formula that says

“ m is the i th member of a ”.

Then $\mathbf{a}(m) = \sum_{i=0}^m \mu(\|\varphi_i(m)\|)$. Also if $m \neq n$ then $\mu(\|\varphi_i(m)\| \cdot \|\varphi_i(n)\|) = 0$. Assume that there exists $k \in \omega$ such that

$$\sum_{m=0}^{n_k-1} \mathbf{a}(m) > k.$$

Then

$$\sum_{m=0}^{n_k-1} \sum_{i=1}^m \mu(\|\varphi_i(m)\|) > k.$$

Therefore

$$\sum_{m=0}^{n_k-1} \sum_{i=1}^k \mu(\|\varphi_i(m)\|) > k \quad (\text{because } m < n_k)$$

and hence

$$\sum_{i=1}^k \sum_{m=0}^{n_k-1} \mu(\|\varphi_i(m)\|) > k.$$

And thus there exists i such that $\sum_{m=0}^{n_k-1} \mu(\|\varphi_i(m)\|) > 1$ and this implies that there exists $n \neq m$ such that

$$\mu(\|\varphi_i(m)\| \cdot \|\varphi_i(n)\|) \neq \emptyset,$$

a contradiction. \square

From now on we fix $M \models ZFC$. Let P in M be the ω_2 -iteration of Mathias reals. Each Mathias real adds a sequence of natural numbers. $P(0)$ is the first coordinate of P . Let $\langle \mathbf{n}_i : i < \omega \rangle$ be a P -name for the sequence added by $P(0)$. Let \mathbf{R} be a P -name for a product of random reals (i.e., a measure algebra). If $H \subseteq P$ is generic over M then $\mathbf{R}[H]$ is the realization of \mathbf{R} in $M[H]$. Let $\mathbf{a} \in M[H]$ be an $\mathbf{R}[H]$ -name for a sequence of natural numbers such that

$$M[H] \models \text{“} \Vdash_{\mathbf{R}[H]} \text{“} |\mathbf{n}_k[H] \cap \mathbf{a}| < k \text{”} \text{”}.$$

Therefore, by 2.2,

$$M[H] \models \sum_{m=0}^{n_k[H]-1} \mathbf{a}(m) \leq k.$$

The function $\mathbf{a}(\cdot) : \omega \rightarrow [0, 1]$, defined by \mathbf{a} , lies in $M[H]$, so it has a P -name. Let \mathbf{f} be such a name (we omit this relation with \mathbf{a} because it is clear).

2.3. Lemma. *Let $p \in P$, and $\varepsilon > 0$ given; then there exists $w_0 \subseteq w_1 \subseteq \dots \subseteq w_n \subseteq \dots$, p_1^n, p_2^n , $k_0 < k_1 < \dots < k_n < \dots$, $B_i^0 \subseteq \dots \subseteq B_i^n \subseteq \dots$, $i = 1, 2$, satisfying*

- (a) *for every n , $w_n \subseteq \omega_2$; $k_n \in \omega$; $p_i^n \in P$, $i = 1, 2$; $B_i^n \in [\omega]^{<\omega}$, $i = 1, 2$; $B_1^n \cap B_2^n = B_1^0 \cap B_2^0$, $\bigcup B_1^n \cup \bigcup B_2^n = \omega$.*
- (b) $\bigcup_n \{\text{Dom}(p_1^n) \cup \text{Dom}(p_2^n)\} = \bigcup_n w_n$.
- (c) *For every n , $p_i^n \leq_{w_n}^n p_i^{n+1}$, $i = 1, 2$ (see [Ba, §7]).*
- (d) $p_i^0 = p$ for $i = 1, 2$.
- (e) *If m is even, then for every $k > k_m$ there exists $q_{w_{m+1}}^{m+1} \geq p_1^m$ such that*
 - (i) $q \Vdash_P \text{“} \sum_{l=k_m}^k \mathbf{f}(l) < \varepsilon/10^m \text{”}$,
 - (ii) $q \Vdash_P \text{“} \sum_{l < k_m} \mathbf{1}_{B_1^m} \mathbf{f}(l) < \varepsilon \left(\frac{1}{10} + \frac{1}{10^2} + \dots + \frac{1}{10^{m-1}} \right) \text{”}$,
 - (iii) $p_2^m \Vdash_P \text{“} \sum_{l < k_m} \mathbf{1}_{B_2^m} \mathbf{f}(l) < \varepsilon \left(\frac{1}{10} + \dots + \frac{1}{10^{m-1}} \right) \text{”}$.
- (f) *If m is odd, then for every $k > k_m$ there exists $q_{w_{m+1}}^{m+1} \geq p_2^m$ such that*
 - (i) $q \Vdash \text{“} \sum_{l=k_m}^k \mathbf{f}(l) < \varepsilon/10^m \text{”}$,
 - (ii) $q \Vdash \text{“} \sum_{l < k_m} \mathbf{1}_{B_2^m} \mathbf{f}(l) < \varepsilon \left(\frac{1}{10} + \dots + \frac{1}{10^{m-1}} \right) \text{”}$,
 - (iii) $p_1^m \Vdash \sum_{l < k_m} \mathbf{1}_{B_1^m} \mathbf{f}(l) < \varepsilon \left(\frac{1}{10} + \dots + \frac{1}{10^{m-1}} \right) \text{”}$.
- (g) $0 \in w_0$.

Proof. By induction on m .

$m = n + 1$: By the symmetry of (e) and (f) without loss of generality n is even. In this stage we need to take care of the relation $\leq_{w_n}^n$.

We define $w_{n+1} \supseteq w_n$ such that w_{n+1} contains the first n numbers of each $\text{sup}(p_1^j) \cup \text{sup}(p_2^j)$, for $j \leq n$, in some fixed enumeration of $\text{sup}(p_1^j) \cup \text{sup}(p_2^j)$.

For every natural number ζ , there exists $p_2^{n,\zeta}$, $\langle f^{\zeta,l} : l < 2^{|w_{n+1}| \times n+1} \rangle$ such that

- (α) $p_2^n \leq_{w_{n+1}}^{n+1} p_2^{n,\zeta} \in P$,
- (β) The $n + 2$ -th member of the infinite part of $p_2^{n,\zeta}(0)$ is larger than ζ ,
- (γ) $p_2^{n,\zeta} \Vdash \text{“} \exists l < 2^{|w_{n+1}| \times n+1} \forall \xi < \zeta (|f^{\zeta,l}(\xi) - \mathbf{f}(\xi)| < 2^{-t(\zeta)}) \text{”}$ where $t(\zeta) = 2^{2^{\zeta+n+1+|w_{n+1}|}}$.

(Use [Ba, §9.5] and an approximation to $\mathbf{f}(\xi)$ with error $t(\zeta)$.)

Now let $\langle \zeta_i : i < \omega \rangle$ be such that for every $\xi \in \omega$, and for every $l < 2^{|w_{n+1}| \times n+1}$ the sequence $\langle f^{\zeta_i,l}(\xi) : i < \omega \rangle$ converges to a real number in $[0, 1]$. We call this limit $f^{*,l}$. (Use a diagonal argument and the compactness theorem.)

Claim. For every $k \in \omega$ $\sum_{k < \xi < \omega} f^{*,l}(\xi) \leq n + 2$.

Proof. If not, then there exists $\bar{k} > k$ such that

$$\sum_{k < \xi \leq \bar{k}} f^{*,l}(\xi) > n + 2.$$

Then there exists i such that $\zeta_i > \bar{k}$ and

$$\sum_{k < \xi \leq \zeta_i} f^{\zeta_i,l}(\xi) > n + 2$$

and this is a contradiction to 2.2. \square

Therefore there exists $k_{n+1} > k_n$ such that for each $l < 2^{|w_{n+1}| \times n+1}$ we have that

$$\sum_{k_{n+1} < \xi} f^{*,l}(\xi) < \varepsilon / 10^{n+8}.$$

Then we define

$$\begin{aligned} B_1^{n+1} &= B_1^n \cup [k_n, k_{n+1}), \\ B_2^{n+1} &= B_2^n, \\ p_2^{n+1} &= p_2^n. \end{aligned}$$

Now, by the induction hypothesis, p_1^n has an extension p_1^{n+1} such that

$$p_1^{n+1} \Vdash \sum_{\substack{l \notin B_1^{n+1} \\ l < k_{n+1}}} \mathbf{f}(l) < \varepsilon \left(\frac{1}{10} + \dots + \frac{1}{10^{n-1}} + \frac{1}{10^n} \right).$$

This completes the induction. The reader may check that this works.

Let \mathbf{U} be a P -name of the \mathbf{R} -name for a rapid filter. Then using 2.3 we get the following

Conclusion 1. There exists p_1, p_2, B_1, B_2 such that

$$(*) \quad p_1 \Vdash_P \mu(\|B_2 \in \mathbf{U}\|) < \varepsilon,$$

$$(**) \quad p_2 \Vdash_P \mu(\|B_1 \in \mathbf{U}\|) < \varepsilon,$$

and $B_1 \cap B_2$ is finite, $B_1 \cup B_2 = \omega$.

Using Conclusion 1, we get the following

Conclusion 2. There exists $\delta < \omega_2$, such that for every $\alpha < \delta$ ($\text{cof}(\delta) = \omega_1$) and for every $\langle B_1, B_2 \rangle \in V[G[\alpha]]$ (where $G[\alpha]$ is $\mathbf{G}[P_\alpha]$) if there exists β such that in $V[G[\beta]]$ Conclusion 1 holds for B_1 and B_2 , then there exists $\beta < \delta$ such that in $V[G[\beta]]$ Conclusion 1 holds, and $\text{supp}(p_1) \cup \text{supp}(p_2) \subset \delta$.

Now using Lemma 2.3, working in $V[G[\delta]]$, there exists B_1, B_2, p_1, p_2 such that

$$V[G[\delta]] \models "(*) \wedge (**) \wedge B_1 \cup B_2 = \omega \wedge B_1 \cap B_2 \text{ is finite}."$$

(Remember that $P \cong P_{\delta\omega_2}$.) Then, by the hypothesis on δ , and using the fact that P_δ is proper, we have that there exists $\beta < \delta$ and p'_1, p'_2 such that

$$B_i \in V[G[\beta]], \quad i = 1, 2,$$

and in $V[G[\beta]]$ we have (*), (**) for p'_1, p'_2 respectively.

Without loss of generality $p'_1 \in G[\delta]$. Now $p'_1 \cup p_2$ is a condition and w.l.o.g. $p'_1 \cup p_2 \in G[\omega_2]$. Therefore in $V[G[\omega_2]]$ we have

- (a) $\mu(\|B_1 \in \mathbf{U}[G[\omega_2]]\|) < \varepsilon$,
- (b) $\mu(\|B_2 \in \mathbf{U}[G[\omega_2]]\|) < \varepsilon$.

If ε was chosen small, we can deduce that $\mu(\|B_1 \cap B_2 \in \mathbf{U}[G[\omega_2]]\|) \neq 0$, and this implies that

$$\mu(\|\mathbf{U}[G[\omega_2]] \text{ is not a filter}\|) > 0 \quad (\text{because } B_1 \cap B_2 \text{ is finite}),$$

a contradiction.

REFERENCES

- [Ba] J. Baumgartner, *Iterated forcing*, Surveys in Set Theory (A. R. D. Mathias, ed.), London Math Soc. Lecture Notes Series 87, Cambridge Univ. Press, 1983.
- [Fl] W. Fleissner, *Current research on Q-sets*, Topology, vol. 1, Colloq. Math. Soc. Janos Bolyai, vol. 23, North-Holland, 1980.
- [Ju] H. Judah, *Strong measure zero sets and rapid filters*, J. Symbolic Logic 53 (1988).
- [JSh1] H. Judah and S. Shelah, $\Delta_{\frac{1}{2}}$ -set of reals, Ann. Pure Appl. Logic 42.
- [JSh2] —, *Q-set do not necessarily have strong measure zero*, Proc. Amer. Math. Soc. 102 (1988).
- [Ku] K. Kunen, *Set theory*, North-Holland, 1980.
- [Mi1] A. Miller, *There are no Q-points in Laver's model for the Borel conjecture*, Proc. Amer. Math. Soc. 178 (1980).
- [Mi2] —, *Special subsets of the real line*, Handbook of Set-Theoretic Topology (K. Kunen and J. E. Vaughan, eds.), North-Holland, 1984.
- [Ra] J. Rausonnier, *A mathematical proof of S. Shelah's theorem on the measure problem and related results*, Israel J. Math. 48 (1984).
- [Sh] S. Shelah, *Proper forcing*, Lecture Notes in Math., vol. 940, Springer-Verlag, New York.

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