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PERIODICALLY PERTURBED NONCONSERVATIVE SYSTEMS OF LIÉNARD TYPE

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Dedicated to Professor Dr. R. Reissig on the occasion of his 65th birthday

ABSTRACT. We give sufficient conditions for the solvability of forced, strongly coupled nonlinear vector Liénard equations. These conditions guarantee the existence of periodic solutions for any forcing term. They include sublinear as well as superlinear nonlinearities. They do not require the symmetry of the restoring term. The method of proof makes use of Leray–Schauder degree.

I. INTRODUCTION

The *scalar* forced Liénard differential equation with periodicity boundary conditions:

$$\begin{aligned} u''(t) + f(u(t))u'(t) + g(u(t)) &= e(t), \\ u(0) = u(2\pi), \qquad u'(0) &= u'(2\pi), \end{aligned} \tag{1.1}$$

where $f: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$, and $e: [0, 2\pi] \to \mathbb{R}$ are continuous functions, has been extensively studied in recent years (see e.g. [1, 2, 6–8, 10–13] and references therein).

Starting with a device due to Faure [4], a research effort of Bebernes and Martelli [1], Cesari and Kannan [2], Lazer [7], Mawhin [8], Mawhin and Ward [10], Reissig [12, 13], and others [6, 11], leads to the following interesting *nonresonance* result.

Assume there exists a positive constant a such that

(i) either $f(u) \geq a$ or $f(u) \leq -a$ for all $u \in \mathbb{R}$,
(ii) $\lim_{u\to\infty} g(u) = \infty$ and $\lim_{u\to-\infty} g(u) = -\infty$.

Then (1.1) has at least one solution for *any* forcing term $e(t)$. (Note that assumption (ii) is equivalent to saying

$$\lim_{|u|\to\infty} (g(u)u)|u|^{-1} = \infty.)$$

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Although this result is not new, attempts to generalize it to forced Liénard systems (of arbitrary degree of freedom) have resulted in very partial extensions that imply further restrictions on the function f or g (see e.g. [1, 2, 3, 5] or Remark 1 herein).

The purpose of this paper is to give a full extension of the aforementioned result to *systems* of Liénard differential equations

$$\begin{aligned} u''(t) + \tfrac{d}{dt}[\text{grad } F(u(t))] + g(u(t)) &= p(t, u(t), u'(t)), \\ u(0) = u(2\pi), \qquad u'(0) &= u'(2\pi), \end{aligned} \tag{1.2}$$

where $g: \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function such that $g(u) = \text{grad } G(u)$ for some C^1-function $G: \mathbb{R}^N \to \mathbb{R}$, $F: \mathbb{R}^N \to \mathbb{R}$ is a C^2-function and $p: [0, 2\pi] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies Carathéodory conditions, i.e., $p(\cdot, u, v)$ is measurable for all $u, v \in \mathbb{R}^N$, $p(t, \cdot, \cdot)$ is continuous for a.e. $t \in [0, 2\pi]$. Moreover, we shall assume that there exists a function $e \in L^2([0, 2\pi], \mathbb{R})$ (the usual Lebesgue space) such that

$$|p(t, u, v)| \le e(t) \tag{1.3}$$

for a.e. $t \in [0, 2\pi]$ and all $u, v \in \mathbb{R}^N$. It is obvious that (1.1) can be put in the framework of (1.2) under the same conditions. Indeed, it suffices to consider

$$G(u) = \int_0^u g(s)\,ds \quad \text{and} \quad F(u) = \int_0^u h(s)\,ds$$

with $h(s) = \int_0^s f(x)\,dx$, whenever f and g are given scalar continuous functions.

The proof of our main result (§2) makes use of Leray–Schauder topological degree [9]. On the other hand, we avoid the standard line of proof that relies heavily on the fact that equations considered are scalar [6, 10, 12, 13], or with restoring term weakly coupled [1, 2], or involving the symmetry of the restoring term [3, 5].

Throughout this paper the Hessian matrix of the C^2-function $F: \mathbb{R}^N \to \mathbb{R}$ will be denoted by

$$A(u) = \left(\frac{\partial^2 F}{\partial u_i \partial u_j} \right)(u), \quad \text{for } i, j = 1, \dots, N. \tag{1.4}$$

Hence (1.2) is equivalent to

$$\begin{aligned} u''(t) + A(u(t))u'(t) + g(u(t)) &= p(t, u(t), u'(t)), \\ u(0) = u(2\pi), \qquad u'(0) &= u'(2\pi). \end{aligned} \tag{1.5}$$

II. Main result

Let $F \in C^2(\mathbb{R}^N, \mathbb{R})$, $G \in C^1(\mathbb{R}^N, \mathbb{R})$ with $g(u) = \text{grad } G(u)$, and let $p: [0, 2\pi] \times \mathbb{R}^{2N} \to \mathbb{R}^N$ be a Carathéodory function satisfying condition (1.3).

We have the following

Theorem. *Assume there exists a positive constant* a *such that*

(H_1) *either* $(A(u)v, v) \geq a|v|^2$ *or* $(A(u)v, v) \leq -a|v|^2$ *for all* $u, v \in \mathbb{R}^N$, *where* $A(u)$ *is the Hessian matrix of* F *provided by* (1.4);

(H_2) $\lim_{|u|\to\infty}(g(u), u)|u|^{-1} = \infty$.

Then (1.5) (*or equivalently* (1.2)) *has at least one solution.*

Proof. We consider the following spaces

$$X = C^1([0, 2\pi], \mathbb{R}^N), \qquad Z = L^2([0, 2\pi], \mathbb{R}^N), \qquad Y = L^1([0, 2\pi], \mathbb{R}^N),$$

and the operators L: dom $L \subset X \to Z$, $N: X \to Z$, defined by

$$\begin{gathered} \text{dom}\, L = \{u \in X : u, u' \text{ are absolutely continuous, } u'' \in Z, \text{ and} \\ u(0) = u(2\pi),\ u'(0) = u'(2\pi)\}, \\ Lu = u'', \qquad Nu = g(u(\cdot)), \end{gathered} \tag{2.1}$$

where a and g are as in the statement of the theorem. It is well known that the operator L is a Fredholm mapping of index zero [9, p. 6]. On the other hand, by assumption (H_2), there exist positive constants R, b with $b > |e|_Y$ such that

(H_2') $(g(u), u) \geq b|u|$ for all $u \in \mathbb{R}^N$ with $|u| \geq R$. Without loss of generality, we may assume that

$$n^2 < b < (n+1)^2, \quad \text{for some } n \in \mathbb{N}.$$

Let us consider the homotopy

$$\begin{gathered} u'' + (1-\lambda)bu + \lambda \tfrac{d}{dt}[\text{grad } F(u)] + \lambda g(u) = \lambda p(\cdot, u, u'), \qquad \lambda \in [0, 1], \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \end{gathered} \tag{2.2}$$

When $\lambda = 1$, we have our original problem, and when $\lambda = 0$, (2.2) has only the trivial solution, since $n^2 < b < (n+1)^2$, $n \in \mathbb{N}$.

In an abstract setting (see e.g. [9]), the above homotopy is equivalent to

$$Lu + bu + \lambda(N+M)u = 0, \qquad \lambda \in [0, 1], \tag{2.3}$$

where $M: X \to Z$ is the operator defined by

$$Mu = -bu + \tfrac{d}{dt}[\text{grad } F(u)] - p(\cdot, u, u'). \tag{2.4}$$

Note that (2.3) is equivalent to

$$u + \lambda K(N+M)u = 0, \qquad \lambda \in [0, 1], \tag{2.5}$$

where

$$K = [L + bI]^{-1}.$$

By standard arguments (see e.g. [9]), it can be shown that the operator $K: Z \to Z$ is compact (completely continuous), and $M + N$ is continuous, so that the operator $\lambda K(M+N)$ is completely continuous. Hence, by the property of invariance under homotopy of the Leray–Schauder degree, it suffices to show that the set of all possible solutions of (2.2) is bounded in X (independently of $\lambda \in (0, 1])$. So the existence of at least one solution of (1.2) will follow from the existence property of the degree [9].

Taking the inner product (in Z) of (2.2) with u', and using integration by parts, periodic boundary conditions, and the fact that $g(u) = \operatorname{grad} G(u)$, we obtain

$$\lambda\left(\tfrac{d}{dt}[\operatorname{grad} F(u)],\, u'\right) = \lambda(p(\cdot,\, u,\, u'),\, u'), \qquad \lambda \in (0,\, 1].$$

By assumption (H_1), inequality (1.3), and Cauchy–Schwarz inequality, we have

$$a|u'|_Z^2 \le |e|_Z |u'|_Z$$

which implies that

$$|u'|_Z \le a^{-1}|e|_Z = R_1. \tag{2.6}$$

Firstly, we claim that all possible solutions $u \in X$ of (2.2) such that $|u(t)| \ge R$ for all $t \in [0, 2\pi]$, are bounded in $C([0, 2\pi], \mathbb{R}^N)$ independently of $\lambda \in (0, 1]$. Indeed, by taking the inner product (in Z) of (2.2) with u, and using integration by parts and periodic boundary conditions, we obtain

$$-|u'|_Z^2 + (1-\lambda)b|u|_Z^2 + \lambda(g(u),\, u) = \lambda(p(\cdot,\, u,\, u'),\, u).$$

By assumption (H_2') and inequality (1.3),

$$b|u|_Y \le |u'|_Z^2 + |e|_Y |u|_C$$

which implies by inequality (2.6) that

$$b|u|_Y \le R_1^2 + |e|_Y |u|_C. \tag{2.7}$$

Now, for any $u \in X$ solution of (2.2), we write

$$u = \overline{u} + \tilde{u},$$

where

$$\overline{u} = (2\pi)^{-1}\int_0^{2\pi} u(t)\,dt \quad \text{and} \quad \tilde{u} = u - \overline{u},$$

so that

$$\int_0^{2\pi} \tilde{u}(t)\,dt = 0$$

and the inner product, in Z, $(\overline{u},\, \tilde{u}) = 0$.

By using triangle inequality in (2.7) and inequality (2.6), we obtain

$$2\pi b|\overline{u}| - 2\pi b|\tilde{u}|_C \le R_1^2 + |e|_Y(|\overline{u}| + |\tilde{u}|_C).$$

So that by the Sobolev inequality (see e.g. [14, Corollary 7.7, p. 208]) and (2.6), we get

$$2\pi b|\overline{u}| - 2\pi b 6^{-1/2}\pi^{1/2}R_1 \le R_1^2 + 6^{-1/2}\pi^{1/2}R_1|e|_Y + |e|_Y|\overline{u}|$$

which implies, since $b > |e|_Y$, that

$$|\overline{u}| \leq K \tag{2.8}$$

for some constant $K > 0$ depending only on a, b, and e. Since $|u|_C \leq |\overline{u}| + |\tilde{u}|_C$, we deduce from (2.6), (2.8), and Sobolev inequality, that there exists a constant $R_2 > 0$ depending only on a, b, and e such that $|u|_C \leq R_2$ for all possible solutions of (2.5) which are such that $|u(t)| \geq R$ for all $t \in [0, 2\pi]$, and the claim is proved.

Now, assume that for some $u \in X$ solution of (2.2) there exists $\overline{t} \in [0, 2\pi]$ such that $|u(\overline{t})| < R$. Then standard arguments [1, 6, 10, 12] and Cauchy–Schwarz inequality imply

$$|\overline{u}_i| \leq R + (2\pi)^{1/2}|u_i'|_{L^2}, \qquad 1 \leq i \leq N.$$

So that, by using inequality (2.6) once more, we deduce that

$$|\overline{u}| \leq NR + (2\pi)^{1/2}R_1.$$

Since $|u|_C \leq |\overline{u}| + |\tilde{u}|_C$, it follows from the above inequality, the Sobolev inequality, and inequality (2.6), that $|u|_C$ is bounded.

Therefore, there exists a constant $R_3 > 0$ depending only on a, b, e, N, and R such that

$$|u|_C \leq R_3, \tag{2.9}$$

for all solutions of (2.2).

Taking into account inequalities (1.3), (2.6), and (2.9) and using Cauchy–Schwarz inequality, we deduce from (2.2) that there exists a constant R_4, independent of λ, and depending only on a, b, e, g, and F such that $|u''|_Y \leq R_4$.

Hence, since $u_i'(t)$ must vanish somewhere for $i = 1, \dots, N$, we get

$$|u'|_C \leq R_4. \tag{2.10}$$

Inequalities (2.9) and (2.10) imply that there exists a constant $r \geq R$ such that all possible solutions of (2.2) satisfy $|u|_X < r$, and the proof is complete. □

Remarks. (1) Our main result is an improvement of [3] where it is assumed that G is a C^2-function, satisfying

> $(g(u), u) \geq b|u|^2$ for all $u \in \mathbb{R}^N$ with $|u| \geq R$, and some constant $b > 0$;

and $A(u) = A$ is a symmetric matrix with constant entries such that $(Au, u) \geq a|u|^2$ for all $u \in \mathbb{R}^N$ with $a > 0$ a constant. On the other hand, in contrast to [1, 2, 8], where the resonance situation is considered, we do not impose componentwise conditions on g, so that we consider the case of systems with *strongly* coupled restoring terms.

(2) The approach used herein also works when, in the statement of the theorem, assumption (H_2) is replaced by

$$\lim_{|u|\to\infty} (g(u), u)|u|^{-1} = -\infty.$$

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