A NOTE ON COBORDISM OF SURFACE LINKS IN $S^4$

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Abstract. Sato's idea of the asymmetric linking number is used in cyclic branched coverings to give an invariant of the cobordism of embedded surfaces in the 4-sphere.

In this article we consider the link cobordism of surface links in $S^4$. We work in the smooth category.

Let $L = J \cup K$ be a link in $S^4$, where $J$, $K$ are embedded, oriented connected surfaces. $L$ is called semi-boundary [S] if each component bounds an embedded, orientable 3-manifold in $S^4$ which misses the other component. Sato [S] defined the asymmetric linking number, denoted by $\text{alk}(J, K)$, to be the nonnegative generator of the image of $H_1(K : \mathbb{Z}) \rightarrow H_1(S^4 \setminus J : \mathbb{Z}) \cong \mathbb{Z}$. He proved that a link is semi-boundary iff

$$\text{alk}(J, K) = 0 = \text{alk}(K, J),$$

and being semi-boundary is preserved under link cobordism. We call two surface links $L_0 = J_0 \cup K_0$ and $L_1 = J_1 \cup K_1$ cobordant if there are disjointly embedded, orientable 3-manifolds $C$ and $E$ in $S^4 \times I$ such that $\partial C = J_0 \cup (-J_1)$, $\partial E = K_0 \cup (-K_1)$, and $C$, $E$ are homeomorphic to $J_0 \times I$, $K_0 \times I$, respectively, where we regard $L_i$ as lying in $S^4 \times \{i\}$. A link is called null-cobordant if it is cobordant to the standardly embedded surfaces (which bound disjoint handlebodies) in $S^4$. Thus $\text{alk}$ can be regarded as the first obstruction to links being null-cobordant, and we focus on semi-boundary links from now on. The Sato-Levine invariant was defined [S] for semi-boundary links, and Cochran [C] defined the derived series of this invariant. In this paper we observe that the covering asymmetric linking number can be used as a link cobordism invariant and give examples of links with vanishing Sato-Levine invariant and trivial derivatives which belong to different cobordism classes.

Let $L = J \cup K$ be a 2-component, oriented semi-boundary link. Consider the $n$-fold cyclic branched covering $M$ of $S^4$ along $J$, where $n = p^f$ is a prime power. Then there are $n$ lifts $k_0, \ldots, k_{n-1}$ of $K$ to $M$ since $L$ is disjoint from $J$ and $K$. We can choose $k_i$ so that $k_i$ and $k_j$ intersect transversely and isotropically along $J$. A lift $k_i$ is called an $i$-lift. Let $\text{covalk}(J, K)$ be the covering asymmetric linking number, denoted by $\text{covalk}(J, K)$, to be the nonnegative generator of the image of $H_1(K : \mathbb{Z}) \rightarrow H_1(M) \cong \mathbb{Z}$. He proved that a link is semi-boundary iff

$$\text{covalk}(J, K) = 0 = \text{covalk}(K, J),$$

and being semi-boundary is preserved under link cobordism. We call two surface links $L_0 = J_0 \cup K_0$ and $L_1 = J_1 \cup K_1$ cobordant if there are disjointly embedded, orientable 3-manifolds $C$ and $E$ in $S^4 \times I$ such that $\partial C = J_0 \cup (-J_1)$, $\partial E = K_0 \cup (-K_1)$, and $C$, $E$ are homeomorphic to $J_0 \times I$, $K_0 \times I$, respectively, where we regard $L_i$ as lying in $S^4 \times \{i\}$. A link is called null-cobordant if it is cobordant to the standardly embedded surfaces (which bound disjoint handlebodies) in $S^4$. Thus $\text{covalk}$ can be regarded as the first obstruction to links being null-cobordant, and we focus on semi-boundary links from now on. The Sato-Levine invariant was defined [S] for semi-boundary links, and Cochran [C] defined the derived series of this invariant. In this paper we observe that the covering asymmetric linking number can be used as a link cobordism invariant and give examples of links with vanishing Sato-Levine invariant and trivial derivatives which belong to different cobordism classes.
Lemma 1. \( H_1(M \setminus k_j : \mathbb{Q}) \) is isomorphic to either 0 or \( \mathbb{Q} \). Furthermore, this is a link cobordism invariant. More precisely, let \( L_i = J_i \cup K_i \) \((i = 0, 1)\) be cobordant links and \( M_i, k_j \) be their cyclic branched coverings and the lifts of \( K_i \) respectively \((i = 0, 1, j = 0, \ldots, n - 1)\). Then

\[
H_1(M_0 \setminus k_j^0 : \mathbb{Q}) \cong H_1(M_1 \setminus k_j^1 : \mathbb{Q}).
\]

In particular, this is \( \mathbb{Q} \) if \( L \) is null-cobordant.

The proof is given later. If \( H_1(M \setminus k_j : \mathbb{Q}) = 0 \), then \( L \) is not null-cobordant. Thus we focus on links with \( H_1(M \setminus k_j : \mathbb{Q}) = \mathbb{Q} \). Consider the following homomorphism

\[
H_1(k_j : \mathbb{Q}) \to H_1(M \setminus k_0 : \mathbb{Q}) \cong \mathbb{Q}.
\]

Definition 2. Define \( \xi_j^n = 0 \) if this homomorphism is zero, \( \xi_j^n = 1 \) otherwise \((j = 1, \ldots, n - 1)\).

Theorem 3. \( \xi_j^n \in \mathbb{Z}_2, j \in \mathbb{Z}_n \setminus \{0\} \), are link cobordism invariants.

Proof of Lemma 1. Let \( X_n \) be the \( n \)-fold cyclic (unbranched) covering \( (X_n \) denotes the infinite cyclic covering\) of \( X = S^3 \setminus N(J) \). Then we have an exact sequence \([S-S]\) with integral coefficient

\[
\cdots \to H_q(X_\infty) \xrightarrow{t-1} H_q(X_\infty) \to H_q(X_n) \to H_q-1(X_\infty) \to \cdots,
\]

where \( t \) is the homomorphism induced from the generator of covering transformations. Thus we have

\[
\begin{array}{cccc}
H_1(X) & \to & H_0(X_\infty) & \to & H_0(X_n) \\
\| & \cong & t^{-1} & / \\
Z & \to & Z & 0
\end{array}
\]

Hence \((t - 1): H_1(X_\infty : Z) \to H_1(X_\infty : Z)\) is surjective. Therefore \((t^n - 1) = (t - 1)^n : H_1(X_\infty : \mathbb{Z}_p) \to H_1(X_\infty : \mathbb{Z}_p)\) is also surjective \((n = p^r \text{ is a prime power})\). Again using the exact sequence, we have \( H_1(X_n : \mathbb{Z}_p) = \mathbb{Z}_p \). But the lift of the meridian of \( J \) represents a nontrivial element of infinite order in \( H_1(X_n : Z) \). Hence \( H_1(X_n : \mathbb{Q}) = \mathbb{Q} \). Thus we have \( H_1(M : \mathbb{Q}) = 0 \). A Mayer-Vietoris sequence with \( \mathbb{Q} \)-coefficients gives

\[
\begin{array}{cccc}
H_1(\partial N(k_j)) & \to & H_1(N(k_j)) & \oplus & H_1(M \setminus N(k_j)) \\
\| & \| & \| & \| & , \\
\mathbb{Q}^{2g} & \oplus & \mathbb{Q}^{2g} & \to \mathbb{Q}^{2g} & \to 0
\end{array}
\]

where \( g \) is the genus of \( K \). Hence \( H_1(M \setminus k_j : \mathbb{Q}) = 0 \) or \( \mathbb{Q} \).
Let $L_i = J_i \cup K_i$ ($i = 0, 1$) be cobordant links via $C$, $E$, and let $W$ be the $n$-fold cyclic branched covering of $S^4 \times I$ along $C, E_j$ ($j = 0, \ldots, n-1$) be the lifts of $E$ to $W$. Then $\partial E_j = k_j^0 \cup (-k_j^1)$, where $k_j^0$ are the lifts of $K_i$ to $M_i$, the $n$-fold branched covering of $S^3$ along $J_i$. (Note that we have exactly $n$ lifts of $E$ because $L_i$'s are semi-boundary and $E$ is homeomorphic to the product $K_0 \times I$.) The same argument shows that $H_1(W : \mathbb{Q}) = 0$ and $H_1(W \setminus E_0 : \mathbb{Q}) = 0$ or $\mathbb{Q}$.

We need to know the homomorphism $H_2(M_i) \to H_2(W)$. Let $Y = S^4 \setminus \bigcup C$, $Y_n$ (resp. $Y_\infty$) be the $n$-fold (resp. infinite) cyclic covering of $Y$. Let $X^i = S^4 \setminus J_i$, $X^i_n$ (resp. $X^i_\infty$) be the $n$-fold (resp. infinite) cyclic covering of $X^i$. Since $(t-1): H_1(X^i_\infty : \mathbb{Z}) \to H_1(X^i_\infty : \mathbb{Z})$ is surjective, it is an isomorphism (because $H_1(X^i_\infty : \mathbb{Z})$ is a finitely generated module over a Noetherian ring $\Lambda = \mathbb{Z}[t, t^{-1}]$, see [S-S]). Also we have $H_2(Y, X^i) = 0$ since the inclusion induces an isomorphism $H_*(X^i) \cong H_*(Y)$. Therefore we have the following commutative diagram with $\mathbb{Z}$-coefficients:

$$
\begin{array}{cccc}
0 & \to & H_2(X^i_\infty) & \overset{i-1}{\to} & H_2(X^i_\infty) & \to & H_2(X^i) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H_2(Y_\infty) & \overset{i-1}{\to} & H_2(Y_\infty) & \to & H_2(Y) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H_2(Y_n, X^i_n) & \overset{i-1}{\to} & H_2(Y_n, X^i) & \to & 0 \\
\end{array}
$$

Note that $i_*: H_2(X^i) \to H_2(Y)$ (the homomorphism induced from the inclusion map) is an isomorphism, and $H_2(X^i) \cong \mathbb{Z}^{2h}$, where $h$ is the genus of $J_i$.

Consider the following splitting

$$H_2(X^i_\infty : \mathbb{Q}) = F(X^i_\infty : \mathbb{Q}) \oplus T(X^i_\infty : \mathbb{Q})$$

where $F(\ )$, $T(\ )$ denote the free and torsion part of $H_2(\ )$ as a $\Gamma = \Lambda \otimes_{\mathbb{Z}} Q$-module respectively ($\Gamma$ is a PID). Furthermore, $T(X^i_\infty : \mathbb{Q}) = T^0(X^i_\infty : \mathbb{Q}) \oplus T^1(X^i_\infty : \mathbb{Q})$, where

$$T^0(X^i_\infty : \mathbb{Q}) \cong \Gamma/(t-1)^{p_1} \oplus \cdots \oplus \Gamma/(t-1)^{p_r}$$

is the $(t-1)$-summand and $T^1(\ )$ is the $(t-1)$-free summand.

Comparing the above sequence to the following sequence

$$0 \to \Gamma/(t-1) \to \Gamma/(t-1)^{p_k} \overset{i-1}{\to} \Gamma/(t-1)^{p_k} \to \Gamma/(t-1) \to 0,$$

where $\Gamma/(t-1) \cong \mathbb{Q}$, we conclude that $T^0(X^i_\infty) = 0$. (The same is true for $H_2(Y_\infty)$ and $H_2(Y_n, X^i_n)$.)

Hence $T(X^i_\infty : \mathbb{Q}) \cong \Gamma/\lambda_1 \oplus \cdots \oplus \Gamma/\lambda_m$ where $\lambda_j$ is normalized so that $\lambda_j \in \Lambda$ and coefficients are relatively prime ($j = 1, \ldots, m$).
On the other hand, \( \text{Cok}(t - 1 : H_2(X^i_\infty : \mathbb{Z}) \to H_2(X^i_\infty : \mathbb{Z})) \) is isomorphic to \( H_2(X^i_\infty) \otimes_{\Lambda} \mathbb{Z} \), where \( \mathbb{Z} \) is regarded as \( \Lambda \)-module via the augmentation map \( \Lambda \to \mathbb{Z}, t \to 1 \) [S-S]. Since \( H_2(X) \) is torsion free, we have \( \lambda_j(1) = \pm 1, j = 1, \ldots, m \) [S-S]. Then the same argument as Theorem 3 in [Sum] shows that
\[
\text{Cok}(t^n - 1 : T(X^i_\infty : \mathbb{Q}) \to T(X^i_\infty : \mathbb{Q})) = 0 \quad (n = p')
\]
The same is true for \( H_2(Y_\infty, X^i_\infty) \) and we have \( H_2(Y_n, X^i_n : \mathbb{Q}) = 0 \) (since \( H_2(Y_\infty, X^i_\infty) \) is \( \Gamma \)-torsion), and hence \( i_*^*: H_2(X^i_n : \mathbb{Q}) \to H_2(Y_n : \mathbb{Q}) \) is an epimorphism.
Since \( t - 1 \) is an isomorphism on \( T(X^i_\infty) \), the sequence
\[
0 \to F(X^i_\infty) \xrightarrow{t - 1} F(X^i_\infty) \to H_2(X^i) \to 0
\]
shows that \( \text{rank}_\mathbb{F} F(X^i_\infty) = 2g \) where \( g \) is the genus of \( J_i \). The same is true for \( Y_\infty \) and the similar exact sequences for \( t^n - 1 \) show that \( \dim_\mathbb{Q} H_2(X^i_\infty) = 2gn = \dim_\mathbb{Q} H_2(Y_n) \). Since \( i_*^*: H_2(X^i_n) \to H_2(Y_n) \) is an epimorphism of vector spaces of the same dimension, it is an isomorphism.
A Mayer-Vietoris sequence shows that \( i_*^*: H_2(M^i_j) : \mathbb{Q}) \to H_2(W : \mathbb{Q}) \) is an isomorphism. Also we have the following Mayer-Vietoris sequences with \( \mathbb{Q} \)-coefficients:
\[
\begin{align*}
H_2(M^i_j) & \to H_1(\partial N(k^i_j)) \\
\downarrow \cong & \downarrow \cong \\
H_2(W) & \to H_1(\partial N(E_j)) \\
\to H_1(N(k^i_j)) \oplus H_1(M^i_j \setminus \text{Int} N(k^i_j)) & \to 0 \\
\cong \downarrow & \downarrow \\
H_1(N(E_j)) \oplus H_1(W \setminus \text{Int} N(E_j)) & \to 0
\end{align*}
\]
It follows that \( i_*^{i^i}: H_1(M^i_j \setminus k^i_j) : \mathbb{Q}) \to H_1(W \setminus E_j : \mathbb{Q}) \) is an isomorphism for \( i = 0, 1 \) and the lemma follows. Q.E.D.

Properties of Theorem 3. We use the same notation as in the proof of Lemma 1. Consider the following diagram with \( \mathbb{Q} \)-coefficients:
\[
\begin{array}{ccc}
H_1(k^0_j) & \to & H_1(M^0_j \setminus k^0_j) \cong \mathbb{Q} \\
\downarrow & & \downarrow \\
H_1(E_j) & \to & H_1(W \setminus E_0) \cong \mathbb{Q} \\
\uparrow & & \uparrow \\
H_1(k^1_j) & \to & H_1(M^1_j \setminus k^1_j) \cong \mathbb{Q}
\end{array}
\]
Each vertical homomorphism is induced from an inclusion map and an isomorphism in \( \mathbb{Q} \)-coefficients. Hence the top homomorphism is zero iff so is the bottom homomorphism. Therefore \( \xi_j^n \) is well-defined. Q.E.D.

Example. Let \( L_m = K_{m,0} \cup K_{m,1} \) be an “untwisted spun link” indicated in Figure 1, where \( m \) is a positive integer. (Regard \( S^4 \) as \( B^3 \times S^1 \cup S^2 \times B^2 \).
Figure 1

The circle in the 3-ball becomes a torus in $S^4$ after spinning, and the spun arc together with two disks in $S^2 \times B^2$ forms an $S^2$ in $S^4$.) Thus $K_{m,0}$ is homeomorphic to $S^2$, and $K_{m,1}$ to a torus. Furthermore, $K_{m,0}$ and $K_{m,1}$ are unknotted. One can calculate

$$
\kappa^m_j (L_m) = \begin{cases} 
1 & \text{if } j \equiv \pm m \pmod{n}, \\
0 & \text{otherwise,}
\end{cases}
$$

for any prime power $n$. Hence $L_m$ and $L_{m'}$ are not cobordant to each other unless $m = m'$. Note that the Sato-Levine invariant vanishes and Cochran’s derivative is trivial for any $m$.

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