**Q-SETS, SIERPINSKI SETS, AND RAPID FILTERS**

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**Abstract.** In this work we will prove the following:

**Theorem 1.** \(\text{cons}(ZF) \implies \text{cons}(ZFC + \text{there exists a } \mathcal{Q}\text{-set of reals} + \text{there exists a set of reals of cardinality } \aleph_1 \text{ which is not Lebesgue measurable})\).

**Theorem 2.** \(\text{cons}(ZF) \implies \text{cons}(ZFC + 2^\aleph_0 \text{ is arbitrarily larger than } \aleph_2 + \text{there exists a Sierpinski set of cardinality } 2^{\aleph_0} + \text{there are no rapid filters on } \omega)\).

These theorems give answers to questions of Fleissner [Fl] and Judah [Ju].

**0. Introduction**

In this work we will solve two open problems about special sets of the reals. In order to state them we need some definitions.

**0.1. Definition.** A set of reals \(A\) is a \(\mathcal{Q}\)-set iff every subset of \(A\) is a relative \(F_\sigma\), i.e., it is a countable union of relatively closed subsets of \(A\).

\(\mathcal{Q}\)-sets are very strange: for example \(2^{\aleph_0} < 2^{\aleph_1}\) implies that there are no \(\mathcal{Q}\)-sets of cardinality \(\aleph_1\). Also \(\mathcal{Q}\)-sets have universal measure zero, but they do not necessarily have strong measure zero (see [Fl, JSh2, Mi2]).

In [Fl] it is asked if the existence of a \(\mathcal{Q}\)-set of cardinality \(\aleph_1\) implies that every \(\aleph_1\)-set of reals is of Lebesgue measure zero. Our first theorem answers this question negatively by showing

**Theorem.** \(\text{cons}(ZF) \implies \text{cons}(ZFC + \text{there exists a } \mathcal{Q}\text{-set of reals} + \text{there exists a set of reals of cardinality } \aleph_1 \text{ which is not Lebesgue measurable})\).

We show this theorem as follows. We begin by forcing a set \(A\) of reals of cardinality \(\aleph_1\), and then we force, with a countable support iteration of length \(\omega_1\), making \(A\) a \(\mathcal{Q}\)-set in the generic extension. We prove that this composition of forcing notions satisfies the Sacks property (studied in [Sh]) and, in the end
of the section, we prove that if a forcing notion has the Sacks property then in
the generic extension the old reals have outer measure one. Clearly this implies,
if we begin from \( L \), that in our generic extension there exists an uncountable
\( Q \)-set and a \( \aleph_1 \)-set of reals which is not Lebesgue measurable.

0.2. Definition. (a) A set of reals \( A \) is a Sierpinski set iff for every measure
zero set \( M \), \( A \cap M \) is countable.

(b) \([\omega]^{\omega} = \{x : x \subseteq \omega \land |x| = \aleph_0 \} ; \; [\omega]^{<\omega} = \{x : x \subseteq \omega \land |x| < \aleph_0 \} \).

(c) A subset \( F \subseteq [\omega]^{\omega} \) is a rapid filter iff

(i) \((\forall x, y \in F)(x \cap y \in [\omega]^{\omega}) \) and \((\forall x \forall y)(x \in F \land x \subseteq y \rightarrow y \in F) \),

(ii) \((\forall f \in [\omega]^{\omega} \exists x \in F)(\forall n \in \omega)(|f(n) \cap x| < n) \).

Clearly, if the Sierpinski set has the cardinality of the continuum then the
real line cannot be the union of less than \( 2^{\aleph_0} \)-many measure zero sets.

In [Ju] it was remarked that if the reals are not the union of less than \( 2^{\aleph_0} \)-many meager sets then there exists a rapid filter on \( \omega \). Therefore it was asked: if the reals are not the union of less than \( 2^{\aleph_0} \)-many measure zero sets then does there exist a rapid filter on \( \omega \)? The next theorem will answer this question negatively.

**Theorem.** \( \text{cons}(ZF) \) implies \( \text{cons}(ZFC + 2^{\aleph_0} \text{ is arbitrarily larger than } \aleph_2 + \text{there exists a Sierpinski set of cardinality } 2^{\aleph_0} + \text{there are no rapid filters on } \omega ) \).

This theorem has some applications. For example, the existence of a Sierpinski set of cardinality \( 2^{\aleph_0} \) implies that every \( \Delta_1^1 \)-set of reals is measurable (see [JSh1]); also in this model \( \omega_1^L = \omega_1 \), and therefore, we get a model for
“Every \( \Delta_1^1 \)-set of reals is Lebesgue measurable + \( \omega_1^L = \omega_1 \) + there is no rapid filter on \( \omega \).” This says that it is impossible to improve the following result of Raisonnier [Ra]:

“If every \( \Sigma_2^1 \)-set of reals is Lebesgue measurable and \( \omega_1^L = \omega_1 \) then there is a rapid filter on \( \omega \).”

We prove this theorem in \( \S 2 \). The model is gotten by adding \( \omega_2 \)-many Mathias reals and afterward adding random reals. It was remarked by A. Miller in [Mi1] that in the model obtained by iterating \( \omega_2 \)-Mathias reals over \( L \) there is no rapid filter on \( \omega \).

We assume that the reader knows the material given in [Ba], about countable
support iterated forcing and forcing notion satisfying the Axiom A (for the
notation). The rest of the notation is standard.

1. **Q-sets**

In this section we build a model of set theory where there exists a \( Q \)-set of
reals and there exists an outer measure one set of reals of cardinality \( \aleph_1 \). This
is the model given in 1.6. For the basic definitions the reader may consult the
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introduction (§0) and Fleissner [Fl]. We also need some definitions used in the construction.

1.1. Definition. $\vec{A} = \langle a_i, A_i : i < \omega_1 \rangle$ is a suitable sequence if and only if
(a) $A_i \in [\omega]^\omega$ for every $i \in \omega_1$;
(b) if $i < j < \omega_1$ then $A_i \subseteq^* A_j$ ($\exists n(A_i - n \subseteq A_j)$) and $A_j - A_i \in [\omega]^\omega$;
(c) $a_i \in [A_{i+1} - A_i]^\omega$ for every $i \in \omega_1$.

1.2. Definition. For $\vec{A} = \langle a_i, A_i : i < \omega_1 \rangle$ suitable, and $X \subseteq \omega_1$ we define the partially ordered set $P(\vec{A}, X)$ by stipulating that $h$ belongs to $P(\vec{A}, X)$ if and only if
(i) $h$ is a partial function from $\omega$ to $\{0, 1\}$;
(ii) there exists $i = i(h)$ such that $\text{Dom } h \subseteq^* A_i$ (take such $i$ minimal);
(iii) for every $j < i(h)$ we have
\[
\begin{align*}
a_j \subseteq^* & \text{Dom}(h), \\
& \text{if } j \in X \text{ then } a_j \subseteq^* h^{-1}(\{1\}), \\
& \text{if } j \notin X \text{ then } a_j \subseteq^* h^{-1}(\{0\}).
\end{align*}
\]

For $h_1, h_2 \in P(\vec{A}, X)$ we set $h_1 \leq h_2$ if and only if $h_1 \subseteq h_2$.

1.3. Lemma. If $\vec{A} = \langle a_i, A_i : i < \omega_1 \rangle$ and $X \subseteq \omega_1$, $P(\vec{A}, X)$ are as in 1.2 and $h \in P(\vec{A}, X)$, hence $i(h) = \alpha$ is well defined, $\alpha < \beta < \omega_1$, then there exists $h^* \in P(\vec{A}, X)$ such that
\[
\begin{align*}
h \subseteq h^* \quad \text{and} \quad i(h^*) & \geq \beta.
\end{align*}
\]

Proof. There exists $g : [\alpha, \beta) \rightarrow \omega$ such that
(a) $\alpha \leq \gamma < \beta$ implies $(\text{Dom } h) \cap a_\gamma \subseteq g(\gamma) \supseteq a_\gamma - A_\beta$;
(b) $\alpha \leq \gamma < \delta < \beta$ implies $(a_\gamma - g(\gamma)) \cap (a_\delta - g(\delta)) = \varnothing$ (simply let $\langle \gamma_l : l < l^* \leq \omega \rangle = [\alpha, \beta)$ and construct $g(\gamma_l)$ by induction on $l$).

Now $\text{Dom } h^* = (\text{Dom } h) \cup \bigcup_{\gamma \in [\alpha, \beta]} (a_\gamma - g(\gamma))$ and
\[
\begin{align*}
h^*(n) & = \begin{cases} 
 h(n) & \text{if } n \in \text{Dom } h, \\
 0 & \text{if } n \in a_\gamma - g(\gamma) \text{ and } \gamma \notin X, \\
 1 & \text{if } n \in a_\gamma - g(\gamma) \text{ and } \gamma \in X. 
\end{cases}
\end{align*}
\]

1.4. Lemma. Let $V$ be a model of ZFC satisfying
(i) $\vec{A} = \langle a_i, A_i : i < \omega_1 \rangle$ is suitable, $\vec{A} \in V$;
(ii) for every $X \subseteq \omega_1$ there exists $M \subseteq V$ such that $X \in M$, $\vec{A} \in M$, and therefore, $P(\vec{A}, X)$ is definable in $M$;
(iii) there exists $G \in V$ such that $G \subseteq P(\vec{A}, X) \cap M$ and $G$ is generic over $M$.

Then $B(\vec{A}) = \{ f \in \mathcal{P}^\omega : (\exists i < \omega_1)(\text{char}(a_i) = f) \}$ is a $Q$-set in $V$. 

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**Proof.** Use 1.3 and the hypothesis. □

1.5. Definition. Let \( \overline{Q} = \langle P_j; Q_j : i < \omega_2, j < \omega_2 \rangle \) be a countable support iterated forcing system satisfying

(a) \( Q_0 = \langle \{ \langle a_i, A_i : i < \alpha \rangle : a < \omega_1 \text{ and } \langle a_i, A_i : i < \alpha \rangle \text{ is an initial segment of a suitable sequence } \} \rangle \).

Let \( \overline{A} \) be the \( Q_0 \)-name of the suitable sequence generated by the \( Q_0 \)-generic object.

(b) Let \( 0 < i < \omega_2 \); then there exists a \( P_i \)-name \( X \) such that

\( \models_{P_i} \text{"} X \subseteq \omega_1 \text{ and } Q_i = P(\overline{A}, X) \text{"} \).

(c) If \( i < \omega_2 \) and \( X \) is a \( P_i \)-name such that

\( \models_{P_i} \text{"} X \subseteq \omega_1 \text{"} \)

then there exists \( j \geq i \) and \( Y \) a \( P_j \)-name satisfying

\( \models_{P_j} \text{"} X[G \upharpoonright j] = Y \text{ and } Q_j = P(\overline{A}, Y) \text{"} \).

1.6. Theorem. Let \( P\omega_2 \) be the directed limit of the iterated forcing system \( \overline{Q} \) defined in 1.5. Let \( G \subseteq P\omega_2 \) be generic over \( V \models \text{"} GCH \text{"} \). Then the following holds:

(a) For every \( i < \omega_2 \)

\( \models_{P_i} \text{"} Q_i \text{ satisfies } \kappa_2 \text{ - c.c.} \text{"} \).

Therefore \( P\omega_2 \) satisfies \( \kappa_2 \text{ - c.c.} \).

(b) \( P\omega_2 \) is a Proper Forcing notion, moreover \( P\omega_2 \) satisfies the Sacks property. Therefore \( V[G] \models \text{"} 2^\omega \cap V \text{ has outer measure one} \text{"} \) (see 1.8).

(c) If \( V[G] \) we have

\( B(\overline{A}[G]) \text{ is a } Q \text{-set.} \)

**Proof.** (a) easy; (c) use 1.4. The proof of (b) is sharp:

In this work we say that a forcing notion \( P \) satisfies the Sacks property iff

\( (\forall f \in V^P \forall p \in P) \text{ (if } p \models \text{"} f \in V \text{"} \text{ then } (\exists q \geq p \exists g \in V \cap \omega \forall \forall q \models \text{"} f(n) \in g(n) \forall n \in \omega)(|g(n)| \leq 2^n \forall n \in \omega) \).)

Let \( \chi \) be sufficiently large, and \( p \in P\omega_2 \). Let \( N \) be such that

\( N < \langle H(\chi), \varepsilon, <^* \rangle \text{ (} <^* \text{ is some fixed well order),} \)

\( p \in N, \quad \overline{Q}, P\omega_2 \in N, \quad \|N\| = \kappa_0. \)

Set \( \delta = N \cap \omega_1 \), and let \( \langle w_n : n < \omega \rangle \) be such that \( \bigcup \{ w_n : n < \omega \} = N \cap \omega_2 - \{0\} \)

\( w_n \subseteq w_{n+1}, \quad \|w_n\| = n. \)

Also let \( \langle \tau_n : n < \omega \rangle \) be an enumeration of the \( P\omega_2 \)-names of ordinal numbers that belong to \( N \). Let \( \langle \alpha_n : n < \omega \rangle \) be such that \( \alpha_n < \alpha_{n+1} \text{ and } \sup_{n<\omega} \alpha_n = \delta. \)

And fix \( h \in N \) such that \( h \) is a \( P\omega_2 \)-name of a function from \( \omega \) to \( V \).
We will choose, by induction on $\omega$, conditions $p_n \in P\omega_2$ and finite sets $u_n$ such that

(i) $p \leq p_n \leq p_{n+1} \in P\omega_2 \cap N$;
(ii) $p_{n+1} \Vdash "\tau_n \in N \cap \text{Ord}"$;
(iii) there exists $b_n \in [V]^{\omega_2}$ such that $p_{n+1} \Vdash "h(n) \in b_n"$;
(iv) $u_n \subseteq u_{n+1} \subseteq \omega$ and $\|u_n\| = n$;
(v) if $\beta \in w_{m+1} - w_m$ and $m \leq n$ then $p_n[\beta] \Vdash "\text{Dom}(p_n(\beta)) \cap u_n \subseteq u_{m+1}"$;
(vi) $\alpha_n < \lg(p_n(0))$ and $u_n \cap A_{\alpha_m} \subseteq u_{m+1}$, for $m < n$.

Claim. (*) implies (b).

Proof of the claim. Let $p_\omega$ be defined by $p_\omega(0) = (a_j, A_j : i < \delta) \cup \{A_\delta\}$ when for every $n$

$$p_n(0) = (a_j, A_j : i < \lg(p_n(0))) \quad \text{and} \quad A_\delta = \omega - \bigcup_{n \in \omega} u_n.$$  

If $\beta \in \bigcup_n \text{Dom}(p_n) - \{0\}$ then, from the hypothesis, for $\langle p_n(\beta) : n < \omega \rangle$ there exists $p_\omega(\beta) = \bigcup_n p_n(\beta)$ such that for every $n \in \omega$ $p_n(\beta) \subseteq p_\omega(\beta)$. (This holds in $V^{p_\omega}$ where $p_\omega[\beta]$ belongs to the generic set.) Then $p_\omega$ is $(N, P\omega_2)$-generic and $p_\omega \Vdash "(\forall n)(h(n) \in b_n)"$. □

Therefore the problem is to show that the inductive construction given in (*) is realizable: suppose that $p_n$ and $u_n$ were given satisfying all conditions of (*).

Let $w_{n+1} = \{\alpha_0 < \alpha_1 < \cdots < \alpha_n < \alpha_{n+1}\}$. We try to extend $p_n$ to a condition satisfying (ii) and (iii):

Notation. If $r \in P(\overline{A}, X)$ and $h : \omega \rightarrow \{0, 1\}$ is finite then $r^{[h]} \in P(\overline{A}, X)$ when $r^{[h]}(i)$ is such that

$$r^{[h]}(i) = \begin{cases} r(i) & \text{if } i \in \text{Dom}(r) - \text{Dom}(h), \\ h(i) & \text{if } i \in \text{Dom}(h). \end{cases}$$

Clearly for every finite $h : \omega \rightarrow \{0, 1\}$ if $r \in P(\overline{A}, X)$ then $r^{[h]} \in P(\overline{A}, X)$.

Let $\langle h_0^\alpha : \alpha \in w_{n+1} : l \leq l_0 \leq 2^{(n+1)^2} \rangle$ be a list of all $\langle h_0^\alpha : \alpha \in w_{n+1} \rangle$ such that for every $\alpha \in w_{m+1} - w_m$

$$h_0^\alpha : u_n - u_{m+1} \rightarrow \{0, 1\}.$$  

Now we choose by induction on $l \leq l_0$, $p_{n,l}$ such that

(a) $p_{n,0} = p_n$. 

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The induction.

for $l = 0$: $p_{n, 0} = p_n$,
for $l + 1$: $p_{n, l}$ was defined satisfying (a), (b), (c), (d), and let $p_{n, l}^*$ be such that

$$p_{n, l+1}(\alpha) = \begin{cases} p_{n, l+1}(\alpha)[h_n] & \text{if } \alpha \in w_n, \\ p_{n, l+1}(\alpha) & \text{if } \alpha \notin w_n, \end{cases}$$

then $p_{n, l+1}'$ forces values for $\tau_i$ ($i \leq n + 1$) and for $h(i)$ ($i \leq n + 1$).

1.7. Lemma. If $w \subseteq \omega_2$ is finite, and $p \in P\omega_2$ and $n < \omega$ then there exists $k \in \omega$ and $q \in P\omega_2$ such that

(a) $n < k < \omega$;
(b) $p \leq q$;
(c) for every $\alpha \in w$

$$q[\alpha] \models \text{"Dom} q(\alpha) \cap [0, n) = \text{Dom} p(\alpha) \cap [0, n)";$$

(d) for every $\alpha \in w$

$$q[\alpha] \models \text{"} k \notin \text{Dom} q(\alpha)."$$

Proof. Let $w = \{\alpha_1 < \alpha_2 < \cdots < \alpha_m\}$. We choose by (decreasing) induction

$q_m, q_{m-1}, \ldots, q_1, q_0$

satisfying

(i) $q_m = p$,
(ii) $q_i[\alpha_i] \leq q_{i-1} \in P_{\alpha_i},$
(iii) \( q_{l-1} \) forces less than \( 2^{n \times m} \)-many possible values for the following names of ordinals (i.e., satisfies condition (c)):

(A) \( \gamma_i \) such that

\[
\text{Dom}(q_i(\alpha_i)) = [A_{\gamma_i}]^{q_i-1}(0)
\]

(remember that \( Q_0 \) fixes \( A_{\gamma_i} \));

(B) \( k_i \) such that

\[
1 + \max(\text{Dom} q_i(\alpha_i) - [A_{\gamma_i}]^{q_i-1}(0)) < k_i
\]

(use the above construction in order to get such \( q_{l-1} \)).

Therefore there exists \( k^i_1 \) \((i = 1, \ldots, t_i < \omega)\); \( \gamma^i_j \) \((i = 1, \ldots, s_i < \omega)\) such that

\[
q_i \forces \exists \gamma_i (\exists i < t_i \exists j < s_i)(\gamma_i = \gamma^i_j \land k_i = k^i_j).\]

Now let

\[
\overline{\gamma}_i = \max\{1, \gamma^i_1, \ldots, \gamma^i_{s_i}\},
\]

\[
\overline{k}_i = \max\{1, k^i_1, \ldots, k^i_{s_i}\}
\]

and

\[
q = \bigcup_{l=0}^m q_l, \quad \text{i.e., if } \alpha_0 = -1 \text{ and } \alpha_{m+1} = \omega_2
\]

then

\[
q(\alpha) = q_i(\alpha) \quad \text{where } \alpha_{l-1} \leq \alpha < \alpha_l.
\]

Then we choose \( k \) such that

\[
k > \overline{k}_i(l \leq m), \quad k \notin A_{\gamma_0} \cup A_{\gamma_1} \cup \cdots \cup A_{\gamma_m}.
\]

This concludes the proof of part (b) of Theorem 1.6. \( \Box \)

1.8. **Theorem.** If \( V \) is a model for ZFC and \( P \in V \) is a forcing notion satisfying the Sacks property then \( \forces \mu^* (2^\omega \cap V) = 1^* \).

**Proof.** Suppose that there exists \( p \in P \) such that \( p \forces \mu (2^\omega \cap V) = 0 \). Then there exists \( (I_n : n < \omega) \) such that for every \( n \in \omega \) \( I_n \) is a P-name of a rational interval, and

\[
0 \forces \sum \mu(I_n) < \infty,
\]

\[
p \forces 2^\omega \cap V \subseteq \bigcap_n \bigcup_{m \geq n} I_n.
\]

Then there exists \( g \) such that \( g \) is a P-name of a function from \( \omega \) to \( \omega \) and satisfying

\[
0 \forces \sum_{m > g(n)} \mu(I_n) < \frac{1}{2^{n^2+n}}.
\]
Fix \( q \geq p \) and \( g \in \omega^\omega \), given by the Sacks property such that
\[
q \vdash \forall n (g(n) < g(n)).
\]
Again using the Sacks property, let \( r > q \) and \( \langle (I_{j,i}^k : g(i) \leq k < g(i+1)) : j < 2^i \rangle : i < \omega \) satisfying for every \( i < \omega \), there exists \( j < 2^i \)
\[
r \vdash \forall k \in [g(i), g(i+1))(I_k = I_{j,i}^k)
\]
and under this notation we define
\[
J_i = \bigcup_{j < 2^i} \bigcup_{g(i) \leq k < g(i+1)} I_{j,i}^k.
\]
Then for every \( i \)
\[
\mu(J_i) \leq 2^i \cdot \frac{1}{2^{2^{i+1}}} = \frac{1}{2^i}.
\]
Therefore \( \sum_{i \in \omega} \mu(J_i) < \infty \) and this implies that there exists \( x \in 2^\omega \) such that \( x \notin \bigcap_i \bigcup_{j \geq i} J_j \), and by the hypothesis this implies that
\[
r \vdash \neg x \in P \cup j(J_j) > \aleph_1
\]
a contradiction. \( \Box \)

It may be possible that these ideas help to solve the question in [JSh]: remember that the Sacks property implies the Laver property.

2. Rapid filters and Sierpinski sets

2.0. Theorem. \( \text{cons}(ZF) \Rightarrow \text{cons}(ZFC + \text{there exists a Sierpinski set} + 2^{\aleph_0} \text{ is a regular cardinal} + \text{there are no rapid filters on } \omega) \).

Proof. We begin with \( V = L \) and let \( P \) be the \( \omega_1 \)-iteration of Mathias reals with countable support and let \( R \) be a \( P \)-name of the product of \( \aleph_0 \) random reals (i.e., the measure algebra). Then we will prove that if \( G \subseteq P \ast R \) then
\[
(*) \quad V[G] \models \text{"there are no rapid filters on } \omega\text{"}.
\]
Clearly this is enough in order to obtain the inclusion of the theorem. The proof of \((*)\) will take the remainder of this section.

2.1. Definition. Let \( a \) be an \( R \)-name of a set of ordinals; then we define
\[
a(n) = \mu(|n \in a|).
\]
Then \( a(\,\cdot\, \,\, ) \) is a function from ordinals to \([0, 1]\).

2.2. Fact. Let \( M \) be a model of \( ZFC \) and let \( R \) be any product of random reals, i.e., a measure algebra. Let \( \langle n_i : i < \omega \rangle \) be in \( M \) an increasing sequence of natural numbers. Assume that \( a \) is in \( M^R \) and
\[
M \models \text{"} \vdash \forall k \in \omega (|n_k \cap a| < k) \text{"
\]
Then for every $k \in \omega$

$$M \models \sum_{m=0}^{n_k-1} a(m) \leq k.$$  

**Proof.** We will work in $M$. Let $\varphi_i(m)$ be the formula that says

"$m$ is the $i$th member of $a$".

Then $a(m) = \sum_{i=0}^{n_k-1} \mu(\|\varphi_i(m)\|)$. Also if $m \neq n$ then $\mu(\|\varphi_i(m)\| \cdot \|\varphi_i(n)\|) = 0$.

Assume that there exists $k \in \omega$ such that

$$\sum_{m=0}^{n_k-1} a(m) > k.$$  

Then

$$\sum_{m=0}^{n_k-1} \sum_{i=1}^{n_i} \mu(\|\varphi_i(m)\|) > k.$$  

Therefore

$$\sum_{i=1}^{n_k} \sum_{m=0}^{n_k-1} \mu(\|\varphi_i(m)\|) > k \quad \text{(because $m < n_k$)}$$  

and hence

$$\sum_{i=1}^{n_k} \sum_{m=0}^{n_k-1} \mu(\|\varphi_i(m)\|) > k.$$  

And thus there exists $i$ such that $\sum_{m=0}^{n_k-1} \mu(\|\varphi_i(m)\|) > 1$ and this implies that there exists $n \neq m$ such that

$$\mu(\|\varphi_i(m)\| \cdot \|\varphi_i(n)\|) \neq \emptyset,$$

a contradiction. \(\square\)

From now on we fix $M \models ZFC$. Let $P$ in $M$ be the $\omega_2$-iteration of Mathias reals. Each Mathias real adds a sequence of natural numbers. $P(0)$ is the first coordinate of $P$. Let $\langle n_i : i < \omega \rangle$ be a $P$-name for the sequence added by $P(0)$. Let $R$ be a $P$-name for a product of random reals (i.e., a measure algebra). If $H \subseteq P$ is generic over $M$ then $R[H]$ is the realization of $R$ in $M[H]$. Let $a \in M[H]$ be an $R[H]$-name for a sequence of natural numbers such that

$$M[H] \models \|R[H] \|^n_k [H] \cap a] < k\).$$

Therefore, by 2.2,

$$M[H] \models \sum_{m=0}^{n_k[H]-1} a(m) \leq k.$$  

The function $a(\cdot) : \omega \to [0, 1]$, defined by $a$, lies in $M[H]$, so it has a $P$-name. Let $f$ be such a name (we omit this relation with $a$ because it is clear).
2.3. Lemma. Let \( p \in P \), and \( \varepsilon > 0 \) given; then there exists \( w_0 \subseteq w_1 \subseteq \cdots \subseteq w_n \subseteq \cdots \), \( p_1^n, p_2^n \), \( k_0 < k_1 < \cdots < k_n < \cdots \), \( B_i^0 \subseteq \cdots \subseteq B_i^n \subseteq \cdots \), \( i = 1, 2 \), satisfying

(a) for every \( n \), \( w_n \subseteq \omega_2 \); \( k_n \in \omega \); \( p_i^n \in P \), \( i = 1, 2 \); \( B_i^n \in [\omega]^{<\omega} \), \( i = 1, 2 \); \( B_1^n \cap B_2^n = B_1^0 \cap B_2^0 \), \( \bigcup B_1^n \cup B_2^n = \omega \).

(b) \( \bigcup \{\text{Dom}(p_1^n) \cup \text{Dom}(p_2^n)\} = \bigcup_n w_n \).

(c) For every \( n \), \( p_i^n \leq_{w_n} p_i^{n+1} \), \( i = 1, 2 \) (see [Ba, §7]).

(d) \( p_i^0 = p \) for \( i = 1, 2 \).

(e) If \( m \) is even, then for every \( k > k_m \) there exists \( q_{w_{m+1}}^{m+1} \geq_{p_1^m} p_1^m \) such that

(i) \( q \models \sum_{l=k_m}^k f(l) < \varepsilon / 10^m \),

(ii) \( q \models \sum_{l \in B_1^n} f(l) < \varepsilon \left( \frac{1}{10} + \frac{1}{10^m} + \cdots + \frac{1}{10^{m-r}} \right) \),

(iii) \( p_2^m \models \sum_{l \in B_2^n} f(l) < \varepsilon \left( \frac{1}{10} + \cdots + \frac{1}{10^{m-r}} \right) \).

(f) If \( m \) is odd, then for every \( k > k_m \) there exists \( q_{w_{m+1}}^{m+1} \geq_{p_2^{m+1}} p_2^m \) such that

(i) \( q \models \sum_{l=k_m}^k f(l) < \varepsilon / 10^m \),

(ii) \( q \models \sum_{l \in B_2^n} f(l) < \varepsilon \left( \frac{1}{10} + \cdots + \frac{1}{10^{m-r}} \right) \),

(iii) \( p_1^m \models \sum_{l \in B_1^n} f(l) < \varepsilon \left( \frac{1}{10} + \cdots + \frac{1}{10^{m-r}} \right) \).

(g) \( 0 \in w_0 \).

Proof. By induction on \( m \).

\( m = n + 1 \): By the symmetry of (e) and (f) without loss of generality \( n \) is even. In this stage we need to take care of the relation \( \leq_{w_n}^{n} \).

We define \( w_{n+1} \supseteq w_n \) such that \( w_{n+1} \) contains the first \( n \) numbers of each \( \text{sup}(p_1^n) \cup \text{sup}(p_2^n) \), for \( j \leq n \), in some fixed enumeration of \( \text{sup}(p_1^n) \cup \text{sup}(p_2^n) \).

For every natural number \( \zeta \), there exists \( p_2^{n, \zeta} \), \( \langle f_\zeta^{n,l} : l < 2^{|w_{n+1}| \times n+1} \rangle \) such that

\( \alpha \) \( p_2^n \leq_{w_{n+1}} p_2^{n, \zeta} \in P \),

\( \beta \) The \( n + 2 \)-th member of the infinite part of \( p_2^{n, \zeta}(0) \) is larger than \( \zeta \),

\( \gamma \) \( p_2^{n, \zeta} \models \exists l < 2^{|w_{n+1}| \times n+1} \forall \xi < \zeta (|f_\zeta^{n,l}(\xi) - f(\xi)| < 2^{-t(\xi)}) \),

\( \text{(Use [Ba, §9.5] and an approximation to } f(\xi) \text{ with error } t(\zeta).) \)

Now let \( \langle \zeta_i : i < \omega \rangle \) be such that for every \( \xi \in \omega \), and for every \( l < 2^{|w_{n+1}| \times n+1} \) the sequence \( \langle f_\zeta^{n,l}(\xi) : i < \omega \rangle \) converges to a real number in \([0, 1]\).

We call this limit \( f^{n,l} \). (Use a diagonal argument and the compactness theorem.)

Claim. For every \( k \in \omega \) \( \sum_{k < \zeta < \omega} f^{n,l}(\xi) \leq n + 2 \).
Proof. If not, then there exists \( \bar{k} > k \) such that
\[
\sum_{k < \xi \leq \bar{k}} f^*_{\xi, l}(x) > n + 2.
\]
Then there exists \( i \) such that \( \zeta_i > \bar{k} \) and
\[
\sum_{k < \xi \leq \bar{k}} f^*_{\xi, l}(x) > n + 2
\]
and this is a contradiction to 2.2. \( \square \)

Therefore there exists \( k_{n+1} > k_n \) such that for each \( l < 2^{\omega_{n+1} \times n+1} \) we have that
\[
\sum_{k_{n+1} < \xi} f^*_{\xi, l}(x) < \epsilon / 10^{n+8}.
\]

Then we define
\[
egin{align*}
B_1^{n+1} &= B_1^n \cup [k_n, k_{n+1}) , \\
B_2^{n+1} &= B_2^n , \\
P_2^{n+1} &= P_2^n.
\end{align*}
\]
Now, by the induction hypothesis, \( p^n \) has an extension \( p^{n+1} \) such that
\[
p_1^{n+1} \models \sum_{l \in B_1^{n+1}, l < k_{n+1}} f(l) < \epsilon \left( \frac{1}{10} + \cdots + \frac{1}{10^{n-1}} + \frac{1}{10^n} \right).
\]

This completes the induction. The reader may check that this works.

Let \( U \) be a \( P \)-name of the \( R \)-name for a rapid filter. Then using 2.3 we get the following

**Conclusion 1.** There exists \( p_1, p_2, B_1, B_2 \) such that

\[
\begin{align*}
&(*) \quad p_1 \models \mu(B_2 \in U) < \epsilon , \\
&(**) \quad p_2 \models \mu(B_1 \in U) < \epsilon ,
\end{align*}
\]
and \( B_1 \cap B_2 \) is finite, \( B_1 \cup B_2 = \omega \).

Using Conclusion 1, we get the following

**Conclusion 2.** There exists \( \delta < \omega_2 \), such that for every \( \alpha < \delta(\text{cof}(\delta) = \omega_1) \) and for every \( (B_1, B_2) \in V[G[\alpha]] \) (where \( G[\alpha] \) is \( G[P_\alpha] \)) if there exists \( \beta \) such that in \( V[G[\beta]] \) Conclusion 1 holds for \( B_1 \) and \( B_2 \), then there exists \( \beta < \delta \) such that in \( V[G[\beta]] \) Conclusion 1 holds, and \( \text{supp}(p_1) \cup \text{supp}(p_2) \subset \delta \).

Now using Lemma 2.3, working in \( V[G[\delta]] \), there exists \( B_1, B_2, p_1, p_2 \) such that
\[
V[G[\delta]] \models "(*) \land (**) \land B_1 \cup B_2 = \omega \land B_1 \cap B_2 \text{ is finite}".
\]
(Remember that $P \equiv P_{\delta \omega_2}$.) Then, by the hypothesis on $\delta$, and using the fact that $P_\delta$ is proper, we have that there exists $\beta < \delta$ and $p'_1, p'_2$ such that

$$B_i \in V[G[\beta]], \quad i = 1, 2,$$

and in $V[G[\beta]]$ we have $(\ast), (\ast\ast)$ for $p'_1, p'_2$ respectively.

Without loss of generality $p'_1 \in G[\delta]$. Now $p'_1 \cup p'_2$ is a condition and w.l.o.g. $p'_1 \cup p'_2 \in G[\omega_2]$. Therefore in $V[G[\omega_2]]$ we have

(a) $\mu(||B_1 \in U[G[\omega_2]]||) < \varepsilon$,
(b) $\mu(||B_2 \in U[G[\omega_2]]||) < \varepsilon$.

If $\varepsilon$ was chosen small, we can deduce that $\mu(||B_1 \cap B_2 \in U[G[\omega_2]]||) \neq 0$, and this implies that

$$\mu(||U[G[\omega_2]] \text{ is not a filter}||) > 0 \quad (\text{because } B_1 \cap B_2 \text{ is finite}),$$

a contradiction.

References


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