PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 111, Number 3, March 1991

REGULARITY OF THE $\bar{\partial}$-NEUMANN PROBLEM

SO-CHIN CHEN

(Communicated by Clifford J. Earle, Jr.)

ABSTRACT. In this paper we prove that locally there is no obstruction to global regularity for the $\bar{\partial}$-Neumann problem. By this we mean the following: Let D be a smoothly bounded pseudoconvex domain in $\mathbf{C}^n$, $n \geq 2$, and let $p \in \mathbf{D}$. Given any $m \in \mathbf{Z}^+$, one can construct a smoothly bounded pseudoconvex subdomain $D_m \subseteq D$ such that $bD_m \cap bD$ contains an open neighborhood of p in bD and the $\bar{\partial}$-Neumann problem on D_m is globally regular up to order m in the sense of the Sobolev norm.

1. INTRODUCTION

Let D be a smoothly bounded pseudoconvex domain in $\mathbf{C}^n$, $n \geq 2$, and let p be a boundary point. The $\bar{\partial}$-Neumann problem on the domain D is concerned with the existence and the regularity of the solution u to the following equation:

$$Q(u, v) = (\bar{\partial}u, \bar{\partial}v) + (\bar{\partial}^* u, \bar{\partial}^* v) = (f, v), \tag{1.1}$$

for all $v \in \mathscr{D}_{p,q}(D)$ with given $f \in L^2_{p,q}(D)$, where $\mathscr{D}_{p,q}(D)$ is defined in our Main Theorem. Namely, given $f \in W^k_{p,q}(D)$ for any $k \in \mathbf{Z}^+$, let $u \in L^2_{p,q}(D)$ be the solution to the equation (1.1). Then we ask: is $u \in W^k_{p,q}(D)$ and $\|u\|_k \leq C_k\|f\|_k$ for some constant $C_k > 0$? The solution of this problem implies that condition R holds on D. Here condition R means that the Bergman projection on D maps $C^\infty(\bar{D})$ into itself. Then by a well-known fact (e.g., see [2]) that one can reduce the classification problem between smoothly bounded pseudoconvex domains to simply examining the boundary invariants of the domains. This is one of the main consequences of the $\bar{\partial}$-Neumann problem. For more details, see [10].

Next, a few words about the history of the $\bar{\partial}$-Neumann problem are in order. This problem was proved affirmatively by Kohn [11], [11A], [12], [13] and Catlin [5] when the domain D is of finite type in the sense of D'Angelo [9]. In fact they obtained much stronger local results; i.e., subelliptic estimates of order ε, $0 < \varepsilon \leq \frac{1}{2}$. However, it is not hard to see that the singularity of the solution to

Received by the editors June 7, 1989 and, in revised form, February 27, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 32F20, 35N15.
Key words and phrases. Pseudoconvex domains, $\bar{\partial}$-Neumann problems.

the $\bar{\partial}$-equation can propagate along the complex variety sitting in the boundary. Hence in general there is no hope of obtaining local regularity results for any weakly pseudoconvex domain.

Catlin [6] first introduced the notion called property P. He showed that if a smoothly bounded pseudoconvex domain satisfies property P, then global regularity of the $\bar{\partial}$-Neumann problem holds. However, an analytic variety is still not allowed to exist on the boundary of domains with property P. Recently Boas [4] showed, by using the localized version of property P, that the $\bar{\partial}$-Neumann problem is globally regular if the domains satisfy the requirement that the Hausdorff two-dimensional measure of the set where the Levi-form degenerates to infinite type is equal to zero. Also, Chen was able to prove [7] that if the domain D is circular and satisfies some transversal condition on the boundary, then global regularity of the $\bar{\partial}$-Neumann problem holds on D. Recently the author proved [8] that the $\bar{\partial}$-Neumann problem is globally regular on a class of weakly pseudoconvex domains with Levi-flat boundaries.

For arbitrary pseudoconvex domains this problem is still quite open. However, the main result proved in this paper shows that the local geometry of the boundary presents no obstruction to the global regularity of the $\bar{\partial}$-Neumann problem.

Main Theorem. *Locally there is no obstruction to global regularity for the $\bar{\partial}$-Neumann problem. By this statement we mean the following: Given any $m \in \mathbf{Z}^+$, there is a smoothly bounded pseudoconvex subdomain $D_m \subseteq D$ such that $bD_m \cap bD$ contains an open neighborhood of p in bD and such that the $\bar{\partial}$-Neumann problem is globally regular up to order m on D_m in the sense of Sobolev norm; i.e., given $f \in W^k_{p,q}(D_m)$, $k \le m$, let $u \in L^2_{p,q}(D_m)$ be the solution to the $\bar{\partial}$-Neumann problem,*

$$Q(u, v) = (\bar{\partial} u, \bar{\partial} v) + (\bar{\partial}^* u, \bar{\partial}^* v) = (f, v) \tag{1.1}$$

for all $v \in \mathscr{D}_{p,q}(D_m)$. Then we have

(i) *$u \in W^k_{p,q}(D_m)$, and the following estimate holds:*
(ii) *$\|u\|_k \le C_k \|f\|_k$, for some constant $C_k > 0$,*

where $W^k_{p,q}(D_m)$ denotes the Sobolev space of order k of (p, q)-forms on D_m and $\mathscr{D}_{p,q}(D_m)$ denotes the space of all smooth (p, q)-forms on $\bar{D}_m$ that satisfy the Neumann boundary conditions (e.g., see [10]).

Since global regularity for the $\bar{\partial}$-Neumann problem implies condition R, from the Main Theorem we obtain immediately the following corollary:

Corollary (D. Barrett [1]). *Locally there is no obstruction to condition R. By this statement we mean the following: Let D be a smoothly bounded pseudoconvex domain in $\mathbf{C}^n$ and let $p \in bD$. Then for every positive integer m there is a smoothly bounded subdomain $D_m \subseteq D$ such that $bD_m \cap bD$ contains an open neighborhood of p in bD and the Bergman projection for D_m maps $W^k(D_m)$*

boundedly onto $W^k(D_m) \cap H(D_m)$ *for all* $k \le m$, *where* $H(D_m)$ *denotes the space of holomorphic functions on* D_m.

We hope that the result presented here can provide some insight into this problem.

Finally the author would like to thank R. M. Range for many helpful discussions during the preparations of this paper.

2. Proof of the main theorem

The formulation of the $\bar\partial$-Neumann problem is now well known; for instance, see [10]. So we omit it.

Let $D \subset \mathbf{C}^n$, $n \ge 2$, be a smoothly bounded pseudoconvex domain with defining function r. Let $p \in bD$ be the origin in $\mathbf{C}^n$. For any given $m \in \mathbf{Z}^+$, we are going to show that there exists a smoothly bounded pseudoconvex subdomain $D_m \subseteq D$ such that $bD_m \cap bD$ contains an open neighborhood of p in bD and that the $\bar\partial$-Neumann problem on D_m is globally regular up to order m measured in the Sobolev norm.

Consider the ball $\mathbf{B}_0$ with center p and radius ρ_0, $0 < \rho_0 < 1/(4\sqrt{e})$, such that there exists a $C_0 > 0$, depending on m, satisfying

$$\sup_{z \in \bar D \cap \bar{\mathbf{B}}_0} \left(\sum_{|\alpha| \le m+4} \left| \frac{\partial^\alpha r}{\partial x^\alpha} \right| \right) \le C_0,$$

where the x's are the real coordinates of the underlying space $\mathbf{R}^{2n}$ of $\mathbf{C}^n$. Put $\widetilde{C} = C_0 + \varepsilon$ for some small $\varepsilon > 0$. Consider a decreasing sequence $\{\rho_j\}_{j=1}^\infty$ with $\rho_1 < \rho_0$ and $\rho_j \to 0$. Denote by $\mathbf{B}_j$ the ball with center p and radius ρ_j. Now it is standard (e.g., see [3]) that one can construct a subdomain $D_j \subseteq \mathbf{B}_j$ for each j such that

(i) $D_j \subseteq D$ and $bD_j \cap bD$ contains an open neighborhood of p in bD,
(ii) D_j is a smoothly bounded pseudoconvex domain,
(iii) the points of $bD_j - bD$ are strictly pseudoconvex, and
(iv) the defining function r_j for D_j agrees to infinite order with r on $bD_j \cap bD$.

Hence by continuity there exists a decreasing sequence of open neighborhoods $\{V_j\}_{j=1}^\infty$ such that

(i) $bD_j \cap bD \subseteq V_j \subseteq \bar V_j \subseteq \mathbf{B}_j$ for all j,
(ii) the following estimate holds uniformly for all j,

$$\sup_{z \in \bar D_j \cap \bar V_j} \left(\sum_{|\alpha| \le m+4} \left| \frac{\partial^\alpha (r_j - r)}{\partial x^\alpha} \right| \right) \le \varepsilon. \tag{2.1}$$

In particular, this implies that

$$\sup_{z \in \bar D_j \cap \bar V_j} \left(\sum_{|\alpha| \le m+4} \left| \frac{\partial^\alpha r_j}{\partial x^\alpha} \right| \right) \le \widetilde{C}. \tag{2.1$'$}$$

Now it is routine to construct a smooth bounded plurisubharmonic function on $\overline{\mathbf{B}}_j$ with large Hessian.

Lemma 2.2. *On* $\mathbf{B}_j$ *for any given* $M > 0$, $M \le (e\rho_j^2)^{-1}$, *there exists a smooth plurisubharmonic function* $\lambda_j \in C^\infty(\overline{\mathbf{B}}_j)$, $0 \le \lambda_j \le 1$, *such that for all* $z \in \overline{\mathbf{B}}_j$ *and all* $t \in \mathbf{C}^n$, *we have*

$$\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \lambda_j}{\partial z_\alpha \partial \overline{z}_\beta}(z) t_\alpha \overline{t}_\beta \ge M|t|^2.$$

Proof. Let $\lambda_j = e^{(eM)(|z_1|^2+\cdots+|z_n|^2)-1}$.

Then by using this sort of plurisubharmonic function, D. Catlin proved [6] that a compactness estimate holds on D_j up to certain order.

Lemma 2.3 (Catlin). *On* D_j *we have, for all* $M \le (e\rho_j^2)^{-1}$,

(i) $\|f\|^2 \le \frac{40}{M}(\|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2) + \|\zeta_M f\|_{-1}^2$, $\zeta_M \in C_0^\infty(D_j)$, *if* $f \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*)$, *where* $\| \ \|_{-1}$ *denotes the Sobolev norm of order* -1. *In particular, if* $M = (e\rho_j^2)^{-1}$, *we have*

$$\|f\|^2 \le (40e)\rho_j^2(\|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2) + \|\zeta_j f\|_{-1}^2, \qquad \textit{for some } \zeta_j \in C_0^\infty(D_j),$$

and

(ii) *a weaker estimate holds on* D_j, *namely,*

$$\|f\|^2 \le \|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2,$$

for all $f \in \mathscr{D}_{p,q}(D_j)$ *and all* j.

Estimate (ii) is valid because $\rho_j \le \frac{1}{(4\sqrt{e})}$ for all j. (See the proof in [6].)

Now we are ready to prove our Main Theorem. The idea is essentially contained in Kohn-Nirenberg [14], so we simply outline the proof here. We first apply the elliptic regularization technique developed in [14] to the form Q to make it elliptic, namely, define the new form Q_δ for $0 < \delta < 1$ on D_j as follows:

(2.4)

$$\begin{aligned} Q_\delta(u_{j\delta}, v) &= Q(u_{j\delta}, v) + \delta \sum_{i=1}^{n} \left\{ \left(\frac{\partial}{\partial z_i} u_{j\delta}, \frac{\partial}{\partial z_i} v \right) + \left(\frac{\partial}{\partial \overline{z}_i} u_{j\delta}, \frac{\partial}{\partial \overline{z}_i} v \right) \right\} \\ &= (f_j, v), \end{aligned}$$

for all $v \in \mathscr{D}_{p,q}(D_j)$ with given $f_j \in C^\infty_{p,q}(\overline{D}_j)$. Here $u_{j\delta} \in L^2_{p,q}(D_j)$ is the solution to equation (2.4). Since an elliptic-type estimate holds for Q_δ, we see that $u_{j\delta} \in C^\infty_{p,q}(\overline{D}_j)$ for all $\delta > 0$. The main point of the proof is to obtain a uniform a priori estimate in δ for $u_{j\delta}$ in terms of f_j. So we introduce a cut-off function φ_j for each j with $0 \le \varphi_j \le 1$ such that $\varphi_j \equiv 1$ in some open neighborhood of $bD_j \cap bD$ and such that φ_j is supported in V_j. Denote by $D_t(k)$ any tangential differential operator of order k, and by $D(k)$ any differential operator of order k that may involve the normal differentiation. Then the proof of the a priori estimate will be accomplished by induction on k. That is, we are going to show that the following estimate (2.5) holds for all q, $0 \le q \le m$. Here we put

$$I_q = \sum_{\beta\le q} \|\varphi_j D(\beta) u_{j\delta}\|^2 + \sum_{\beta\le q} Q_\delta(\varphi_j D_t(\beta) u_{j\delta}, \varphi_j D_t(\beta) u_{j\delta}) \le C(q, j)\|f_j\|_q^2, \tag{2.5}$$

where the summation runs over a finite basis of differential operators of order β, and the constant $C(q, j)$ in general depends on q and j but will be independent of $\delta > 0$. The initial step $q = 0$ is easy to check simply by observing the following fact:

(2.6) The derivative of the cut-off function is supported at strongly pseudoconvex points of D_j.

Therefore we assume that the estimate (2.5) is valid for all $q \le k-1$. Then we prove the case $q = k$. First we estimate the second terms in (2.5). By commuting $\varphi_j D_t(\beta)$ with $\bar\partial$, $\bar\partial^*$, $\frac{\partial}{\partial z_i}$ or $\frac{\partial}{\partial \bar z_i}$, $i = 1, \dots, n$, we get, for $\beta = k$:

$$Q_\delta(\varphi_j D_t(k) u_{j\delta}, \varphi_j D_t(k) u_{j\delta}) = (\varphi_j D_t(k) f_j, \varphi_j D_t(k) u_{j\delta}) + E. \tag{2.7}$$

The error terms E can be handled as follows. First we observe that the commutators can be written in the following way:

$$[\bar\partial, D_t(k)] = \sum_{\ell=1}^{k} a_\ell(z) D(1) D_t(k-\ell), \tag{2.8}$$

and

$$[D_t^*(k), \bar\partial] = \sum_{\ell=1}^{k} D_t^*(k-\ell) a_\ell(z) D(1), \tag{2.9}$$

where $D_t^*(k)$ is the adjoint of $D_t(k)$, and $a_\ell(z)$ denotes a smooth function that results from taking commutators and from integration by parts. Similar equations also hold if $\bar\partial$ is replaced by $\bar\partial^*$, $\frac{\partial}{\partial z_i}$ or $\frac{\partial}{\partial \bar z_i}$, $i = 1, \dots, n$. It is clear that $a_\ell(z)$ involves only the derivatives of the defining function r_j up to certain order. Then we apply another fact:

(2.10) The estimate (2.1) (or (2.1)$'$) is uniform for all j up to order m.

Hence for each given $m \in \mathbf{Z}^+$, there exists a constant $C(m) > 0$ such that

$$\sup_{z\in \operatorname{supp} \varphi_j} \{|a_\ell(z)|\} \le C(m). \tag{2.11}$$

The constant $C(m)$ depends on m, but it will be independent of j by (2.10).

Next we have the following well-known trick:

$$|AB| \le \frac{1}{C}|A|^2 + C|B|^2, \tag{2.12}$$

for any $C > 0$. Therefore we can make the coefficient $1/C$ as small as we wish by choosing large C.

If we put these together, it is not hard to see that, by the induction hypotheses, the error terms E can be estimated by

$$\begin{aligned}|E| \le C(j)\|f_j\|_k^2 + \frac{C_1(m)}{C_2}(Q_\delta(\varphi_j D_t(k)u_{j\delta}, \varphi_j D_t(k)u_{j\delta}) \\ + \|\varphi_j D_t(k)u_{j\delta}\|^2) + C_3(m)\|D(1)\varphi_j D_t(k-1)u_{j\delta}\|^2,\end{aligned} \tag{2.13}$$

where $C_1(m)$ and $C_3(m)$ are uniform in j and $C_3(m)$ may depend on the constant $C_2 > 0$ that will be determined later.

Now we use the fact that Q is noncharacteristic to the boundary, so one can estimate

(2.14)

$$\begin{aligned}\|D(1)\varphi_j D_t(k-1)u_{j\delta}\|^2 &\le C_4\Bigg(Q(\varphi_j D_t(k-1)u_{j\delta}, \varphi_j D_t(k-1)u_{j\delta}) \\ &\quad + \sum_{\ell=1}^{2n-1}\|\varphi_j X_\ell D_t(k-1)u_{j\delta}\|^2 + \|\varphi_j D_t(k-1)u_{j\delta}\|^2\Bigg) \\ &\le C(j)\|f_j\|_k^2 + C_4\sum_{\ell=1}^{2n-1}\|\varphi_j X_\ell D_t(k-1)u_{j\delta}\|^2,\end{aligned}$$

where $X_1, \dots, X_{2n-1}$ is a basis for the tangent space, and by (2.10) the constant C_4 is also uniform in j.

So if we choose C_2 large enough say, $C_2 \ge 5C_1(m)$, we get

$$\begin{aligned}&Q_\delta(\varphi_j D_t(k)u_{j\delta}, \varphi_j D_t(k)u_{j\delta}) \\ &\quad \le C(j)\|f_j\|_k^2 + C_4 \cdot C_3(m)\sum_{\ell=1}^{2n-1}\|\varphi_j X_\ell D_t(k-1)u_{j\delta}\|^2.\end{aligned} \tag{2.15}$$

Now we consider a finite spanning set of the $D_t(k)$. By using Lemma 2.3(i), we obtain

$$\begin{aligned}&\sum_{\text{finite sum}}\|\varphi_j D_t(k)u_{j\delta}\|^2 \\ &\quad \le C(j)\|f_j\|_k^2 + (40e)\rho_j^2 \cdot \sum_{\text{finite sum}} Q_\delta(\varphi_j D_t(k)u_{j\delta}, \varphi_j D_t(k)u_{j\delta}) \\ &\quad \le C(j)\|f_j\|_k^2 + (40e)\rho_j^2 \cdot C_4 C_3(m)\sum\sum_{\ell=1}^{2n-1}\|\varphi_j X_\ell D_t(k-1)u_{j\delta}\|^2.\end{aligned} \tag{2.16}$$

Therefore, if we choose j large enough (i.e., ρ_j sufficiently small), one can absorb the second term on the right-hand side of (2.16) by the left-hand side, and get

$$\sum_{\text{finite sum}}\|\varphi_j D_t(k)u_{j\delta}\|^2 \le C(j)\|f_j\|_k^2. \tag{2.17}$$

Equations (2.15) and (2.17) give the desired estimates for the tangential differential operator $D_t(k)$ of order k. The next step is to estimate the normal derivatives of $u_{j\delta}$. This can be done easily simply by observing again that Q is noncharacteristic to the boundary, and by applying estimate (2.14).

Finally, we choose another cut-off function ψ_j for each j, $0 \le \psi_j \le 1$, such that $\varphi_j + \psi_j \equiv 1$ on $\overline{D}_j$. By noting that ψ_j is supported at strictly pseudoconvex points of D_j, one can obtain the desired a priori estimate on D_j; i.e.,

$$\|u_{j\delta}\|_k^2 \le C(j)\|f_j\|_k^2, \tag{2.18}$$

for all $k \le m$, and the constant $C(j)$ is independent of δ. Then it follows from the standard theorem (e.g., see [14] or [10]) that we have our main result.

References

1. D. E. Barrett, *Regularity of the Bergman projection and local geometry of domains*, Duke Math. J. **53** (1986), 333–343.
2. S. Bell, *Biholomorphic mappings and the $\bar{\partial}$-problem*, Ann. of Math. (2) **114** (1981), 103–113.
3. ——, *Differentiability of the Bergman kernel and pseudo-local estimates*, Math. Z. **192** (1986), 467–472.
4. H. P. Boas, *Small sets of infinite type are benign for the $\bar{\partial}$-Neumann problem*, Proc. Amer. Math. Soc. **103** (1988), 569–578.
5. D. Catlin, *Subelliptic estimates for the $\bar{\partial}$-Neumann problem on pseudoconvex domains*, Ann. of Math. (2) **126** (1987), 131–191.
6. ——, *Global regularity of the $\bar{\partial}$-Neumann problem*, in Proc. Sympos. Pure Math., vol. 41, Amer. Math. Soc., Providence, RI, 1984.
7. S.-C. Chen, *Global regularity of the $\bar{\partial}$-Neumann problem on circular domains*, Math. Ann. **285** (1989), 1–12.
8. ——, *Global regularity of the $\bar{\partial}$-Neumann problem: a sufficient condition*, preprint.
9. J. D'Angelo, *Real hypersurfaces, order of contact, and applications*, Ann. of Math. (2) **115** (1982), 615–637.
10. G. B. Folland and J. J. Kohn, *The Neumann problem for the Cauchy-Riemann complex*, Ann. of Math. Studies, vol. 75, Princeton Univ. Press, Princeton, NJ, 1972.
11. J. J. Kohn, *Harmonic integrals on strongly pseudoconvex manifolds* I, Ann. of Math. (2) **78** (1963), 112–148.

11A. ——, *Harmonic integrals on strongly pseudoconvex manifolds* II, Ann. of Math. (2) **79** (1964), 450–472.

12. ——, *Boundary behavior of $\bar{\partial}$ on weakly pseudo-convex manifolds of dimension two*, J. Differential Geom. **6** (1972), 523–542.
13. ——, *Subellipticity of the $\bar{\partial}$-Neumann problem on pseudo-convex domains: sufficient conditions*, Acta Math. **142** (1979), 79–122.
14. J. J. Kohn and L. Nirenberg, *Non-coercive boundary value problems*, Comm. Pure Appl. Math. **18** (1965), 443–492.

Department of Mathematics, State University of New York at Albany, Albany, New York 12222