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CE-EQUIVALENCE, UV^k-EQUIVALENCE AND DIMENSION OF COMPACTA

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ABSTRACT. It is shown that for each $k > 0$ there exists a finite-dimensional continuum X which is not UV^k-equivalent, and therefore not CE-equivalent, to any continuum Y such that the dimension of Y is equal to the shape dimension of X.

A map of compacta is *cell-like* $(=\mathrm{CE})$ if all point-inverses have trivial shape. On the class $\mathbf{CM}_f$ of finite-dimensional compacta, the CE maps generate an equivalence relation known as CE-*equivalence.* To be precise, $X, Y \in \mathbf{CM}_f$ are said to be CE-equivalent if there exist spaces $X_1 = X, X_2, \ldots, X_{2n}, X_{2n+1} = Y$ in $\mathbf{CM}_f$ and CE maps $X_{2i} \to X_{2i\pm 1}$, $i = 1, \ldots, n$. CE-equivalence implies shape equivalence, but the converse fails to be true (see S. Ferry [2] or our paper [5]). This leaves the general problem to decide whether a given *invariant* of CE-equivalence is a shape invariant or not. This paper is concerned with a very natural *dimension invariant.* For $X \in \mathbf{CM}_f$, the CE-*dimension* of X is defined as the number

$$\text{CE-dim}\, X = \min\{\dim Y | Y \text{is CE-equivalent to } X\}.$$

Obviously, $\text{CE-dim}\, X \geq \mathrm{Sd}\, X$ $(=$ *shape dimension* of X; see e.g. [4]). Using our notation, we can restate a question which appears in "A list of open problems in shape theory" by J. Dydak and J. Segal:

$$\text{Is it true that CE-dim}\, X = \mathrm{Sd}\, X \text{ for each } X \in \mathbf{CM}_f?$$

In other words, is $\text{CE-dim}\, X$ a shape invariant?

We shall show that the answer is "no." For this purpose it will be convenient to work with the weaker concept of UV^k-*equivalence*, that is, the equivalence relation on the class of *metrizable spaces* generated by the proper UV^k-maps (cf. [3, 5]). Recall that a map is UV^k if all point-inverses are UV^k, that is, have vanishing homotopy pro-groups up to dimension k. The definition of the UV^k-*dimension* of a metrizable space X, abbreviated by $\mathrm{UV}^k\text{-dim}\, X$, can safely be left to the reader. Since CE-equivalence implies UV^k-equivalence, we have $\text{CE-dim}\, X \geq \mathrm{UV}^k\text{-dim}\, X$ for each $X \in \mathbf{CM}_f$.

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Theorem. *Let X_0 be a continuum (= nonempty connected compactum) such that pro-$\pi_1(X_0)$ is not pro-finite. For each $k > 0$, there exists a compactum X_k with the following properties*:

(1) *X_k and X_0 are shape equivalent*;
(2) $\dim X_k = \max(\mathrm{Sd}\, X_0, k+2)$;
(3) UV^k-*dim* $X_k = k+1$.

This shows in particular that the difference CE-dim X − Sd X can be arbitrarily large even within a *fixed* shape equivalence class of finite-dimensional continua.

The proof of the theorem is based on the theory of UV^k-*components* developed in [5]. Clearly, it is no restriction to assume that $\dim X_0 = \mathrm{Sd}\, X_0$. Let P_k denote the polyhedron obtained by attaching the $(k+1)$-cell D^{k+1} to the k-sphere $S^k = \partial D^{k+1}$ by a map $f\colon S^k \to S^k$ of degree 2. By [5, Theorem 6.1], there exists a compactum $X_k \supset X_0$ such that the remainder $X' = X_k \backslash X_0$ is a UV^1-component of X_k homeomorphic to $P_k \times (0, \infty]$. By construction, X_k satisfies (1) and (2) (cf. [5, Proposition 4.1]). Since X' is path-connected, X' clearly is a UV^k-component of X_k. Now assume that UV^k-dim $X_k \le k$. Then there is a compactum Y, $\dim Y \le k$, such that X_k and Y are UV^k-equivalent. By [5, Theorem 2.15] there exists a UV^k-component Y' of Y such that X' and Y' are UV^k-equivalent. From [5, Theorem 1.4] we infer that there are basepoints $x \in X'$ and $y \in Y'$ such that pro-$\pi_i(X', x) \approx$ pro-$\pi_i(Y', y)$ for $i = 0, \dots, k$. This yields pro-$\pi_i(Y', y) \approx 0$ for $i = 0, \dots, k-1$ and pro-$\pi_k(Y', y) \approx \mathbb{Z}_2$. The Hurewicz isomorphism theorem in shape theory (see e.g. [4, Chapter II §4.1, Theorem 1]) implies then pro-$H_k(Y') \approx \mathbb{Z}_2$. On the other hand, $\mathrm{Sd}\, Y' \le \dim Y' \le \dim Y \le k$, that is, Y' has an HPol-expansion $Y' \to \{Y_\alpha\}$ where all Y_α are polyhedra of dimension $\le k$. Therefore pro-$H_k(Y')$ can be represented by an inverse system of free abelian groups, which cannot be isomorphic to $\mathbb{Z}_2$. This contradiction proves UV^k-dim $X_k \ge k+1$. The equation (3) follows now from Ferry's observation that UV^k-dim $X \le k+1$ for *any* compactum X (see [3, Proposition 1.10]).

Remarks. (1) Let $\mathbf{C}_{\mathrm{profin}}$ denote the class of continua X such that pro-$\pi_1(X)$ is pro-finite. Our theorem leaves the question whether CE-dim X = Sd X for each finite-dimensional $X \in \mathbf{C}_{\mathrm{profin}}$. However, we have a complete picture regarding UV^k-dim X for $X \in \mathbf{C}_{\mathrm{profin}}$. On $\mathbf{C}_{\mathrm{profin}}$, shape equivalence implies UV^k-equivalence (see [3, Theorem 2]); hence UV^k-dim $X \le \mathrm{Sd}\, X$. Moreover, for compacta of dimension $\le k$, UV^k-equivalence implies shape equivalence (see [3, Theorem 1]). We infer that UV^k-dim $X = \mathrm{Sd}\, X$ whenever $\mathrm{Sd}\, X \le k$: Choose a compactum Y shape equivalent to X with $\dim Y = \mathrm{Sd}\, X$ and a compactum Z UV^k-equivalent to X with $\dim Z = \mathrm{UV}^k$-dim X; then Y

and Z must be UV^k-equivalent of dimension $\leq k$, whence $\mathrm{Sd}\, X = \mathrm{Sd}\, Z \leq \dim Z = UV^k\text{-}\dim X$. On the other hand, if $\mathrm{Sd}\, X > k$, $UV^k\text{-}\dim X$ can take any of the values $0, 2, 3, \ldots, k+1$: Let $r > k$ and define $Y_0 = S^r$, $Y_j = S^r \vee S^j$ for $j = 2, \ldots, k$, $Y_{k+1} = S^r \vee P_k$ ($\vee$ denotes one-point union and P_k is taken from the above proof). Then all $Y_i \in \mathbf{C}_{\mathrm{profin}}$, $\mathrm{Sd}\, Y_i = r$, but $UV^k\text{-}dim Y_i = i$. Note that neither $UV^k\text{-}\dim X = 1$ nor $\mathrm{Sd}\, X = 1$ is possible when $X \in \mathbf{C}_{\mathrm{profin}}$.

(2) The counterexamples given in our theorem are not *locally connected.* However, for each $k > 0$ R. Daverman and G. Venema have constructed an LC^{k-1} continuum X_k of dimension $k+1$ which is shape equivalent to S^1 but not CE-equivalent to S^1 (see [1]). Such an X_k has $\mathrm{Sd}\, X_k = 1$ but is not CE-equivalent to *any* locally connected one-dimensional compactum (since shape equivalence and CE-equivalence are the same on locally connected one-dimensional compacta, see [1]).

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