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BASIS OF WAVELETS AND ATOMIC DECOMPOSITIONS OF $H^1(\mathbb{R}^n)$ AND $H^1(\mathbb{R}^n \times \mathbb{R}^n)$

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ABSTRACT. It is shown that, as in the case of $H^1(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$, wavelets provide unconditional basis of $H^1(\mathbb{R}^n \times \mathbb{R}n)$ and $\mathrm{BMO}(\mathbb{R}^n \times \mathbb{R}n)$. Moreover, we show how wavelets can be used to obtain the usual atomic decompositions in $H^1(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n \times \mathbb{R}n)$.

INTRODUCTION

The real Hardy space $H^1(\mathbb{R}^n)$ is defined as

$$H^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : R_i f \in L^1(\mathbb{R}^n)\}$$

with norm

$$\|f\|_{H^1(\mathbb{R}^n)} = \|f\|_1 + \sum_{i=1}^{n} \|R_i f\|_1$$

where R_i is the ith Riesz transform, and can be equivalently characterized by means of certain maximal or square functions. It can also be described in terms of the so-called atomic decompositions, which consist in the decomposition of any function in $H^1(\mathbb{R}^n)$ as a sum of a particularly simple class of functions called atoms.

A normalized atom is, for our purposes, a square integrable function a with support on a cube Q, with vanishing mean and normalized by $|Q|^{1/2}\|a\|_2 = 1$. The well-known atomic decomposition of the Hardy space $H^1(\mathbb{R}^n)$ is given in the following theorem (see [3, 6]).

Theorem 1. *A function f belongs to $H^1(\mathbb{R}^n)$ if and only if there exist sequences $\{a_k\}$ of atoms and $\{\lambda_k\}$ of numbers such that*

$$f = \sum_k \lambda_k a_k \quad \text{and} \quad \sum_k |\lambda_k| < \infty. \tag{1}$$

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Moreover, an equivalent norm on $H^1(\mathbb{R}^n)$ *is given by the infimum of* $\sum_k |\lambda_k|$ *over all possible such decompositions. That is, there exists a constant* C *such that*

$$\frac{1}{C}\inf\sum_k |\lambda_k| \le \|f\|_{H^1(\mathbb{R}^n)} \le C\inf\sum_k |\lambda_k|. \tag{2}$$

We shall first derive such atomic decompositions in terms of wavelets, by using the fundamental fact that some classes of wavelets provide an unconditional basis of $H^1(\mathbb{R}^n)$. Then we carry out the same program in the setting of product domains, leading us to a new proof of S.-Y. A. Chang and R. Fefferman's result on the atomic decomposition of $H^1(\mathbb{R}^n \times \mathbb{R}^n)$.

In the first part of this paper we give a new proof of Theorem 1 by means of the wavelet decomposition.

The following remark is in order. The classical theory defines atoms as being in L^∞ and such that $|Q|\,\|a\|_\infty = 1$, while the atoms we construct are in all the L^p, $p < \infty$, and in BMO, but not, to our knowledge, in L^∞. We find here a weakness of the basis of wavelets (unless it is a weakness of L^∞): the impossibility of characterizing L^1 and L^∞. On the other hand, our proof is very simple and seems to be able to be implemented numerically. Moreover, in the setting of product spaces, where the known atoms are in L^2 [1], the above inconvenience disappears.

Now let us recall what the wavelet transform is, beginning by the one-dimensional case.

Given an integrable function ψ and a dyadic interval $I = [k2^{-j}, (k+1)2^{-j})$ for $j, k \in \mathbb{Z}$, let $\psi_I(x) = 2^{j/2}\psi(2^j x - k)$, so that ψ_I is in a certain sense localized around I. For certain particular choices of the function ψ, the family $\{\psi_I\}$ is an orthonormal basis of $L^2(\mathbb{R})$, and an unconditional basis of $H^1(\mathbb{R})$ and $\mathrm{BMO}(\mathbb{R})$ for the strong and weak topologies, respectively, called a basis of wavelets [7]. Letting $f = \sum_I \lambda_I \psi_I$, where λ_I is the scalar product of f and ψ_I in $L^2(\mathbb{R})$, an equivalent norm on $H^1(\mathbb{R})$ is given by the following integral:

$$\int_{\mathbb{R}} \left(\sum_I |\lambda_I|^2 \frac{1}{|I|}\chi_I(x) \right)^{1/2} dx, \tag{3}$$

where χ_I is the characteristic function of I. On $\mathrm{BMO}(\mathbb{R})$ we have

$$\|f\|_{\mathrm{BMO}} \sim \sup_I \left(\frac{1}{|I|} \sum_{J \subset I} |\lambda_J|^2 \right)^{1/2}. \tag{4}$$

For the n-dimensional case we need $2^n - 1$ wavelets to start with, denoted by ψ^ε, $\varepsilon = 1, \dots, 2^n - 1$, and set $\psi_Q^\varepsilon = 2^{nj/2}\psi^\varepsilon(2^j x - k)$ where $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, and Q is the dyadic cube $[k_1 2^{-j}, (k_1 + 1)2^{-j}) \times \cdots \times [k_n 2^{-j}, (k_n + 1)2^{-j})$.

Denoting as before by λ_Q^ε the scalar product of f and ψ_Q^ε in $L^2(\mathbb{R}^n)$ we have:

$$\|f\|_{H^1(\mathbb{R}^n)} \sim \int_{\mathbb{R}^n} \left(\sum_Q \sum_\varepsilon |\lambda_Q^\varepsilon|^2 \frac{1}{|Q|} \chi_Q(x) \right)^{1/2} dx \tag{5}$$

and

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} \sim \sup_Q \left(\frac{1}{|Q|} \sum_{Q' \subset Q} \sum_\varepsilon |\lambda_{Q'}^\varepsilon|^2 \right)^{1/2}. \tag{6}$$

From now on we choose smooth (at least C^1) wavelets ψ^ε with vanishing mean and compact support. That such a choice is possible is proved in [3]. There exists then a constant $\gamma > 0$ such that the support of ψ_Q^ε is contained in γQ, the cube with the same center as Q dilated by a factor of γ. The functions $|Q|^{-1/2}\psi_Q^\varepsilon$ are then atoms, but the wavelet decomposition $f = \sum_{Q,\varepsilon} \lambda_Q^\varepsilon \psi_Q^\varepsilon$ is not an atomic decomposition. Otherwise we would have

$$\|f\|_{H^1(\mathbb{R}^n)} \sim \sum_{Q,\varepsilon} |\lambda_Q^\varepsilon| \, |Q|^{1/2}$$

which is certainly not the case. The space for which the above series defines a norm is $B_1^{0,1}(\mathbb{R}^n)$, strictly contained in $H^1(\mathbb{R}^n)$ [8]. Our atoms will then be wavelet packets.

1. Atomic decomposition of $H^1(\mathbb{R}^n)$ with wavelets

In this section we shall give a new proof of the existence of an atomic decomposition for any $f \in H^1(\mathbb{R}^n)$ starting from its wavelet decomposition. The way of arranging wavelets into atoms is given by the square function

$$(Af)(x) = \left(\sum_Q \sum_\varepsilon |\lambda_Q^\varepsilon|^2 \frac{1}{|Q|} \chi_Q(x) \right)^{1/2} \tag{7}$$

which plays a similar role to that of a maximal function. The geometry of dyadic cubes and the strong localization imposed by the terms χ_Q make it specially easy to handle. All the information needed about the function f is carried by the modulus of the wavelet coefficients λ_Q^ε. The proof is divided into three steps.

First step: preliminary results. Let $f \in H^1(\mathbb{R}^n)$ be such that $\|Af\|_1 = 1$. For each dyadic cube Q let

$$L_Q = \left(\sum_{Q \subset Q'} \sum_\varepsilon |\lambda_{Q'}^\varepsilon|^2 \frac{1}{|Q'|} \right)^{1/2}.$$

Then, for $x \in \mathbb{R}^n$

$$(Af)(x) = \left(\sum_{x \in Q'} \sum_{\varepsilon} |\lambda_{Q'}^{\varepsilon}|^2 \frac{1}{|Q'|} \right)^{1/2} = \sup_{x \in Q} \left(\sum_{Q \subset Q'} \sum_{\varepsilon} |\lambda_{Q'}^{\varepsilon}|^2 \frac{1}{|Q'|} \right)^{1/2};$$

that is,

$$(Af)(x) = \sup_{x \in Q} L_Q.$$

This way of writing Af is an idea borrowed from A.V. Maslov [7].

Now let $\lambda > 0$ and $\Omega = \{ x : A(f)(x) > \lambda \}$. Since $\|Af\|_1 = 1$, $|\Omega| \leq 1/\lambda$. If $x \in \Omega$, there is a dyadic cube Q such that $L_Q > \lambda$, so that $Q \subset \Omega$ and $|Q| \leq 1/\lambda$. Thus there exists a maximal dyadic cube $Q(x)$ such that $x \in Q(x) \subset \Omega$ and $L_Q(x) > \lambda$. It follows that Ω is the union of the $Q(x)$. The cubes $Q(x)$ being dyadic and maximal, if $Q(x) \cap Q(y) \neq \varnothing$ then $Q(x) = Q(y)$. It follows that only a countable number of them are different, and in fact disjoint. We can write

$$\Omega = \bigcup_j Q_j \qquad \text{and} \qquad \sum_j |Q_j| \leq \frac{1}{\lambda}. \tag{9}$$

Second step: construction of the atomic decomposition of f. Given $k \in \mathbb{Z}$, let $\Omega_k = \{ x : Af(x) > 2^k \}$. From (9) we write

$$\Omega_k = \bigcup_j Q_{k,j} \qquad \text{and} \qquad \sum_j |Q_{k,j}| \leq 2^{-k},$$

and by the geometry of dyadic cubes each $Q_{k+1,i}$ is contained in a unique $Q_{k,j}$.

If for a given Q, $\sum_{\varepsilon} |\lambda_Q^{\varepsilon}|^2 \neq 0$, then $Q \subset \Omega_k$ for some k, and we can write

$$f = \sum_{Q \subset \bigcup_{k,j} Q_{k,j}} \sum_{\varepsilon} \lambda_Q^{\varepsilon} \psi_Q^{\varepsilon}.$$

For each $k \in \mathbb{Z}$ and $j \in \mathbb{N}$ consider the family $F_{k,j}$ of all dyadic cubes $Q \subset Q_{k,j}$ which are contained in no $Q_{k+1,i}$ and let

$$a_{k,j} = \sum_{Q \in F_{k,j}} \sum_{\varepsilon} \lambda_Q^{\varepsilon} \psi_Q^{\varepsilon} \qquad (= 0 \text{ if } Q_{k,j} = Q_{k+1,i} \text{ for some } i). \tag{10}$$

Since the wavelets ψ_Q^{ε} have compact support contained in γQ, the $a_{k,j}$ so defined are nonnormalized atoms. We have formally $f = \sum_{k,j} a_{k,j}$. In fact, the series converges absolutely in $H^1(\mathbb{R}^n)$.

Third step: end of proof. We prove first of all that

$$A(a_{k,j})(x) = \left(\sum_{x \in Q \in F_{k,j}} \sum_{\varepsilon} |\lambda_Q^{\varepsilon}|^2 \frac{1}{|Q|} \right)^{1/2} \leq 2^{k+1}, \tag{11}$$

which is the most subtle part of the argument, the difficulty coming from the fact that the condition $Q \not\subset Q_{k+1,i}$ for all i is weaker than $Q \cap Q_{k+1,i} = \varnothing$.

Let $x \in Q_{k,j}$. If $x \notin \Omega_{k+1}$ there is nothing to prove since clearly $A(a_{k,j})(x) \leq A(f)(x) \leq 2^{k+1}$. Assume then that $x \in \Omega_{k+1}$. There exists a maximal dyadic cube $Q(x)$ such that $x \in Q(x)$ and $L_{Q(x)} > 2^{k+1}$. Let $Q'(x)$ be the unique dyadic cube containing $Q(x)$ and twice its diameter. We must have $L_{Q'(x)} \leq 2^{k+1}$. Let then Q be any dyadic cube such that $x \in Q \subset Q_{k,j}$ and $Q \not\subset Q_{k+1,i}$ for all i. Since $Q \cap Q(x) \neq \varnothing$, we must have $Q \subseteq Q(x)$ or $Q'(x) \subset Q$. The first possibility being excluded because $Q(x) \subset \Omega_{k+1}$ we have $Q'(x) \subset Q$ and

$$A(a_{k,j})(x) \leq L_{Q'(x)} \leq 2^{k+1}.$$

Moreover, the support of $A(a_{k,j})$ is contained in $Q_{k,j}$. Thus, since $\|Af\|_1 \sim \|f\|_{H^1(\mathbb{R}^n)}$, for each fixed k

$$\sum_j \|a_{k,j}\|_{H^1(\mathbb{R}^n)} \leq C \sum_j \|A(a_{k,j})\|_1 \leq C \sum_j 2^{k+1} |Q_{k,j}| = C 2^{k+1} |\Omega_k|,$$

and since $Af \in L^1$,

$$\sum_{k,j} \|a_{k,j}\|_{H^1(\mathbb{R}^n)} \leq \sum_k 2^{k+1} |\Omega_k| \leq C \|Af\|_1,$$

giving the desired atomic decomposition (with nonnormalized atoms). Since $A(a_{k,j}) \in L^\infty$, $a_{k,j}$ belongs to BMO and hence to all L^p for $p < \infty$.

2. Atomic decompositions of $H^1(\mathbb{R}^n \times \mathbb{R}^n)$ and wavelet basis

In order to avoid complicated notations we will restrict ourselves to the case of the biplane; that is, $n = 1$. The space $H^1(\mathbb{R} \times \mathbb{R})$ is defined as

$$H^1(\mathbb{R} \times \mathbb{R}) = \{f \in L^1(\mathbb{R}^2) : H_1 f, H_2 f, H_1 H_2 f \in L^1(\mathbb{R}^2)\}$$

with norm given by

$$\|f\|_{H^1(\mathbb{R}\times\mathbb{R})} = \|f\|_1 + \|H_1 f\|_1 + \|H_2 f\|_1 + \|H_1 H_2 f\|_1$$

where H_i is the Hilbert transform in the direction of the ith coordinate.

S.-Y. A. Chang and R. Fefferman gave the atomic decomposition of the above space, for which they needed two types of atoms (see [1, 2]) that we now describe.

A normalized rectangular atom is a function a supported in a rectangle $R = I \times J$ and such that

$$a \in L^2, \qquad |R|^{1/2} \|a\|_2 = 1,$$

$$\int_I a(x, x_2)\, dx = \int_J a(x_1, y)\, dy = 0 \quad \text{a.e. } x_1, x_2.$$

Unfortunately these atoms do not generate $H^1(\mathbb{R}\times\mathbb{R})$, the essential reason being that the geometry of dyadic rectangles is much less simple than that of dyadic cubes. If Ω is an open set of finite measure and the R_j are maximal dyadic rectangles in Ω, it is not true that $|\Omega| \sim \sum_j |R_j|$. So an atom is defined by the following conditions:

(1) a is supported on an open set Ω of finite measure;
(2) $a \in L^2$ and is normalized by $|\Omega|^{1/2}\|a\|_2 = 1$;
(3) there are maximal dyadic rectangles $R \subset \Omega$ and rectangular atoms a_R associated to each one of them such that

$$a = \sum_R c_R a_R \qquad \text{and} \qquad \left(\sum_R |c_R|^2 \|a_R\|_2^2\right)^{1/2} \le C\|a\|_2 \tag{12}$$

for an absolute constant C.

Using nonnormalized rectangular atoms a_R, we write (12) as

$$a = \sum_R a_R \qquad \text{and} \qquad \left(\sum_R \|a_R\|_2^2\right)^{1/2} \le C\|a\|_2. \tag{13}$$

S.-Y. A. Chang and R. Fefferman's result is then:

Theorem 2. *A function f belongs to $H^1(\mathbb{R}\times\mathbb{R})$ if and only if there exist sequences $\{a_k\}$ of atoms and $\{\lambda_k\}$ of numbers such that*

$$\sum_k |\lambda_k| < \infty \qquad \text{and} \qquad f = \sum_k \lambda_k a_k.$$

Moreover, an equivalent norm on $H^1(\mathbb{R}\times\mathbb{R})$ is given by the inf *of $\sum_k |\lambda_k|$ over all possible such decompositions.*

We shall now describe the atomic decomposition in terms of wavelets (simplifying as a consequence the proof of Theorem 2). There are no new ideas in what follows, but a new technology. The strategy of the proof is directly inspired by S.-Y. A. Chang and R. Fefferman [1]. We need first of all a basis of wavelets on $H^1(\mathbb{R}\times\mathbb{R})$ and an equivalent norm similar to the one used in the previous section. This is the goal of Theorem 3 below, which we have learned has been independently proved by P. G. Lemarié. Starting with the same wavelets ψ_I, smooth and with compact support, we define for $R = I \times J$ a dyadic rectangle, $\psi_R = \psi_I \otimes \psi_J$. To avoid confusion, we will always use the letter R for rectangles and the letters I and J for intervals.

Theorem 3. *The family $\{\psi_R\}$ where R runs over all dyadic rectangles is an unconditional basis of $H^1(\mathbb{R}\times\mathbb{R})$.*

This means, the linear independence of the $\{\psi_R\}$ being obvious, that there is equivalence between

$$f \in H^1(\mathbb{R}\times\mathbb{R}) \tag{14}$$

and

$$f = \sum_R \lambda_R \psi_R, \qquad \int_{\mathbb{R}\times\mathbb{R}} \left(\sum_R |\lambda_R|^2 \frac{1}{|R|} \chi_R \right)^{1/2} < \infty. \tag{15}$$

The preceding integral defines the desired equivalent norm on $H^1(\mathbb{R}\times\mathbb{R})$.

Let us prove now that (14) implies (15). We let G be the group of all sequences $\alpha = \{\alpha_R\}$ indexed by the set of dyadic rectangles and such that $\alpha_R = \pm 1 \quad \forall R$. Given $\alpha \in G$ we define T_α to be the operator with kernel $\sum_R \alpha_R \tilde{\psi}_R(x)\psi_R(y)$ where $\tilde{\psi}$ is the original wavelet ψ translated and dilated so as to have $\chi_{[0,1)\times[0,1)} \le C\tilde{\psi}$. Then, because of the smoothness of ψ and the compactness of its support, T_α is a Calderón-Zygmund operator in the sense of J. L. Journé [5], with bounds uniform in α. It follows that T_α sends $H^1(\mathbb{R}\times\mathbb{R})$ into $L^1(\mathbb{R}^2)$ uniformly. Applying this result to f we obtain:

$$\int_{\mathbb{R}^2} \left| \sum_R \alpha_R \lambda_R \psi_R \right| \le C\|f\|_{H^1(\mathbb{R}\times\mathbb{R})}.$$

Integrating over G with respect to Haar measure and using Khintchin's inequality we get

$$\int_{\mathbb{R}\times\mathbb{R}} \left(\sum_R |\lambda_R|^2 |\tilde{\psi}_R|^2 \right)^{1/2} \le C\|f\|_{H^1(\mathbb{R}\times\mathbb{R})}$$

from where we deduce, since $\frac{1}{|R|}\chi_R \le C|\tilde{\psi}_R|^2$:

$$\int_{\mathbb{R}\times\mathbb{R}} \left(\sum_R |\lambda_R|^2 \frac{1}{|R|} \chi_R \right)^{1/2} \le C\|f\|_{H^1(\mathbb{R}\times\mathbb{R})}.$$

We do not yet finish the proof of Theorem 3, and turn our attention to the construction of an atomic decomposition of f, starting from (15). An essential role is played by the square function

$$A(f) = \left(\sum_R |\lambda_R|^2 \frac{1}{|R|} \chi_R \right)^{1/2}.$$

For $k \in \mathbb{Z}$ let

$$\Omega_k = \{ x \in \mathbb{R}^2 : Af(x) > 2^k \},$$

$\mathbf{R}_k = \{R \text{ dyadic rectangle: } |R \cap \Omega_k| > \frac{1}{2}|R| \text{ and } |R \cap \Omega_{k+1}| \le \frac{1}{2}|R|\}$.

Then if $\lambda_R \neq 0$, there is a unique k such that $R \in \mathbf{R}_k$. It follows that $f = \sum_k a_k$ where

$$a_k = \sum_{R\in\mathbf{R}_k} \lambda_R \psi_R.$$

Let us show that each a_k is a (nonnormalized) atom. To begin with, a_k is supported on the open set $\Omega_k^* = \{ x \in \mathbb{R}^2 : M^*(\chi_{\Omega_k}) > 1/2 \}$ where M^*

is the strong maximal operator on $\mathbb{R}^2$. Thus Ω_k^* is of finite measure and $|\Omega_k^*| \leq C|\Omega_k|$ for a constant C.

Next we prove that $a_k \in L^2$ estimating $\|A(a_k)\|_2$, which equals $\|a_k\|_2$. We have

$$\begin{aligned}\int_{\mathbb{R}^2} A(a_k)^2 &= \int_{\Omega_k^* \setminus \Omega_{k+1}} A(a_k)^2 + \int_{\Omega_{k+1}} A(a_k)^2 \\ &\leq 4^{k+1}|\Omega_k^*| + \sum_{R \in \mathbf{R}_k} |\lambda_R|^2 \frac{|R \cap \Omega_{k+1}|}{|R|} \\ &\leq 4^{k+1}|\Omega_k^*| + \frac{1}{2}\sum_{R \in \mathbf{R}_k} |\lambda_R|^2.\end{aligned}$$

Since the last sum is precisely $\|A(a_k)\|_2^2$, we get $a_k \in L^2$ and

$$\|a_k\|_2 \leq C 2^k |\Omega_k|^{1/2}. \tag{17}$$

Now, to prove that $f = \sum_k a_k$ is a nonnormalized atomic decomposition, we have to estimate $\sum_k |\Omega_k|^{1/2}\|a_k\|_2$. But, because of (17), we have

$$\sum_k |\Omega_k|^{1/2}\|a_k\|_2 \leq C \sum_k 2^k |\Omega_k| \leq C\|Af\|_1.$$

This concludes the proof of the atomic decomposition.

Now, repeating the arguments in [2] we can see that an atom is in $H^1(\mathbb{R}\times\mathbb{R})$, from where it immediately follows that the existence of an atomic decomposition for a function f implies $f \in H^1(\mathbb{R} \times \mathbb{R})$.

It follows from the preceding arguments that atoms could be defined as functions a satisfying the following three conditions:

$$a = \sum_{R \subset \Omega} c_R \psi_R \qquad \text{where } \Omega \text{ is an open set of finite measure,} \tag{18}$$

$$|\Omega| \sum_{R \subset \Omega} |c_R|^2 = 1, \tag{19}$$

$$|R \cap \{x \in \mathbb{R}^2 : A(a) > 2|\Omega|^{-1}\}| \leq \frac{1}{2}|R| \quad \text{whenever } c_R \neq 0. \tag{20}$$

This class of atoms is included in the one defined by Chang and Fefferman, and generates equally $H^1(\mathbb{R} \times \mathbb{R})$. Moreover we have

Proposition 1. *The atoms in the class defined above are in* $\mathrm{BMO}(\mathbb{R} \times \mathbb{R})$, *and hence in* L^p *for* $p < \infty$.

This is a consequence of the following characterization of $\mathrm{BMO}(\mathbb{R} \times \mathbb{R})$.

Theorem 4. *The* $\{\psi_R\}$ *form a weak unconditional basis of* $\mathrm{BMO}(\mathbb{R} \times \mathbb{R})$, *and* $f \in \mathrm{BMO}(\mathbb{R} \times \mathbb{R})$ *if and only if*

$$\sup_{\Omega} \frac{1}{|\Omega|} \sum_{R \subset \Omega} |\lambda_R|^2 = C^2 < \infty, \tag{21}$$

where $f = \sum_R \lambda_R \psi_R$ *and the* sup *is taken over all open sets of finite measure.*

Moreover, the constant C *defines a norm equivalent to the usual one on* $\mathrm{BMO}(\mathbb{R} \times \mathbb{R})$.

Proof. Let $f \in \mathrm{BMO}(\mathbb{R} \times \mathbb{R})$ and write it as $f = \sum_R \lambda_R \psi_R$ (in the sense of distributions). Given an open set of finite measure Ω, let $a = \sum_{R \subset \Omega} \lambda_R \psi_R$. We have $|\langle a, f \rangle| \leq C \|a\|_{H^1}$, where $\langle , \rangle$ denotes the duality bracket between H^1 and BMO. It follows that

$$\sum_{R \subset \Omega} |\lambda_R|^2 \leq C \int_{\mathbb{R}^2} \left(\sum_{R \subset \Omega} |\lambda_R|^2 \frac{1}{|R|} \chi_R \right)^{1/2}.$$

Since the expression under the integral sign is supported on Ω, an application of the Cauchy-Schwarz inequality gives

$$\begin{aligned} \sum_{R \subset \Omega} |\lambda_R|^2 &\leq C|\Omega|^{1/2} \left(\int_{\mathbb{R}^2} \sum_{R \subset \Omega} |\lambda_R|^2 \frac{1}{|R|} \chi_R \right)^{1/2} \\ &= C|\Omega|^{1/2} \left(\sum_{R \subset \Omega} |\lambda_R|^2 \right)^{1/2}. \end{aligned}$$

We finally get

$$\sum_{R \subset \Omega} |\lambda_R|^2 \leq C^2 |\Omega|.$$

(In fact, a is an atom).

Conversely, let $f = \sum_R \lambda_R \psi_R$ satisfy (21).To show $f \in \mathrm{BMO}(\mathbb{R} \times \mathbb{R})$ we test it against atoms in the class defined above. If $a = \sum_R c_R \psi_R$ is such an atom, that is, it satisfies (18), (19), and (20), then

$$|\langle a, f \rangle| = \left| \sum_{R \subset \Omega} c_R \lambda_R \right| \leq \left(\sum_{R \subset \Omega} |c_R|^2 \right)^{1/2} \left(\sum_{R \subset \Omega} |\lambda_R|^2 \right)^{1/2} \leq C.$$

This ends the proof of Theorem 4.

To prove Proposition 1, we take an atom a as above and for any given open set ω of finite measure we define

$$A_\omega(a) = \left(\sum_{R \subset \omega} |c_R|^2 \frac{1}{|R|} \chi_R \right)^{1/2}.$$

Reasoning as in the proof of (17) and noting that $A_\omega \leq A(\omega)$ we have

$$\begin{aligned} \sum_{R \subset \omega} |c_R|^2 = \int_\omega A_\omega(a)^2 &= \int_{\omega \cap \{A_\omega(a) \leq 2|\Omega|^{-1}\}} + \int_{\omega \cap \{A_\omega(a) > 2|\Omega|^{-1}\}} \\ &\leq 4|\omega||\Omega|^{-2} + \sum_{R \subset \omega} |c_R|^2 \frac{|R \cap \omega \cap \{A(a) > 2\,|\Omega|^{-1}\}|}{|R|} \\ &\leq 4|\omega||\Omega|^{-2} + \frac{1}{2} \sum_{R \subset \omega} |c_R|^2, \end{aligned}$$

from which $\sum_{R\subset\omega}|c_R|^2 \le 8|\omega||\Omega|^{-2}$. Thus $f \in \mathrm{BMO}(\mathbb{R}\times\mathbb{R})$ and by interpolation $f \in L^p$ for $1 \le p < \infty$.

We finish now with two remarks. The first one is that the same arguments can be carried over for $H^p(\mathbb{R}\times\mathbb{R})$, $0 < p < 1$, and its dual. More precisely, we choose the wavelet $\psi \in C^{[2/p-1/2]}$ with $[2/p - 3/2]$ moments equal to zero (such a choice is always possible), and define a p-atom as before, but with condition (19) changed to:

$$(19') \qquad |\Omega|^{2/p-1}\sum_{R\subset\Omega}|c_R|^2 \le 1.$$

Then we can prove:

Theorem 5. *Let* $0 < p \le 1$. *The following are equivalent*:

(1) $f \in H^p(\mathbb{R}\times\mathbb{R})$;

(2) $f = \sum_k \lambda_k a_k$ *where the* a_k *are* p*-atoms and* $\sum_k |\lambda_k|^p < \infty$;

(3) $\int A(f)^p < \infty$.

Moreover, a distribution $g = \sum_R \mu_R \psi_R$ *is in the dual of* $H^p(\mathbb{R}\times\mathbb{R})$ *if and only if*

$$\left(\frac{1}{|\Omega|}\sum_{R\subset\Omega}|\mu_R|^2\right)^{1/2} \le C|\Omega|^{1/p-1}$$

for all Ω *open of finite measure.*

Finally, let us remark that Theorems 3–5 are independent of the chosen wavelet basis, as long as the usual conditions of smoothness and size at infinity are satisfied (see [9]). In fact, if $\{\varphi_I\}$ is another orthogonal basis of wavelets on $L^2(\mathbb{R})$ and $\varphi_R = \varphi_I \otimes \varphi_J$ for $R = I\times J$, the operator T of kernel $\sum_R \varphi_R(x)\psi_R(y)$ is continuous from $H^p(\mathbb{R}\times\mathbb{R})$ into itself (see [4]), as well as T^*. Since TT^* is by construction the identity operator, T changes from basis $\{\psi_R\}$ to $\{\varphi_R\}$. It is even possible to choose wavelets without compact support. In that case, it is only the function $A(a)$ that has a localized support for an atom a.

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