MEASURES ON BOOLEAN ALGEBRAS

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(Communicated by Andreas R. Blass)

Abstract. We give, under some set-theoretical assumptions, an example of complete, ccc, weakly \((\omega, \infty)\)-distributive, countably generated Boolean algebra without any strictly positive Maharam submeasure.

The problem of the existence of a complete, ccc, weakly \((\omega, \infty)\)-distributive Boolean algebra is an old one. Maharam [M] solved it assuming existence of a Suslin line; see also [V3] and [F].

We use the topological definitions as in [E], the Boolean measure definitions as in [F], and the set-theoretical ones as in [J]. We also use the standard notation, in particular \(\forall_k^{\infty}\), MA, \(\neg\text{CH}\) abbreviate “for all but finitely many \(k\)’s,” Martin’s Axiom, and negation of the continuum hypothesis. The symbols \(\wedge\), \(\vee\), \(\Delta\) denote infimum, supremum and symmetric difference in Boolean algebras; by \(\mathcal{P}(\lambda)\) we mean the power set of \(\lambda\).

For \(f, g \in \omega\omega\), we say that \(g \leq^* f\) iff the set \(\{n \in \omega | f(n) < g(n)\}\) is finite and let (see [D])

\[ b := \min\{|\mathcal{H}| | \mathcal{H} \subseteq \omega \omega \text{ and } \neg(\exists f \in \omega \omega)(\forall g \in \mathcal{H})g \leq^* f\}. \]

We shall say that the sequence \(\{x_n\}\) of subsets of cardinal \(\lambda\) converges to a subsets \(x\) of \(\lambda\) if and only if \(\bigcap_{k \in \omega} \bigcup_{n \geq k} (x \triangle x_n) = \emptyset\) (in symbols, \(x_n \rightarrow x\)).

The following properties of \(\rightarrow\) convergence in the Boolean algebra \(\mathcal{P}(\lambda)\) are easy to verify:

(L0) If \(x_n \rightarrow x\) and \(x_n \rightarrow y\), then \(x = y\).

(L1) If \(x_n = x\) for all \(n\), then \(x_n \rightarrow x\).

(L2) If \(x_n \rightarrow x\), then any subsequence also converges to \(x\).

(L3) If \(x_n \not\rightarrow x\) (i.e., it is false that \(x_n \rightarrow x\)), then there is a subsequence \(y_m\) of \(x_n\) such that, for any subsequence \(z_p\) of \(y_m\), \(z_p \not\rightarrow x\).

It means that the pair \((\mathcal{P}(\lambda), \rightarrow)\) is an \(L^*\) (see [E] for definitions of an \(L^*\) space and of an \(S^*\) space). But in some models of set theory, we have more.

Received by the editors November 20, 1989 and, in revised form, March 26, 1990.

1980 Mathematics Subject Classification (1985 Revision). Primary 03E50; Secondary 06F10, 06F30, 28A60.

Partially supported by Spoleczny Komitet Nauki.

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0002-9939/91 $1.00 + .25 per page
Proposition 1. If \( \lambda < b \), then the pair \((\mathcal{P}(\lambda), \to)\) satisfies the following diagonal, Fréchet's condition:

If \( x_n \to x \) for \( n \to \infty \) and for each \( n \), \( x_{n,k} \to x_n \) for \( k \to \infty \),
then there exists a sequence of numbers \( g(n) \) such that \( x_{n,g(n)} \to x \) for \( n \to \infty \).

It means that the pair \((\mathcal{P}(\lambda), \to)\) is an \( S^* \) space.

Proof. Let \( x_n \to x \). Then

\[
    x = \bigcap_{k} \bigcup_{n \geq k} x_n = \bigcup_{k} \bigcap_{n \geq k} x_n.
\]

So for each \( \alpha \in \lambda \) there exists \( m_\alpha \) such that

\[
    (\forall n \geq m_\alpha) \alpha \in x_n \quad \text{or} \quad (\forall n \geq m_\alpha) \alpha \notin x_n.
\]

Because \( x_{n,k} \to x_k \) so

\[
    (\forall n \geq m_\alpha) \forall k \alpha \in x_{n,k} \quad \text{or} \quad (\forall n \geq m_\alpha) \forall k \alpha \notin x_{n,k},
\]

and hence

\[
    (\forall n \geq m_\alpha) \exists g_\alpha(n) (\forall k \geq g_\alpha(n)) \alpha \in x_{n,k} \quad \text{or} \quad (\forall k \geq g_\alpha(n)) \alpha \notin x_{n,k}.
\]

So for fixed \( \alpha \) we may obtain a function \( g_\alpha : \omega \to \omega \). If \( \lambda < b \), then there is \( g : \omega \to \omega \) such that \( g_\alpha < g \) for each \( \alpha < \lambda \). It is easy to see that \( x_{n,g(n)} \to x \).

Into the set \( \mathcal{P}(\lambda) \), we introduce the following topology \( \tau \): we call a set \( U \) open if whenever \( x \in U \) and \( x_n \to x \) then \( x_n \in U \) for \( n \) sufficiently large.

Lemma 1. (i) \((\mathcal{P}(\lambda), \tau)\) is a \( T_1 \), a sequential topological space and the \( \to \) convergence is the same as topological convergence.

(ii) If \( \lambda < b \), then \((\mathcal{P}(\lambda), \tau)\) is a Fréchet space.

Proof. For (i), see [Ki]; for (ii), [E, pp. 90–91] and Proposition 1.

We give some other topological properties of \((\mathcal{P}(\lambda), \tau)\).

Proposition 2. The space \((\mathcal{P}(\lambda), \tau)\) is

(i) homogeneous,
(ii) Hausdorff,
(iii) not regular for \( \lambda > \omega \), and
(iv) sequentially compact for \( \lambda < 2^\omega \) and under MA.

Proof. For (i) and (ii), see [S].

(iii) Let \( C \) denote the set of all countable subsets of \( \omega_1 \). The set \( C \) is closed. We show that for any open neighborhood \( U \) of \( \omega_1 \in \mathcal{P}(\lambda) \) the intersection \( C \cap \text{cl} U \neq \emptyset \).

We use an Ulam matrix. Let \( f_\alpha : \omega \to \alpha + 1 \), for \( \alpha < \omega_1 \), be surjections. We define

\[
    A_{\alpha,n} := \{ \xi \in \omega_1 | f_\xi(n) = \alpha \}.
\]
The matrix \( \{A_{\alpha,n}\}_{\alpha<\omega_1} \) has the following two properties: first, \( \bigcup_{n<\omega} A_{\alpha,n} = \{\xi|\alpha \leq \xi < \omega_1\} \); and second, if \( \alpha < \beta < \omega_1 \), then \( A_{\alpha,n} \cap A_{\beta,n} = \emptyset \).

For each \( \alpha < \omega_1 \) we have:

\[
\omega_1 = \alpha \cup \bigcup_{n<\omega} A_{\alpha,n} = \bigcup_{n<\omega} \left( \alpha \cup \bigcup_{n \leq k} A_{\alpha,n} \right).
\]

Let \( B_{\alpha,k} := \alpha \cup \bigcup_{n \leq k} A_{\alpha,n} \). The set \( U \) is open in \( \mathcal{P}(\lambda) \), so for each \( \alpha < \omega_1 \) there exists \( k_\alpha < \omega \) such that \( B_{\alpha,k_\alpha} \in U \). There are \( k < \omega \) and \( S \subseteq \omega_1 \) of cardinality \( \omega \) such that for each \( \alpha \in S \) we have \( k_\alpha = k \).

Let \( \alpha_0 < \alpha_1 < \alpha_2 < \cdots \) be a sequence of elements of \( S \) and \( \beta := \sup \alpha_n \); let \( b_n := B_{\alpha_n,k} \). We claim that \( b_n \to \beta \). The nonobvious inclusion is \( \bigcap_{k} \bigcup_{n \geq k} b_n \subseteq \beta \). Suppose this inclusion is not true. Then there exists \( \xi \geq \beta \) such that for infinitely many \( n \) we have \( \xi \in \bigcup_{l \leq k} A_{\alpha_l,l} \) and there exists \( l_0 \) such that \( \xi \in A_{\alpha_{l_0},l_0} \) for infinitely many \( n \). It is not possible because \( A_{\alpha_{l_0},l_\alpha} = \emptyset \) for each \( \alpha \neq \alpha_0 \).

It follows that \( \beta \in \text{cl} U \). Since \( \beta \in C \) as well, we obtain the result.

(iv) If \( \text{MA} \) holds then for \( \lambda < 2^{\omega_1} \), the set \( \mathcal{P}(\lambda) \) with Tychonoff topology (which is obviously weaker than \( \tau \)) is a sequentially compact space; see [M-S].

**Corollary 1.** The space \( (\mathcal{P}(\lambda), \tau) \) with operation \( \Delta \) is not a topological group for \( \lambda > \omega \).

**Proof.** It follows from Proposition 2(iii). \( \square \)

**Remark 1.** Corollary 1 answers (without any set-theoretical assumptions) the question posed by Savelev [S].

Let \( I \subseteq \mathcal{P}(\lambda) \) be a \( \sigma \)-ideal such that the quotient algebra \( \mathcal{P}(\lambda)/I \) satisfies the countable chain condition (ccc for short). In the complete Boolean algebra \( \mathcal{P}(\lambda)/I \), we induce a topology \( \tau \) by the following convergence:

\[
x_n \Rightarrow x \text{ iff } \bigvee_{k<\omega} \bigwedge_{n \geq k} (x \triangle x_n) = 0,
\]

i.e., in the same way as \( \tau \) by \( \to \) on \( \mathcal{P}(\lambda) \).

The \( \Rightarrow \) convergence satisfies conditions \((\text{L0}), (\text{L1}), (\text{L2})\).

The relation between \( \Rightarrow \) convergence and the topological convergence for the topology \( \tau \) is given by the following:

**Lemma 2.** The topology \( \tau \) is the same as the topology induced by (topological) convergence in \( \tau \).

The sequence \( x_n \) topologically converges to \( x \) iff every subsequence \( y_m \) of \( x_n \) has subsequence \( z_p \) such that \( z_p \Rightarrow x \).

**Proof.** See [Du]. \( \square \)
Lemma 3. Topology $\tau$ equals the quotient topology of $\tau$ and the natural mapping is open.

Corollary 2. If $\lambda < \beta$ then $(\mathcal{P}(\lambda)/I, \tau)$ is a Fréchet space.

By a Maharam submeasure on complete Boolean algebra $\mathcal{A}$, we mean a function $\mu: \mathcal{A} \to [0, 1]$ such that

(a) $\mu(x) = 0$ iff $x = 0$;

(b) if $x \leq y$, then $\mu(x) \leq \mu(y)$;

(c) $\mu(x \lor y) \leq \mu(x) + \mu(y)$; and

(d) if $\bigwedge_{k \in \omega} \bigvee_{n \geq k} (x_n \Delta x) = 0$, then $\lim_{n \to \infty} \mu(x_n) = \mu(x)$.

Similarly, as in the case of measure, we have:

Proposition 3. Let $D$ be a subspace of the reals of uncountable cardinality $< 2^\omega$. Let $I$ be a $\sigma$-ideal of $\mathcal{P}(D)$ such that $\mathcal{P}(D)/I$ is ccc. If MA holds, there is no Maharam submeasure on $\mathcal{P}(D)/I$.

Proof. It is sufficient to prove the nonexistence of a nontrivial, nonnegative function $\mu$ on $\mathcal{P}(D)$ which is zero on points and satisfies as (b), (c), and (d) as in the definition of Maharam submeasure.

The set $D$ is a $Q$-set; i.e., every subset of $D$ is $G_\delta$ in $D$. If $Y \subseteq D$, then $Y = \bigcap_{n \in \omega} G_n$, where $G_n$ is a monotonically decreasing sequence of open subsets of $D$. Then $\mu(G_n) \to \mu(Y)$, and hence $\mu(Y) = 0$ iff, for each $\epsilon > 0$, there is an open set $G \supseteq Y$ and $\mu(G) < \epsilon$. Now we may repeat the classical proof of the statement ‘MA implies the nonexistence of measure on sets with cardinality $< 2^\omega$’ (for example see [J, pp. 563–564]). □

Now we can prove the main theorem of this paper.

Theorem 1. If Con($ZFC+$ there exists a measurable cardinal) then Con($ZFC+$ MA $\neg$ CH $+$ there exists a complete, weakly $(\omega, \infty)$-distributive, ccc, atomless Boolean algebra without any Maharam submeasure).

Proof. Let $M$ be a countable transitive model with measurable cardinal $\kappa$, and let $I$ be a nonprincipal $\kappa$-complete prime ideal over $\kappa$. By forcing it to satisfy ccc, we may obtain model $M[G]$ for MA and $\kappa < 2^\omega$ (see [Ku]). Then in $M[G]$ ideal $J$, defined as

$$x \in J \iff x \subseteq y \text{ for some } y \in I,$$

is a $\sigma$-saturated, $\kappa$-complete ideal over $\kappa$ (see [J, p. 425]).

In $M[G]$ the Boolean algebra $\mathcal{P}(\kappa)/J$ is complete, atomless, satisfies ccc and (by Corollary 2 and Lemma 2) Fréchet diagonal condition $(L4)$ for $\Rightarrow$ convergence. This implies weak $(\omega, \infty)$-distributivity of $\mathcal{P}(\kappa)/J$ (see [V2]). By Proposition 3, on $\mathcal{P}(\kappa)/J$ there is no Maharam submeasure. □

Remark 2.

- In the model considered above, there is no Suslin line, because MA $\neg$ CH is true.
In the Boolean \( \mathcal{P}(\kappa)/J \), there is no finite-additive, strictly positive measure (see Kelley [1959]).

The method used in the proof of Proposition 3 above shows additionally that \( \mathcal{P}(\kappa)/J \) is countably generated.

The weak \((\omega, \infty)\)-distributivity of algebra \( \mathcal{P}(\kappa)/J \) had been proved independently and by different method by A. Kamburelis.

ACKNOWLEDGMENT

The author is indebted to Dr. J. Tryba for several helpful discussions.

REFERENCES


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