**Q-sets, Sierpinski Sets, and Rapid Filters**

HAIM JUDAH AND SAHARON SHELAH

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**Abstract.** In this work we will prove the following:

**Theorem 1.** \( \text{cons}(ZF) \) implies \( \text{cons}(ZFC + \text{there exists a } Q\text{-set of reals} + \text{there exists a set of reals of cardinality } \aleph_1 \text{ which is not Lebesgue measurable}) \).

**Theorem 2.** \( \text{cons}(ZF) \) implies \( \text{cons}(ZFC + 2^{\aleph_0} \text{ is arbitrarily larger than } \aleph_2 + \text{there exists a Sierpinski set of cardinality } 2^{\aleph_0} + \text{there are no rapid filters on } \omega^1) \).

These theorems give answers to questions of Fleissner [Fl] and Judah [Ju].

**0. Introduction**

In this work we will solve two open problems about special sets of the reals. In order to state them we need some definitions.

**0.1. Definition.** A set of reals \( A \) is a \( Q \)-set iff every subset of \( A \) is a relative \( F_\sigma \), i.e., it is a countable union of relatively closed subsets of \( A \).

\( Q \)-sets are very strange: for example \( 2^{\aleph_0} < 2^{\aleph_1} \) implies that there are no \( Q \)-sets of cardinality \( \aleph_1 \). Also \( Q \)-sets have universal measure zero, but they do not necessarily have strong measure zero (see [Fl, JSh2, Mi2]).

In [Fl] it is asked if the existence of a \( Q \)-set of cardinality \( \aleph_1 \) implies that every \( \aleph_1 \)-set of reals is of Lebesgue measure zero. Our first theorem answers this question negatively by showing

**Theorem.** \( \text{cons}(ZF) \) implies \( \text{cons}(ZFC + \text{there exists a } Q\text{-set of reals} + \text{there exists a set of reals of cardinality } \aleph_1 \text{ which is not Lebesgue measurable}) \).

We show this theorem as follows. We begin by forcing a set \( A \) of reals of cardinality \( \aleph_1 \), and then we force, with a countable support iteration of length \( \omega_1 \), making \( A \) a \( Q \)-set in the generic extension. We prove that this composition of forcing notions satisfies the Sacks property (studied in [Sh]) and, in the end
of the section, we prove that if a forcing notion has the Sacks property then in
the generic extension the old reals have outer measure one. Clearly this implies,
if we begin from $L$, that in our generic extension there exists an uncountable
$Q$-set and a $\aleph_1$-set of reals which is not Lebesgue measurable.

0.2. Definition. (a) A set of reals $A$ is a Sierpinski set iff for every measure
zero set $M$, $A \cap M$ is countable.

(b) $[\omega]^\omega = \{x : x \subseteq \omega \land |x| = \aleph_0\}$; $[\omega]^{<\omega} = \{x : x \subseteq \omega \land |x| < \aleph_0\}$.

(c) A subset $F \subseteq [\omega]^\omega$ is a rapid filter iff

(i) $(\forall x, y \in F)(x \cap y \in [\omega]^\omega)$ and $(\forall x \forall y)(x \in F \land x \subseteq y \rightarrow y \in F)$,

(ii) $(\forall f \in [\omega]^\omega \exists x \in F)(\forall n \in \omega)(|f(n) \cap x| < n)$.

Clearly, if the Sierpinski set has the cardinality of the continuum then the
real line cannot be the union of less than $2^{\aleph_0}$-many measure zero sets.

In [Ju] it was remarked that if the reals are not the union of less than $2^{\aleph_0}$-
many meager sets then there exists a rapid filter on $\omega$. Therefore it was asked:
if the reals are not the union of less than $2^{\aleph_0}$-many measure zero sets then does
there exist a rapid filter on $\omega$? The next theorem will answer this question
negatively.

Theorem. $\text{cons}(ZF) \implies \text{cons}(ZFC + 2^{\aleph_0}$ is arbitrarily larger than $\aleph_2 +$
there exists a Sierpinski set of cardinality $2^{\aleph_0}$ + there are no rapid filters on
$\omega$).

This theorem has some applications. For example, the existence of a Sier-
pinski set of cardinality $2^{\aleph_0}$ implies that every $\Delta^1_2$-set of reals is measurable
(see [JSh1]); also in this model $\omega^L_1 = \omega_1$, and therefore, we get a model for
"Every $\Delta^1_2$-set of reals is Lebesgue measurable $+\omega^L_1 = \omega_1 +$ there is no rapid
filter on $\omega$." This says that it is impossible to improve the following result of
Raisonnier [Ra]:

"If every $\Sigma^1_2$-set of reals is Lebesgue measurable and $\omega^L_1 = \omega_1$ then there is
a rapid filter on $\omega$."

We prove this theorem in §2. The model is gotten by adding $\omega_2$-many Math-
ias reals and afterward adding random reals. It was remarked by A. Miller in
[Mi1] that in the model obtained by iterating $\omega_2$-Mathias reals over $L$ there is
no rapid filter on $\omega$.

We assume that the reader knows the material given in [Ba], about countable
support iterated forcing and forcing notion satisfying the Axiom A (for the
notation). The rest of the notation is standard.

1. $Q$-sets

In this section we build a model of set theory where there exists a $Q$-set of
reals and there exists an outer measure one set of reals of cardinality $\aleph_1$. This
is the model given in 1.6. For the basic definitions the reader may consult the
introduction (§0) and Fleissner [Fl]. We also need some definitions used in the
construction.

1.1. Definition. \( \mathcal{A} = \langle a_i, A_i : i < \omega_1 \rangle \) is a suitable sequence if and only if
(a) \( A_i \in [\omega]^{\omega} \) for every \( i < \omega_1 \);
(b) if \( i < j < \omega_1 \) then \( A_i \subseteq A_j \) (\( \exists n(A_i - n \subseteq A_j) \)) and \( A_j - A_i \in [\omega]^{\omega} \);
(c) \( a_i \in [A_{i+1} - A_i]^{\omega} \) for every \( i < \omega_1 \).

1.2. Definition. For \( \mathcal{A} = \langle a_i, A_i : i < \omega_1 \rangle \) suitable, and \( X \subseteq \omega_1 \) we define the
partially ordered set \( P(\mathcal{A}, X) \) by stipulating that \( h \) belongs to \( P(\mathcal{A}, X) \) if and
only if

(i) \( h \) is a partial function from \( \omega \) to \( \{0, 1\} \);
(ii) there exists \( i = i(h) \) such that
\( \text{Dom} \ h \subseteq A_i \) (take such \( i \) minimal);
(iii) for every \( j < i(h) \) we have
\[
\begin{align*}
\text{if } j \in X & \text{ then } a_j \subseteq h^{-1}(\{1\}) , \\
\text{if } j \notin X & \text{ then } a_j \subseteq h^{-1}(\{0\}) .
\end{align*}
\]

For \( h_1, h_2 \in P(\mathcal{A}, X) \) we set \( h_1 \leq h_2 \) if and only if \( h_1 \subseteq h_2 \).

1.3. Lemma. If \( \mathcal{A} = \langle a_i, A_i : i < \omega_1 \rangle \) and \( X \subseteq \omega_1 \), \( P(\mathcal{A}, X) \) are as in 1.2 and
\( h \in P(\mathcal{A}, X) \), hence \( i(h) = \alpha \) is well defined, \( \alpha < \beta < \omega_1 \), then there exists
\( h^* \in P(\mathcal{A}, X) \) such that

\( h \subseteq h^* \) and \( i(h^*) \geq \beta \).

Proof. There exists \( g : [\alpha, \beta) \rightarrow \omega \) such that

(a) \( \alpha \leq \gamma < \beta \) implies \( (\text{Dom} \ h) \cap a_\gamma \subseteq g(\gamma) \supseteq a_\gamma - A_\beta \);

(b) \( \alpha \leq \gamma < \delta < \beta \) implies \( (a_\gamma - g(\gamma)) \cap (a_\delta - g(\delta)) = \emptyset \) (simply let
\( (\gamma_i : l < l^* \leq \omega) = [\alpha, \beta) \) and construct \( g(\gamma_l) \) by induction on \( l \)).

Now \( \text{Dom} \ h^* = (\text{Dom} \ h) \cup \bigcup_{\gamma \in [\alpha, \beta]} (a_\gamma - g(\gamma)) \) and
\[
h^*(n) = \begin{cases} 
  h(n) & \text{if } n \in \text{Dom} \ h , \\
  0 & \text{if } n \in a_\gamma - g(\gamma) \text{ and } \gamma \notin X , \\
  1 & \text{if } n \in a_\gamma - g(\gamma) \text{ and } \gamma \in X .
\end{cases}
\]

1.4. Lemma. Let \( V \) be a model of \( \text{ZFC} \) satisfying

(i) \( \mathcal{A} = \langle a_i, A_i : i < \omega_1 \rangle \) is suitable, \( \mathcal{A} \in V \);

(ii) for every \( X \subseteq \omega_1 \) there exists \( M \subseteq V \) such that \( X \in M \), \( \mathcal{A} \in M \), and
therefore, \( P(\mathcal{A}, X) \) is definable in \( M \);

(iii) there exists \( G \in V \) such that \( G \subseteq P(\mathcal{A}, X) \cap M \) and \( G \) is generic over \( M \).

Then \( B(\mathcal{A}) = \{ f \in 2^{\omega} : (\exists i < \omega_1)(\text{char}(a_i) = f) \} \) is a \( Q \)-set in \( V \).
Proof. Use 1.3 and the hypothesis. □

1.5. Definition. Let $\overline{Q} = \langle P_i; Q_j : i < \omega_2, j < \omega_2 \rangle$ be a countable support iterated forcing system satisfying

(a) $Q_0 = \langle \{ (a_i, A_i : i < \alpha) : a < \omega_1 \text{ and } \langle a_i, A_i : i < \alpha \rangle \text{ is an initial segment of a suitable sequence } \}, \subseteq \rangle$.

Let $\overline{A}$ be the $Q_0$-name of the suitable sequence generated by the $Q_0$-generic object.

(b) Let $0 < i < \omega_2$; then there exists a $P_i$-name $X$ such that

$\Vdash_{P_i} " X \subseteq \omega_1 \text{ and } Q_i = P(\overline{A}, X)".$

(c) If $i < \omega_2$ and $X$ is a $P_i$-name such that

$\Delta^P_i \ " X \upharpoonright i \" = Y \text{ and } Q_j = P(\overline{A}, Y).$

1.6. Theorem. Let $\omega_2$ be the directed limit of the iterated forcing system $\overline{Q}$ defined in 1.5. Let $G \subseteq \omega_2$ be generic over $V \models " \text{GCH}"$. Then the following holds:

(a) For every $i < \omega_2$

$\Vdash_{P_i} " Q_i \text{ satisfies } \kappa_2 - \text{c.c.}".$

Therefore $\omega_2$ satisfies $\kappa_2 - \text{c.c.}$.

(b) $\omega_2$ is a Proper Forcing notion, moreover $\omega_2$ satisfies the Sacks property. Therefore $V[G] \models " 2^\omega \cap V \text{ has outer measure one}"$ (see 1.8).

(c) If $V[G]$ we have

$B(\overline{A}[G])$ is a $Q$-set.

Proof. (a) easy; (c) use 1.4. The proof of (b) is sharp:

(In this work we say that a forcing notion $P$ satisfies the Sacks property iff $(\forall f \in V^P \forall p \in P)$ (if $p \Vdash " f \in^P V"$ then $(\exists g \geq p \exists g \in V \cap^P \omega) (q \Vdash " f(\omega) \in g(n)")$ and $(\forall n \in \omega)(|g(n)| \leq 2^{\omega^2})

$)

Let $\chi$ be sufficiently large, and $p \in \omega_2$. Let $N$ be such that

$N < \langle H(\chi), \varepsilon, \leq^* \rangle$ ($\leq^*$ is some fixed well order),

$\langle \overline{Q}, \omega_2 \rangle \in N, \quad ||N|| = \aleph_0.$

Set $\delta = N \cap \omega_1$, and let $\langle w_n : n < \omega \rangle$ be such that $\bigcup \{ w_n : n < \omega \} = N \cap \omega_2 - \{0\}$

$w_n \subseteq w_{n+1}, \quad |w_n| = n.$

Also let $\langle \tau_n : n < \omega \rangle$ be an enumeration of the $\omega_2$-names of ordinal numbers that belong to $N$. Let $\langle \alpha_n : n < \omega \rangle$ be such that $\alpha_n < \alpha_{n+1}$ and $\sup_{n<\omega} \alpha_n = \delta$.

And fix $h \in N$ such that $h$ is a $\omega_2$-name of a function from $\omega$ to $V$. 

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We will choose, by induction on $\omega$, conditions $p_n \in P\omega_2$ and finite sets $u_n$ such that

(i) $p \leq p_n \leq p_{n+1} \in P\omega_2 \cap N$;

(ii) $p_{n+1} \Vdash "\tau_n \in N \cap \text{Ord}"$;

(iii) there exists $b_n \in [V]^{2^{\omega_2}}$ such that

$$p_{n+1} \Vdash \langle h(n) \in b_n \rangle$$;

(iv) $u_n \subseteq u_{n+1} \subseteq \omega$ and $\|u_n\| = n$;

(v) if $\beta \in w_{m+1} - w_m$ and $m \leq n$ then

$$p_n [\beta] \Vdash \langle \text{Dom}(p_n(\beta)) \cap u_n \subseteq u_{m+1} \rangle$$;

(vi) $\alpha_n < lg(p_n(0))$ and $u_n \cap A_{\alpha_m} \subseteq u_{m+1}$, for $m < n$.

Claim. (*) implies (b).

Proof of the claim. Let $p_\omega$ be defined by

$$p_\omega(0) = \langle a_j, A_j : i < \delta \rangle \cup \langle A_\delta \rangle$$

when for every $n$

$$p_n(0) = \langle a_j, A_j : i < lg(p_n(0)) \rangle \quad \text{and} \quad A_\delta = \omega - \bigcup_{n \in \omega} u_n.$$If $\beta \in \bigcup_n \text{Dom}(p_n) - \{0\}$ then, from the hypothesis, for $\langle p_n(\beta) : n < \omega \rangle$ there exists $p_\omega(\beta) = \bigcup_n p_n(\beta)$ such that for every $n \in \omega$ $p_n(\beta) \subseteq p_\omega(\beta)$. (This holds in $\mathcal{M}_\beta$ where $p_\omega[\beta]$ belongs to the generic set.) Then $p_\omega$ is $\langle N, P\omega_2 \rangle$-generic and $p_\omega \Vdash \langle \forall n \rangle (h(n) \in b_n)$. $\square$

Therefore the problem is to show that the inductive construction given in (*) is realizable: suppose that $p_n$ and $u_n$ were given satisfying all conditions of (*).

Let $w_{n+1} = \{\alpha_0 < \alpha_1 < \cdots < \alpha_n < \alpha_{n+1}\}$. We try to extend $p_n$ to a condition satisfying (ii) and (iii):

Notation. If $r \in P(\overline{A}, X)$ and $h : \omega \to \{0, 1\}$ is finite then $r^{[h]} \in P(\overline{A}, X)$ when $r^{[h]}$ is such that

$$r^{[h]}(i) = \begin{cases} r(i) & \text{if } i \in \text{Dom}(r) - \text{Dom}(h), \\ h(i) & \text{if } i \in \text{Dom}(h). \end{cases}$$

Clearly for every finite $h : \omega \to \{0, 1\}$ if $r \in P(\overline{A}, X)$ then $r^{[h]} \in P(\overline{A}, X)$.

Let $\langle h_\alpha : \alpha \in w_{n+1} : l \leq l_0 \leq 2^{(n+1)^2} \rangle$ be a list of all $\langle h_\alpha : \alpha \in w_{n+1} \rangle$ such that for every $\alpha \in w_{m+1} - w_m$

$$h_\alpha : u_n - u_{m+1} \to \{0, 1\}.$$Now we choose by induction on $l \leq l_0$, $p_{n,l}$ such that

(a) $p_{n,0} = p_n$. 

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(b) $p_{n,l} \leq p_{n,l+1} \in P_{\omega_2} \cap N$.

(c) For every $\alpha \in w_{m+1} - w_m$, $m < n$ we have

$$p_{n,l}[\alpha \Vdash \text{"Dom } p_{n,l}(\alpha) \cap u_n \subseteq u_{m+1}"].$$ 

(d) If $p_{n,l+1}'$ is such that

$$p_{n,l+1}'(\alpha) = \begin{cases} p_{n,l+1}(\alpha)^{[h_\alpha]} & \text{if } \alpha \in w_n, \\ p_{n,l+1}(\alpha) & \text{if } \alpha \notin w_n, \end{cases}$$

then $p_{n,l+1}'$ forces values for $\tau_i$ ($i \leq n+1$) and for $h(i)$ ($i \leq n+1$).

The induction.

for $l = 0$: $p_{n,0} = p_n$,

for $l + 1$: $p_{n,l}$ was defined satisfying (a), (b), (c), (d), and let $p_{n,l}'$ be such that

$$p_{n,l}'(\alpha) = \begin{cases} p_{n,l}(\alpha)^{[h_\alpha]} & \text{if } \alpha \in w_n, \\ p_{n,l}(\alpha) & \text{if } \alpha \notin w_n. \end{cases}$$

There exists $q_{n,l} \geq p_{n,l}'$ such that $q_{n,l}$ forces values for $\tau_i$ ($i \leq n+1$), $h(i)$ ($i \leq n+1$). Fix such $q_{n,l}$ in $N$, and then we define $p_{n,l+1}$ satisfying

$$p_{n,l+1}(\alpha) = \begin{cases} q_{n,l}(\alpha)[(\omega - \text{Dom } h_\alpha)] & \text{if } \alpha \in w_n, \\ q_{n,l}(\alpha) & \text{if } \alpha \notin w_n. \end{cases}$$

Clearly $p_{n,l+1}$ satisfies (a), (b), (c), and (d). $P_{n,l_0}$ is almost $p_{n+1}$, only we need to extend it in order to find the $u_{n+1}$ required. Clearly $p_{n,l_0}$ fix $2^{n^2}$-many possible values to $h(n)$. The next lemma is exactly what we need.

1.7. Lemma. If $w \subseteq \omega_2$ is finite, and $p \in P_{\omega_2}$ and $n < \omega$ then there exists $k \in \omega$ and $q \in P_{\omega_2}$ such that

(a) $n < k < \omega$;

(b) $p \leq q$;

(c) for every $\alpha \in w$

$$q[\alpha \Vdash \text{"Dom } q(\alpha) \cap [0, n) = \text{Dom } p(\alpha) \cap [0, n"]];$$

(d) for every $\alpha \in w$

$$q[\alpha \Vdash \text{"} k \notin \text{ Dom } q(\alpha)"].$$

Proof. Let $w = \{\alpha_1 < \alpha_2 < \cdots < \alpha_m\}$. We choose by (decreasing) induction

$$q_m, q_{m-1}, \ldots, q_1, q_0$$

satisfying

(i) $q_m = p$,

(ii) $q_i[\alpha_i \leq q_{i-1} \in P_{\alpha_i}$,
(iii) \( q_{l-1} \) forces less than \( 2^{n \times m} \)-many possible values for the following names of ordinals (i.e., satisfies condition (c)):

(A) \( \gamma_i \) such that

\[
\text{Dom}(q_i(\alpha_i)) = \star [A_{\eta_i}]^{q_{i-1}(0)}
\]

(remember that \( Q_0 \) fixes \( A_{\eta_i} \));

(B) \( k_i \) such that

\[
1 + \max(\text{Dom} q_i(\alpha_i) - [A_{\eta_i}]^{q_{i-1}(0)}) < k_i
\]

(use the above construction in order to get such \( q_{l-1} \)).

Therefore there exists \( k_i^j \) \((i = 1, \ldots, t, < \omega)\); \( \gamma_i^j \) \((i = 1, \ldots, s_i < \omega)\) such that

\[
q_i \models_p "(\exists i < t, \exists j < s_i)(\gamma_i = \gamma_i^j \land k_i = k_i^j)"
\]

Now let

\[
\bar{\gamma}_i = \max\{1, \gamma_i^1, \ldots, \gamma_i^{s_i}\},
\]

\[
\bar{k}_i = \max\{1, k_i^1, \ldots, k_i^{s_i}\}
\]

and

\[
q = \bigcup_{l=0}^{m} q_i,
\]

\(\text{i.e., if } \alpha_0 = -1 \text{ and } \alpha_{m+1} = \omega_2\)

then

\[
q(\alpha) = q_i(\alpha) \text{ where } \alpha_{l-1} \leq \alpha < \alpha_l.
\]

Then we choose \( k \) such that

\[
k > \bar{k}_i(l \leq m), \quad k \not\in A_{\gamma_0} \cup A_{\gamma_1} \cup \cdots \cup A_{\gamma_m}.
\]

This concludes the proof of part (b) of Theorem 1.6. \( \square \)

1.8. Theorem. If \( V \) is a model for ZFC and \( P \in V \) is a forcing notion satisfying the Sacks property then \( \models_p "\mu(2^\omega \cap V) = 1". \)

Proof. Suppose that there exists \( p \in P \) such that \( p \models_p "\mu(2^\omega \cap V) = 0". \) Then there exists \( (I_n : n < \omega) \) such that for every \( n \in \omega \) \( I_n \) is a \( P \)-name of a rational interval, and

\[
0 \models_p "\sum \mu(I_n) < \infty",
\]

\[
p \models_p "2^\omega \cap V \subseteq \bigcap_{n} \bigcup_{m \geq n} I_n".
\]

Then there exists \( g \) such that \( g \) is a \( P \)-name of a function from \( \omega \) to \( \omega \) and satisfying

\[
0 \models_p "\sum_{m > g(n)} \mu(I_n) < \frac{1}{2^{n^2+n}}.".
\]
Fix $q \geq p$ and $g \in \omega^\omega$, given by the Sacks property such that
\[ q \vdash \forall n (g(n) < g(n)). \]
Again using the Sacks property, let $r \geq q$ and \[ \langle \langle I_{j,i}^k : g(i) \leq k < g(i+1) : j < 2^i \rangle : i < \omega \rangle \]
satisfying for every $i < \omega$, there exists $j < 2^i$
\[ r \vdash \forall k \in [g(i), g(i+1)) (I_k = I_{j,i}^k) \]
and under this notation we define
\[ J_i = \bigcup_{j < 2^i} \bigcup_{g(i) \leq k < g(i+1)} I_{j,i}^k. \]
Then for every $i$
\[ \mu(J_i) \leq 2^i \cdot \frac{1}{2^{i+1}} = \frac{1}{2^i}. \]
Therefore $\sum_{i<\omega} \mu(J_i) < \infty$ and this implies that there exists $x \in 2^\omega$ such that $x \notin \bigcap_i \bigcup_{j \geq i} I_j$, and by the hypothesis this implies that
\[ r \vdash \forall x \notin \bigcap_i \bigcup_{j \geq i} I_j \]
a contradiction. $\Box$

It may be possible that these ideas help to solve the question in [JSh]: remember that the Sacks property implies the Laver property.

2. Rapid filters and Sierpinski sets

2.0. Theorem. $\text{cons}(ZF) \Rightarrow \text{cons}(ZFC + \text{there exists a Sierpinski set} + 2^{\aleph_0} \text{is a regular cardinal} + \text{there are no rapid filters on } \omega)$.

Proof. We begin with $V = L$ and let $P$ be the $\omega_1$-iteration of Mathias reals with countable support and let $R$ be a $P$-name of the product of $\aleph_0$ random reals (i.e., the measure algebra). Then we will prove that if $G \subseteq P \ast R$ then
\[ (\ast) \hspace{1cm} V[G] \models \text{"there are no rapid filters on } \omega". \]
Clearly this is enough in order to obtain the inclusion of the theorem. The proof of $(\ast)$ will take the remainder of this section.

2.1. Definition. Let $\mathfrak{a}$ be an $R$-name of a set of ordinals; then we define
\[ \mathfrak{a}(n) = \mu(||n \in \mathfrak{a}||). \]
Then $\mathfrak{a}(\_)$ is a function from ordinals to $[0, 1]$.

2.2. Fact. Let $M$ be a model of $ZFC$ and let $R$ be any product of random reals, i.e., a measure algebra. Let $\langle n_i : i < \omega \rangle$ be in $M$ an increasing sequence of natural numbers. Assume that $\mathfrak{a}$ is in $M^R$ and
\[ M \models \forall k \in \omega (||n_k \cap \mathfrak{a}|| < k). \]
Then for every $k \in \omega$

$$M \models \sum_{m=0}^{n_k-1} a(m) \leq k.$$  

Proof. We will work in $M$. Let $\varphi_i(m)$ be the formula that says

"$m$ is the $i$th member of $a$".

Then $a(m) = \sum_{i=0}^{m} \mu(\|\varphi_i(m)\|)$. Also if $m \neq n$ then $\mu(\|\varphi_i(m)\| \cdot \|\varphi_i(n)\|) = 0$.

Assume that there exists $k \in \omega$ such that

$$\sum_{m=0}^{n_k-1} a(m) > k.$$  

Then

$$\sum_{m=0}^{n_k-1} \sum_{i=1}^{m} \mu(\|\varphi_i(m)\|) > k.$$  

Therefore

$$\sum_{m=0}^{n_k-1} \sum_{i=1}^{k} \mu(\|\varphi_i(m)\|) > k \quad \text{(because $m < n_k$)}$$  

and hence

$$\sum_{i=1}^{k} \sum_{m=0}^{n_k-1} \mu(\|\varphi_i(m)\|) > k.$$  

And thus there exists $i$ such that $\sum_{m=0}^{n_k-1} \mu(\|\varphi_i(m)\|) > 1$ and this implies that there exists $n \neq m$ such that

$$\mu(\|\varphi_i(m)\| \cdot \|\varphi_i(n)\|) \neq \emptyset,$$

a contradiction. $\Box$

From now on we fix $M \models \text{ZFC}$. Let $P$ in $M$ be the $\omega_2$-iteration of Mathias reals. Each Mathias real adds a sequence of natural numbers. $P(0)$ is the first coordinate of $P$. Let $\langle n_i : i < \omega \rangle$ be a $P$-name for the sequence added by $P(0)$. Let $R$ be a $P$-name for a product of random reals (i.e., a measure algebra). If $H \subseteq P$ is generic over $M$ then $R[H]$ is the realization of $R$ in $M[H]$. Let $a \in M[H]$ be an $R[H]$-name for a sequence of natural numbers such that

$$M[H] \models \|R[H]\| \cap |n_k[H] \cap a| < k \quad \text{"}$$  

Therefore, by 2.2,

$$M[H] \models \sum_{m=0}^{n_k[H]-1} a(m) \leq k.$$  

The function $a(\cdot) : \omega \to [0,1]$, defined by $a$, lies in $M[H]$, so it has a $P$-name. Let $f$ be such a name (we omit this relation with $a$ because it is clear).
2.3. Lemma. Let \( p \in P \), and \( \epsilon > 0 \) given; then there exists \( w_0 \subseteq w_1 \subseteq \cdots \subseteq w_n \subseteq \cdots \), \( p_1^n, p_2^n \), \( k_0 < k_1 < \cdots < k_n < \cdots \), \( B_1^0 \subseteq \cdots \subseteq B_i^n \subseteq \cdots \), \( i = 1, 2 \), satisfying

(a) for every \( n \), \( w_n \subseteq \omega_2 \); \( k_n \in \omega \); \( p_1^i \in P \), \( i = 1, 2 \); \( B_i^0 \in [\omega]^{<\omega} \), \( i = 1, 2 \); \( B_1^0 \cap B_2^0 = B_1^n \cap B_2^n \), \( \cup B_1^0 \cup B_2^n = \omega \).

(b) \( \bigcup_i (\text{Dom}(p_1^n) \cup \text{Dom}(p_2^n)) = \bigcup_i w_n \).

(c) For every \( n \), \( p_i^n \leq_w p_i^{n+1} \), \( i = 1, 2 \) (see [Ba, §7]).

(d) \( p_i^0 = p \) for \( i = 1, 2 \).

(e) If \( m \) is even, then for every \( k > k_m \) there exists \( q_{w_{m+1}} \geq p_m^m \) such that

(i) \( q \models \sum_{l=k_m}^k f(l) < \epsilon/10^m \),

(ii) \( q \models \sum_{l \in B_1^m} f(l) < \epsilon \left( \frac{1}{10} + \frac{1}{10^2} + \cdots + \frac{1}{10^{m-1}} \right) \),

(iii) \( p_2^m \models \sum_{l \not\in B_2^m} f(l) < \epsilon \left( \frac{1}{10} + \cdots + \frac{1}{10^{m-1}} \right) \).

(f) If \( m \) is odd, then for every \( k > k_m \) there exists \( q_{w_{m+1}} \geq p_m^m \) such that

(i) \( q \models \sum_{l=k_m}^k f(l) < \epsilon/10^m \),

(ii) \( q \models \sum_{l \in B_2^m} f(l) < \epsilon \left( \frac{1}{10} + \cdots + \frac{1}{10^{m-1}} \right) \),

(iii) \( p_1^m \models \sum_{l \not\in B_1^m} f(l) < \epsilon \left( \frac{1}{10} + \cdots + \frac{1}{10^{m-1}} \right) \).

(g) \( 0 \in w_0 \).

Proof. By induction on \( m \).

\( m = n + 1 \): By the symmetry of (e) and (f) without loss of generality \( n \) is even. In this stage we need to take care of the relation \( \leq_{w_n} \).

We define \( w_{n+1} \supseteq w_n \) such that \( w_{n+1} \) contains the first \( n \) numbers of each \( \sup(p_1^i) \cup \sup(p_2^i) \), for \( j \leq n \), in some fixed enumeration of \( \sup(p_1^i) \cup \sup(p_2^i) \).

For every natural number \( \zeta \), there exists \( p_2^{n,\zeta} \), \( (f_{\zeta, l} : l < 2^{w_{n+1}}, \zeta) \) such that

(\( \alpha \)) \( p_2^n \leq_{w_{n+1}} p_2^{n, \zeta} \in P \),

(\( \beta \)) The \( n + 2 \)-th member of the infinite part of \( p_2^{n, \zeta}(0) \) is larger than \( \zeta \),

(\( \gamma \)) \( p_2^{n, \zeta} \models \exists \zeta < 2^{w_{n+1}} \forall \xi < \zeta (|f_{\zeta, l}(\xi) - f(\xi)| < 2^{-t(\zeta)}) \) where \( t(\zeta) = 2^{2^{\zeta+1}+|w_{n+1}|} \).

(Use [Ba, §9.5] and an approximation to \( f(\xi) \) with error \( t(\zeta) \).)

Now let \( (\zeta_i : i < \omega) \) be such that for every \( \xi \in \omega \), and for every \( l < 2^{w_{n+1}+1} \), the sequence \( (f_{\zeta_i, l}(\xi) : i < \omega) \) converges to a real number in \([0, 1]\).

We call this limit \( f_{\zeta}^{*, l} \). (Use a diagonal argument and the compactness theorem.)

Claim. For every \( k \in \omega \) \( \sum_{k < \xi < \omega} f_{\zeta, l}(\xi) \leq n + 2 \).
Proof. If not, then there exists \( k > k \) such that
\[
\sum_{k < \xi \leq k} f^{*,l}(\xi) > n + 2.
\]
Then there exists \( i \) such that \( \zeta_i > \tilde{k} \) and
\[
\sum_{k < \xi \leq \tilde{k}} f^{*,l}(\xi) > n + 2
\]
and this is a contradiction to 2.2. \( \square \)

Therefore there exists \( k_{n+1} > k_n \) such that for each \( l < 2^{\aleph_n+1} \), we have that
\[
\sum_{k_{n+1} < \xi} f^{*,l}(\xi) < \varepsilon / 10^{n+8}.
\]

Then we define
\[
\begin{align*}
B_1^{n+1} &= B_1^n \cup \{k_n, k_{n+1}\}, \\
B_2^{n+1} &= B_2^n, \\
\mu_{n+1} &= \mu_n.
\end{align*}
\]

Now, by the induction hypothesis, \( p_n^n \) has an extension \( p_1^{n+1} \) such that
\[
p_1^{n+1} \models \sum_{l \in B_1^{n+1}} f(l) < \varepsilon \left( \frac{1}{10} + \cdots + \frac{1}{10^{n-1}} + \frac{1}{10^n} \right).
\]

This completes the induction. The reader may check that this works.

Let \( U \) be a \( P \)-name of the \( R \)-name for a rapid filter. Then using 2.3 we get the following

Conclusion 1. There exists \( p_1, p_2, B_1, B_2 \) such that
\[
\begin{align*}
(\ast) & \quad p_1 \models \mu(\|B_2 \in U\|) < \varepsilon, \\
(\ast\ast) & \quad p_2 \models \mu(\|B_1 \in U\|) < \varepsilon,
\end{align*}
\]
and \( B_1 \cap B_2 \) is finite, \( B_1 \cup B_2 = \omega \).

Using Conclusion 1, we get the following

Conclusion 2. There exists \( \delta < \omega_2 \), such that for every \( \alpha < \delta(\text{cof}(\delta) = \omega_1) \) and for every \( \langle B_1, B_2 \rangle \in V[G[\alpha]] \) (where \( G[\alpha] \) is \( G[P_\alpha] \)) if there exists \( \beta \) such that in \( V[G[\beta]] \) Conclusion 1 holds for \( B_1 \) and \( B_2 \), then there exists \( \beta < \delta \) such that in \( V[G[\beta]] \) Conclusion 1 holds, and \( \text{supp}(p_1) \cup \text{supp}(p_2) \subseteq \delta \).

Now using Lemma 2.3, working in \( V[G[\delta]] \), there exists \( B_1, B_2, p_1, p_2 \) such that
\[
V[G[\delta]] \models "(\ast) \land (\ast\ast) \land B_1 \cup B_2 = \omega \land B_1 \cap B_2 \text{ is finite}".
\]
(Remember that $P \cong P_{\delta \omega_2}$. Then, by the hypothesis on $\delta$, and using the fact that $P_\delta$ is proper, we have that there exists $\beta < \delta$ and $p_1', p_2'$ such that
\[ B_i \in V[G[\beta]], \quad i = 1, 2, \]
and in $V[G[\beta]]$ we have $(*)$, $(**)$ for $p_1'$, $p_2'$ respectively.

Without loss of generality $p_1' \in G[\delta]$. Now $p_1' \cup p_2$ is a condition and w.l.o.g. $p_1' \cup p_2 \in G[\omega_2]$. Therefore in $V[G[\omega_2]]$ we have
\begin{enumerate}
  \item $\mu(\|B_1 \in U[G[\omega_2]]\|) < \epsilon,$
  \item $\mu(\|B_2 \in U[G[\omega_2]]\|) < \epsilon.$
\end{enumerate}

If $\epsilon$ was chosen small, we can deduce that $\mu(\|B_1 \cap B_2 \in U[G[\omega_2]]\|) \neq 0$, and this implies that
\[ \mu(\|U[G[\omega_2]] \text{ is not a filter}\|) > 0 \quad (\text{because } B_1 \cap B_2 \text{ is finite},) \]
a contradiction.

References


Department of Mathematics, Bar Ilan University, 52900 Ramat Gan, Israel

Institute of Mathematics, The Hebrew University of Jerusalem, Israel