PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 111, Number 4, April 1991

THE SET OF SECOND ITERATES IS NOWHERE DENSE IN C

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(Communicated by R. Daniel Mauldin)

ABSTRACT. Let C denote the set of continuous functions mapping $[0, 1]$ into itself endowed with sup norm. It is proved that the set $\{f \circ f : f \in C\}$ is nowhere dense in C.

Let C denote the set of continuous functions mapping $[0, 1]$ into itself endowed with sup norm. Humke and Laczkovich [1, 2] investigated the structure of the set $W^2 = \{f \circ f : f \in C\}$. They proved that W^2 is an analytic non-Borel subset of C. They also proved that W^2 is not everywhere dense in C. The author of this paper proved that W^2 is a set of first category [5] and of zero Wiener measure [4]. Our aim here is to prove the

Theorem. *W^2 is nowhere dense in C.*

For $f \in C$ and $n \in \mathbb{N}$ we define $f^n(x)$ by induction:

$$f^n(x) = f(f^{n-1}(x)) \quad \text{with} \quad f^1(x) = f(x).$$

For each $n \in \mathbb{N}$ P_f^n denotes the set of periodic points of f with period n and $\operatorname{Fix}(f^n) = \{x : f^n(x) = x\}$; then $P_f^n = \operatorname{Fix}(f^n) \setminus \bigcup_{k<n} \operatorname{Fix}(f^k)$. The closed interval with endpoints a and b will be denoted by $\langle a, b\rangle$ even if $b < a$. We put

$$B(f, \varepsilon) = \left\{ h \in C : \sup_{x \in [0,1]} |f(x) - h(x)| < \varepsilon \right\}.$$

Proof of theorem. Put

$$K = \{f \in C : \text{ for every } \varepsilon > 0, \text{ there exists } \delta > 0,\ g \in B(f, \varepsilon) \text{ such that } B(g, \delta) \cap W^2 = \varnothing\}.$$

Thus K is the set of all functions f such that W^2 is not dense in any neighbourhood of f. It is enough to show that $K = C$. For every $f \in C$ we define $D_f = \{x : f'(x) \text{ exists}\}$. Put

$$\mathrm{PL} = \{f \in C : f \text{ is piecewise linear; } \operatorname{Fix}(f) \subset D_f \cap (0, 1);\ \text{for every } x \in D_f\ |f'(x)| > 2\}.$$

Received by the editors March 7, 1989 and, in revised form, November 1, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54H20.

It is easy to see that PL is dense in C. We show that PL $\subset K$, from which it follows that $K = C$ since K is closed in C. Let $f \in$ PL and $\varepsilon > 0$ be fixed. We show that there exists $\delta > 0$ and there is $g \in B(f, \varepsilon)$ such that $B(g, \delta) \cap W^2 = \varnothing$. Since $f \in$ PL card(Fix(f)) is odd, assume that $2n + 1 = \mathrm{card}(\mathrm{Fix}(f))$. Furthermore let $q = p_1 \cdot p_2 \cdot \cdots \cdot p_{(n+1)}$ where $5 < p_1 < p_2 < \cdots < p_{(n+1)}$ are prime numbers. Then $\mathrm{Fix}(f^q)$ is a finite set since $f \in$ PL. Write $m = \mathrm{card}(\mathrm{Fix}(f^q) - \mathrm{Fix}(f))$.

Lemma 0. *There exist positive numbers δ_1, δ_2 such that $\varepsilon/2 > \delta_1 > \delta_2 > 0$ and*

(i) *if $x \in \mathrm{Fix}(f)$ and $|t - x| < \delta_1$ then $f'(t)$ exists;*

(ii) *if $h \in B(f, \delta_2)$ then $\mathrm{Fix}(h^q) \subset \bigcup_{x \in \mathrm{Fix}(f^q)} (x - \delta_1, x + \delta_1)$;*

(iii) $$\left.\begin{array}{l} h \in B(f, \delta_2), \\ 0 \le l \le k \le q, \\ x, y \in \mathrm{Fix}(f^q), \\ f^l(x) \ne f^k(y) \end{array}\right\} \Rightarrow h^l((x - \delta_1, x + \delta_1)) \cap h^k((y - \delta_1, y + \delta_1)) = \varnothing.$$

Proof of Lemma 0. Choose $0 < \eta_1$ so that it satisfies the following conditions: $\eta_1 < \varepsilon/2$; $\eta_1 < \min\{|x - y| : x, y \in \mathrm{Fix}(f^q),\ x \ne y\}$, and if $x \in \mathrm{Fix}(f)$ with $|t - x| < \eta_1$ then $t \in D_f$. This is possible because $f \in$ PL.

We define $0 < \eta_2$ so that $\eta_2 < \eta_1/10$ and

$$|x - y| < \eta_2 \Rightarrow \max_{0 \le i \le q} |f^i(x) - f^i(y)| < \eta_1/10$$

for every $x, y \in [0, 1]$. Put $\delta_1 = \eta_2$ and define $0 < \eta_3 < \delta_1$ so that $\eta_3 \le \min\{|f^q(y) - y| : y \in [0, 1] \text{ and } \mathrm{dist}(y, \mathrm{Fix}(f^q)) \ge \delta_1\}$. We know that $\eta_3 > 0$, thus we can choose $0 < \eta_4$ so that $\eta_4 < \eta_3$ and

$$(*) \qquad h \in B(f, \eta_4) \Rightarrow \min\{|h^q(y) - y| : \mathrm{dist}(y, \mathrm{Fix}(f^q)) \ge \delta_1\} > \eta_3/2.$$

Choose $0 < \eta_5$ so that $\eta_5 < \eta_4$ and if $h \in B(f, \eta_5)$ then $\max_{0 \le i \le q} \|f^i - h^i\| < \eta_4$. We put $\delta_2 = \eta_5$.

It is obvious that (i) is fulfilled and that $(*)$ implies (ii). To see that (iii) holds let h, l, k, x, y satisfy the conditions listed in (iii). Let $z_1 \in (x - \delta_1, x + \delta_1)$, $z_2 \in (y - \delta_1, y + \delta_1)$ be arbitrary. We have to show that $h^l(z_1) \ne h^k(z_2)$. We have $|f^l(x) - f^l(z_1)| < \eta_1/10$ and $|f^k(y) - f^k(z_2)| < \eta_1/10$ from the definition of η_2. Since $f^l(x), f^k(y) \in \mathrm{Fix}(f^q)$ and $f^l(x) \ne f^k(y)$, $|f^l(x) - f^k(y)| > \eta_1$ follows from the definition of η_1. Thus $|f^l(z_1) - f^k(z_2)| > 8\eta_1/10$. We know that $|h^l(z_1) - f^l(z_1)| < \eta_2 < \eta_1/10$ and $|h^k(z_2) - f^k(z_2)| < \eta_2 < \eta_1/10$ from the definition of η_5. Thus $8\eta_1/10 < |f^l(z_1) - f^k(z_2)| < |f^l(z_1) - h^l(z_1)| + |h^l(z_1) - h^k(z_2)| + |h^k(z_2) - f^k(z_2)| < \eta_1/10 + |h^l(z_1) - h^k(z_2)| + \eta_1/10$ whence $6\eta_1/10 < |h^l(z_1) - h^k(z_2)|$, which completes the proof.

Let $\mathrm{Fix}(f) = \{a'_1, b'_1, a'_2, b'_2, \ldots, a'_n, b'_n, a'_{(n+1)}\}$ where $a'_i < b'_i < a'_{(i+1)}$ $(i = 1, \ldots, n)$ and put $U_i = (a'_i - \delta_1, a'_i + \delta_1)$, $V_j = (b'_j - \delta_1, b'_j + \delta_1)$ for

every $i = 1, \dots, n+1$, $j = 1, \dots, n$. Furthermore let $\mathrm{Fix}(f^q)\backslash\mathrm{Fix}(f) = \{c_1, \dots, c_m\}$ and put $W_k = (c_k - \delta_1, c_k + \delta_1)$ for every $1 \le k \le m$.

The main steps of the proof

To facilitate the presentation we will briefly outline the elements involved in our proof. There are two basic steps.

Step 1. In order to define a function $g \in B(f, \varepsilon)$ and find a suitable $\delta > 0$ so that $B(g, \delta) \cap W^2 = \varnothing$ we will first describe a technique for construction of a function $\Phi^p_{(I)}$, for a preassigned $p > 5$ and interval $I \subset [0, 1]$. Then we pick intervals $I_i \subset U_i$ for $1 \le i \le n+1$. The function g will differ from f only on $\bigcup_{i=1}^{n+1} U_i$. On each I_i, it will be a special function $'\Phi^{p_i}_{(I_i)}$ derived from the technique. On the remainder of each $U_i\backslash I_i$ g will be defined so that

(i) it is continuous on $[0, 1]$;

(ii) it has no odd order periodic points in $\bigcup_{i=1}^{n+1}(U_i\backslash I_i)$.

Then the δ is chosen so that any function $v \in B(g, \delta)$ will have at least $m+1$ periodic orbits of period p_i in U_i, $1 \le i \le n+1$.

Step 2. We will examine, via 4 lemmas, the possible locations of periodic orbits of period p_i for any function $v \in B(g, \delta)$. It will be shown that the assumptions $v \in B(g, \delta)$ and $v = h^2$ for some $h \in C$ are self-contradictory.

Step 1

Let $|I|$ denote the length of the interval I. Put $\gamma = |I| \cdot [2(2p+1)]^{-1}$ and define the pairwise disjoint closed intervals

$$A_i = [d_i - \gamma, d_i + \gamma] \qquad (i = 0, 1, 2, \dots, p-1)$$

where $d_0 = \gamma$, $d_i = d_{(i-1)} + 4\gamma$ $1 \le i \le p-2$, and $d_{(p-1)} = d_{(p-2)} + 8\gamma$. Write $d' = d_{(p-2)} + 4\gamma$ and denote by Ω the increasing linear function mapping I onto $[d' - \gamma, d' + \gamma]$. Next we define a function Φ^p on $[0, 1]$ which is to be linear on the intervals A_i $(i = 0, 1, \dots, p-1)$, $[d' - \gamma, d' + \gamma]$ and on the intervals contiguous to them: put $\Phi^p(d_i - \gamma) = d_{(i+1)} - 9\gamma/10$, $\Phi^p(d_i + \gamma) = d_{(i+1)} + 9\gamma/10$ $(i = 0, 1, \dots, p-2)$ $\Phi^p(d_{(p-1)} - \gamma) = d_0 + 9\gamma/10$, $\Phi^p(d_{(p-1)} + \gamma) = d_0 - 9\gamma/10$.

We define $\gamma' = \gamma/(2p+1)$ from $\gamma'/|\Omega([0, 1])| = \gamma$ and put

$$\Phi^p(d' - \gamma) = d' - \gamma + 4.1\gamma', \Phi^p(d' + \gamma) = d' - \gamma + 0.1\gamma'.$$

Thus we have defined Φ^p (see Figure 1, p. 1144). It is easy to see that Φ^p has the following properties:

(A) If $\|h - \Phi^p\| < \gamma \cdot (0.9)^p/30$ then there exists a periodic orbit of h with period p in $\bigcup_{i=0}^{p-1}(d_i - \gamma/3, d_i + \gamma/3)$.

(B) If $\|h - \Phi^p\| < \gamma \cdot (0.9)^p/30$ and $\{x_i\}_{i=0}^{p-1}$ is a periodic orbit of h with period p in $\bigcup_{i=0}^{p-1} A_i$ then $\{x_i\}_{i=0}^{p-1} \subset \bigcup_{i=0}^{p-1}(d_i - \gamma/3, d_i + \gamma/3)$.

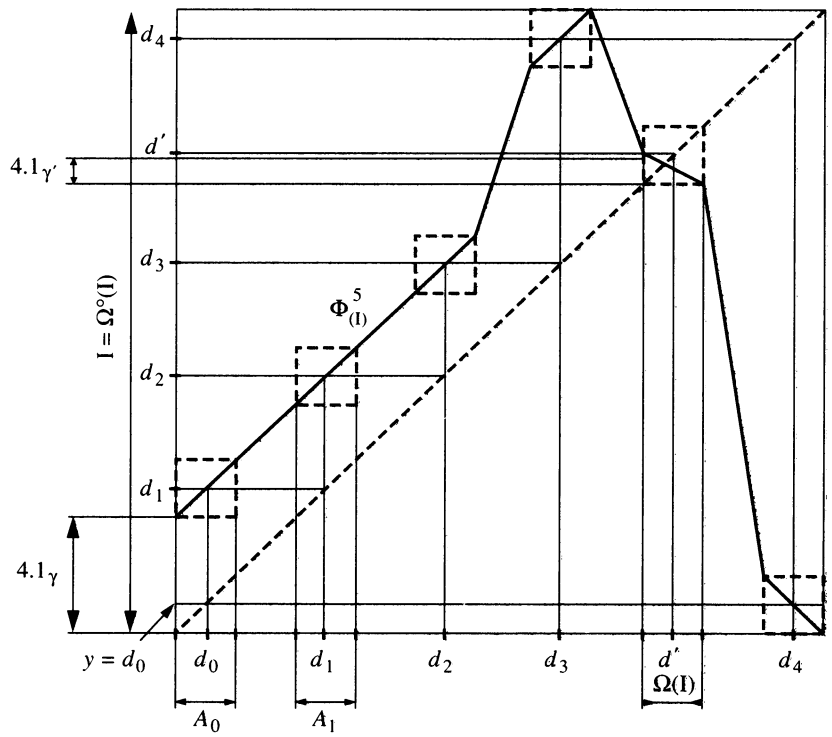

FIGURE 1

(C) If $\{x_i\}_{i=0}^{p-1} \subset I \backslash \Omega(I)$ is a periodic orbit of Φ^p and p is odd then the smallest interval which contains $\{x_i\}_{i=0}^{p-1}$ also contains the point d_{p-2} (see Figure 1).

Proof of (A) *and* (B). Let $d_0 - \gamma < x \leq d_0 - \gamma/3$ and $d_0 + \gamma/3 < y \leq d_0 + \gamma$. Since the absolute value of the slope of Φ^p above A_i $(0 \leq i \leq p-1)$ equals to 0.9 it is implied that if $\|h - \Phi^p\| < \gamma \cdot (0.9)^p/30$ then $h^p(x) > d_0$ and $h^p(y) < d_0$. So the interval (x, y) contains a periodic point of Φ^p with period p (this proves (A)), but x, y are not periodic points of Φ^p with period p so we have proved (B) as well.

Proof of (C). The fact that $\{x_i\}_{i=0}^{p-1}$ is a periodic orbit of Φ^p implies that there exists $0 \leq 1 \leq p-1$ such that $\Phi^p(x_l) < x_l$ whence $x_l > d' + \gamma$. Without loss of generality we may assume that $l = 0$. If $x_k > d_{p-2}$ for every $0 \leq k \leq p-1$ then $x_1, x_3, \ldots, x_{p-2}, x_0 \in [d_{p-2} + \gamma, d' - \gamma]$ which contradicts the fact that $x_0 > d' + \gamma$ (see Figure 1).

For an arbitrary interval I let τ_I denote the similarity transformation mapping $[0, 1]$ onto I. We define

$$\Phi^p_{(I)}(x) = \tau_I(\Phi^p(\tau_I^{-1}(x))) \qquad (x \in I).$$

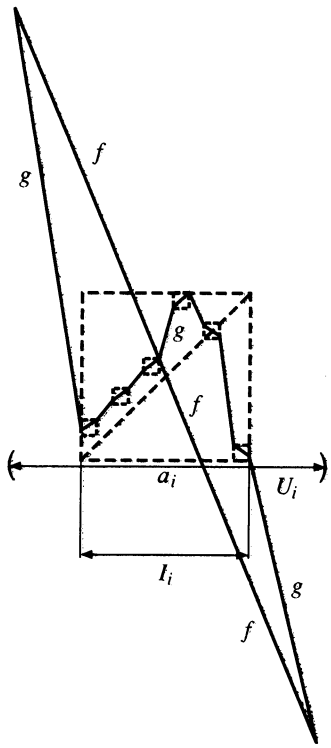

FIGURE 2

Put

$$'\Phi^p_{(I)}(x) = \begin{cases} \Phi^p_{\Omega^i(I)}(x), & \text{if } x \in \Omega^i(I)\backslash\Omega^{i+1}(I) \text{ and } 0 \le i < m; \\ \Phi^p_{\Omega^m(I)}(x), & \text{if } x \in \Omega^m(I). \end{cases}$$

We next define $g \in C$ as follows (see Figure 2). We put $g(x) = f(x)$ if $x \notin \bigcup_{i=1}^{n+1} U_i$.

Let $1 \le i \le n+1$ be fixed. We choose an interval $I_i \subset U_i$ with midpoint a_i such that $|I_i| < \delta_2/(2|f'(a_i)|)$. Let g be linear decreasing on both components of $U_i\backslash I_i$ and

$$g(x) = {}'\Phi^p_{(I_i)}(x) \qquad (x \in I_i).$$

Thus we have $g \in C \cap B(f, \delta_2)$. Put $\delta = \min_{1\le i\le n+1} |\Omega^m(I_i)|/100$. Let $1 \le i \le n+1$ be fixed and let

$$A^i_k = \tau_{I_i}(A_k) \qquad (k = 0, 1, 2, \ldots, p_i - 1).$$

Proposition 1. *We can see that if* $\|h - g\| < \delta$ *then*:

(a) *For every* $1 \le i \le n+1$ h *has at least one periodic orbit with period* p_i *in* $\bigcup_{k=0}^{p_i-1} \Omega^j(A^k_i)$ $(j = 0, 1, \ldots, m)$.

(b) *If* l *is a divisor of* q *then* $P^l_h \cap (U_i\backslash I_i) = \varnothing$ (see Figure 2).

Proof of Proposition 1. (a) It is easy to see (a) is implied by (A) above.

(b) Using Lemma 0 we can see that if $\{x_k\}_{k=0}^{l-1}$ is a periodic orbit of h and $\{x_k\}_{k=0}^{l-1} \cap U_i \neq \emptyset$ then $\{x_k\}_{k=0}^{l-1} \subset U_i$. The same remains valid for I_i because $h(i_i) \subset I_i$, thus if (b) is false then $\{x_k\}_{k=0}^{l-1} \subset U_i \backslash I_i$. However, since l is odd x_0 and x_l are in different components of $U_i \backslash I_i$ and thus $x_0 \neq x_l$.

We now proceed to our Step 2.

Step 2

Suppose that $\psi^2 \in B(g, \delta)$. Let $\eta > 0$ be so small that $\varphi^2 \in B(g, \delta)$ holds for every $\varphi \in B(\psi, \eta)$. In [3, Theorem 1] it is proved that: there exists a residual subset A of C such that for every $\varphi \in A$ and $u, v \in \mathbb{N}$, every neighbourhood of any periodic point of φ with period v contains periodic points of φ with period uv.

Choose $\varphi \in B(\psi, \eta) \cap A$ and put $h = \varphi^2$. Let $1 \leq i \leq n+1$ and $0 \leq j \leq m$ arbitrary. From Proposition 1 we know that $\bigcup_{k=0}^{p_i-1} \Omega^j(A_i^k)$ contains a periodic orbit $\{x_k\}_{k=0}^{p_i-1}$ of h with period p_i. Since $h = \varphi^2$ one can easily see that either there is a periodic orbit $\{a_k\}_{k=0}^{2p_i-1}$ of φ with period $2p_i$ such that $a_{2k} = x_k$ $(k = 0, 1, \ldots, p_i - 1)$ or $\{x_k\}_{k=0}^{p_i-1}$ is a periodic orbit of φ itself. In the latter case, using the previous statement with $v = p_i$ and $u = 2$, we get that there is a periodic orbit $\{a_k\}_{k=0}^{2p_i-1}$ of φ such that $\{a_k\}_{k=0}^{2p_i-1} \subset \bigcup_{k=0}^{p_i-1} \Omega^j(A_i^k)$. Hence we have shown that

Proposition 2. *For every* $1 \leq i \leq n+1$ *and* $0 \leq j \leq m$ *there exists a periodic orbit* $\{a_k\}_{k=0}^{2p_i-1}$ *of* φ *so that*

$$\{a_{2k}\}_{k=0}^{p_i-1} \subset \bigcup_{k=0}^{p_i-1} \Omega^j(A_i^k).$$

Since $\{a_{2k-1}\}_{k=0}^{p_i}$ *is a periodic orbit of* φ *with period* $2p_i$, $\{a_{2k-1}\}_{k=0}^{p_i}$ *is a periodic orbit of* h *with period* p_i. *Thus Lemma* 0 *implies that*

$$\{a_{2k-1}\}_{k=0}^{p_i} \subset \bigcup_{z=1}^{m} W_z \cup \bigcup_{l=1}^{n} V_l \cup \bigcup_{k=1}^{n+1} U_k.$$

We will show this is impossible via the four lemmas.

We use the notation of Proposition 2; *clearly we may assume that* $a_{2k} \in \Omega^j(A_i^k)$ $(0 \leq k \leq p_i - 1)$.

Lemma 1. $\{a_{2k-1}\}_{k=1}^{p_i}$ *is not contained in* U_i.

Proof of Lemma 1. We actually prove more; namely,

(i) $\{a_{2k-1}\}_{k=1}^{p_i}$ is not contained in $\bigcup_{k=0}^{p_i-1} \Omega^j(A_i^k)$,

(ii) $\{a_{2k-1}\}_{k=1}^{p_i}$ is not contained in $\Omega^j(I_i)$ and then

(iii) $\{a_{2k-1}\}_{k=1}^{p_i}$ is not contained in U_i.

Assume first that

$$\{a_{2k-1}\}_{k=1}^{p_i} \subset \bigcup_{k=0}^{p_i-1} \Omega^j(A_i^k).$$

Since $\varphi(a_{2p_i-1}) = a_0$ and $a_0 \in \Omega^j(A_0^i)$, $a_{2p_i-2} \in \Omega^j(A_{p_i-1}^i)$ (see Figure 1) we have $[a_0, a_{2p_i-2}] \subset \varphi(\langle a_{2p_i-1}, a_{2p_i-3}\rangle)$ thus $\varphi(\langle a_{2p_i-1}, a_{2p_i-3}\rangle)$ contains $\{a_{2k-1}\}_{k=1}^{p_i}$ except for (at most) two elements. Thus $\varphi^2(\langle a_{2p_i-1}, a_{2p_i-3}\rangle) \supset \{a_2, a_4, \dots, a_{2p_i-4}\}$ and this implies that

(a) $a_{2p_i-1} \approx a_{2p_i-2}$ or

(b) $a_{2p_i-3} \approx a_{2p_i-2}$ (see Figure 1),

where $x \approx y$ iff there are indices i, j, k such that $x, y \in \Omega^j(A_k^i)$. If (a) then $a_3 \in (a_0, a_{2p_i-1}) \subset \varphi(\langle a_{2p_i-1}, a_{2p_i-2}\rangle)$ thus $a_2 \in \varphi^2(\langle a_{2p_i-1}, a_{2p_i-3}\rangle)$ which is impossible. If (b) then $a_1 \in \langle a_{2p_i-2}, a_{2p_i-1}\rangle \subset \varphi(\langle a_{2p_i-3}, a_{2p_i-2}\rangle)$ thus $a_2 \in \varphi^2(\langle a_{2p_i-3}, a_{2p_i-2}\rangle)$ which is impossible.

Next assume that $\{a_{2k-1}\}_{k=1}^{p_i} \subset \Omega^j(I_i)$. Then from (1) we get $\{a_{2k-1}\}_{k=1}^{p_i} \subset I_i^j = \Omega^j(I_i)\backslash \bigcup_{k=0}^{p_i-1} \Omega^j(A_i^k)$ (see Figure 1). But $a_{2p_i-1} \in I_i^j \subset [a_0, a_{2p_i-2}] \subset \varphi(\langle a_{2p_i-1}, a_{2p_i-3}\rangle)$ and thus $a_4 \in \varphi^2(\langle a_{2p_i-1}, a_{2p_i-2}\rangle) \subset h(I_i^j)$ contradicting $a_0 \approx \Omega^j(d_0)$.

Finally suppose $\{a_{2k-1}\}_{k=1}^{p_i} \subset U_i\backslash\Omega^j(I_i)$. Since p_i is a divisor of q it follows from Proposition 1 that it is enough to show that $\{a_{2k-1}\}_{k=1}^{p_i}$ is not contained in I_i. Suppose that

$$\{a_{2k-1}\}_{k=1}^{p_i} \subset I_i.$$

Then there exists $0 \le 1 \le j$ such that $\{a_{2k-1}\}_{k=1}^{p_i} \subset \Omega^l(I_i)\backslash\Omega^j(I_i)$, this follows from (ii). Since $\{a_{2k-1}\}_{k=1}^{p_i}$ is a periodic orbit of h with period p_i, there are odd numbers $1 \le m_1, m_2 \le 2p_i - 1$ such that $h(a_{m_1}) > a_{m_1}$ and $h(a_{m_2}) < a_{m_2}$. From (C) it follows that

$$a_{2p_i-4} \in [a_{m_1}, a_{m_2}] \subset \varphi(\langle a_0, a_{2p_i-2}\rangle).$$

Therefore $a_{2p_i-5} \in \varphi^2(\langle a_0, a_{2p_i-2}\rangle) \subseteq h(\Omega^j(I_i) \subset \Omega^j(I_i))$, which is a contradiction. This completes the proof of Lemma 1.

Lemma 2. *If $j_1 \ne j$, $0 \le j_1 \le m$ is fixed, and if $\{y_k\}_{k=0}^{p_i-1} \subset \bigcup_{k=0}^{p_i-1} \Omega^{j_1}(A_i^k)$ is a periodic orbit of h and $\{b_l\}_{l=0}^{2p_i-1}$ is a periodic orbit of φ such that $y_k = b_{2k} \in \Omega^{j_1}(A_i^k)$ $(k = 0, 1, \dots, p_i - 1)$, then there do not exist indices $0 \le j_2, j_3 \le 2p_i - 1$, $1 \le j_4 \le m$ such that $a_{j_2}, b_{j_3} \in W_{j_4}$.*

Proof of Lemma 2. Assume that there exists a periodic orbit $\{b_l\}_{l=1}^{2p_i-1}$ of φ such that $b_{2k} \in \Omega^{j_1}(A_i^k)$ $(k = 0, 1, \dots, p_i - 1)$ and $a_{j_2}, b_{j_3} \in W_{j_4}$. We can assume $j < j_1$. Then it follows from Lemma 0 that there are indices j_5, j_6 such that $0 \le j_5 \le 2p_i - 1$, $1 \le j_6 \le m$ and such that $a_1, b_{j_5} \in W_{j_6}$.

Thus, since $\varphi(a_1) = a_2$ and $\varphi(b_{j_5}) \in \Omega^{j_1}(I_1)$, the point a_4 belongs to $\varphi(\langle a_1, b_{j_5}\rangle)$ and this implies a_5 is an element of $\varphi^2(\langle a_1, b_{j_5}\rangle) \subset h(W_{j_6})$. But $a_1 \in W_{j_6}$ which implies that $a_3 \in h(W_{j_6})$ and $a_5 \in h^2(W_{j_6})$. Thus $h^2(W_{j_6}) \cap h(W_{j_6}) \neq \varnothing$ and this contradicts Lemma 0.

It should be noted that Lemma 2 implies that there exists $0 \leq 1 \leq m$ so that for every periodic orbit $\{b_k\}_{k=0}^{2p_i-1}$ of φ if $b_{2k} \in \Omega^l(A_k^i)$ $(k = 0, 1, \dots, p_i - 1)$, then $\{b_{2k-1}\}_{k=0}^{2p_i-1} \cap \bigcup_{z=0}^{m} W_z = \varnothing$.

Lemma 3. *$\{a_{2k-1}\}_{k=1}^{p_i-1}$ is not contained in U_{i_1} where $1 \leq i_1 \leq n+1$ and $i_1 \neq i$.*

Proof of Lemma 3. Assume that $\{a_{2k-1}\}_{k=1}^{p_i} \subset U_{i_1}$. First we suppose that $\{a_{2k-1}\}_{k=1}^{p_i}$ is not contained in $\Omega^{m+1}(I_{i_1})$. Then there exists $0 \leq w \leq m$ such that

$$\{a_{2k-1}\}_{k=1}^{p_i} \subset \Omega^w(I_{i_1}) \backslash \Omega^{w+1}(I_{i_1}).$$

Assume that $\{b_l\}_{l=0}^{2p_i-1}$ is a periodic orbit of φ such that $b_{2l} \in \Omega^w(A_l^{i_1})$ for every $0 \leq 1 \leq p_{i_1} - 1$. Then

$$b_{2p_{i_1}-4} \in \varphi(\langle a_0, a_{2p_i-2}\rangle)$$

(this follows from property (C) of $\Phi^p_{(I)}$). Thus

$$b_{2p_{i_1}-3} \in \varphi^2(\langle a_0, a_{2p_i-2}\rangle) \subset \Omega^j(I_i)$$

and hence that

$$\{b_{2l-1}\}_{l=1}^{p_i-1} \subset \Omega^j(I_i) \backslash \bigcup_{k=0}^{p_i-1} \Omega^j(A_k^i) = I_i^j$$

because $p_{i_1} \neq p_i$ and p_i, p_{i_1} are prime numbers. Further

$$a_{2p_i-1} \in \langle b_0, b_{2p_{i_1}-2}\rangle \subset \varphi(\langle b_{2p_{i_1}-1}, b_{2p_{i_1}-3}\rangle)$$

and thus $a_0 \in \varphi^2(\langle b_{2p_{i_1}-1}, b_{2p_{i_1}-3}\rangle)$. This is a contradiction since $a_0 \notin \varphi^2(I_i^j)$ (see Figure 1).

Next suppose that $\{a_{2k-1}\}_{k=1}^{p_i} \subset \Omega^{m+1}(I_{i_1})$ (see Figure 3). Then for every $0 \leq z \leq m$ we choose a periodic orbit of φ, $\{b_l^z\}_{l=0}^{2p_{i_1}-1}$, so that

$$b_{2l}^z \in \Omega^z(A_l^{i_1}) \qquad (l = 0, 1, \dots, p_{i_1} - 1).$$

It follows from Lemma 2 that there exists $0 \leq z_0 \leq m$ such that $\{b_{2l-1}^{z_0}\}_{l=1}^{p_i}$ does not intersect any W_l. Since $\{b_{2l-1}^{z_0}\}_{l=1}^{p_i}$ is a periodic orbit of h, it intersects some U_k or V_k. Since $h(\Omega^{j+1}(I_i))$ does not contain a_{2l}, $(l = 0, \dots, p_i - 1)$, we get from Lemma 0 and the assumption $\{a_{2k-1}\}_{k=1}^{p_i} \subset \Omega^{m+1}(I_{i_1})$, that $\{b_{2l-1}^{z_0}\}_{l=1}^{p_i} \subset I_i \backslash \Omega^{j+1}(I_i)$. But since $j \leq m$ an analogous case with changing

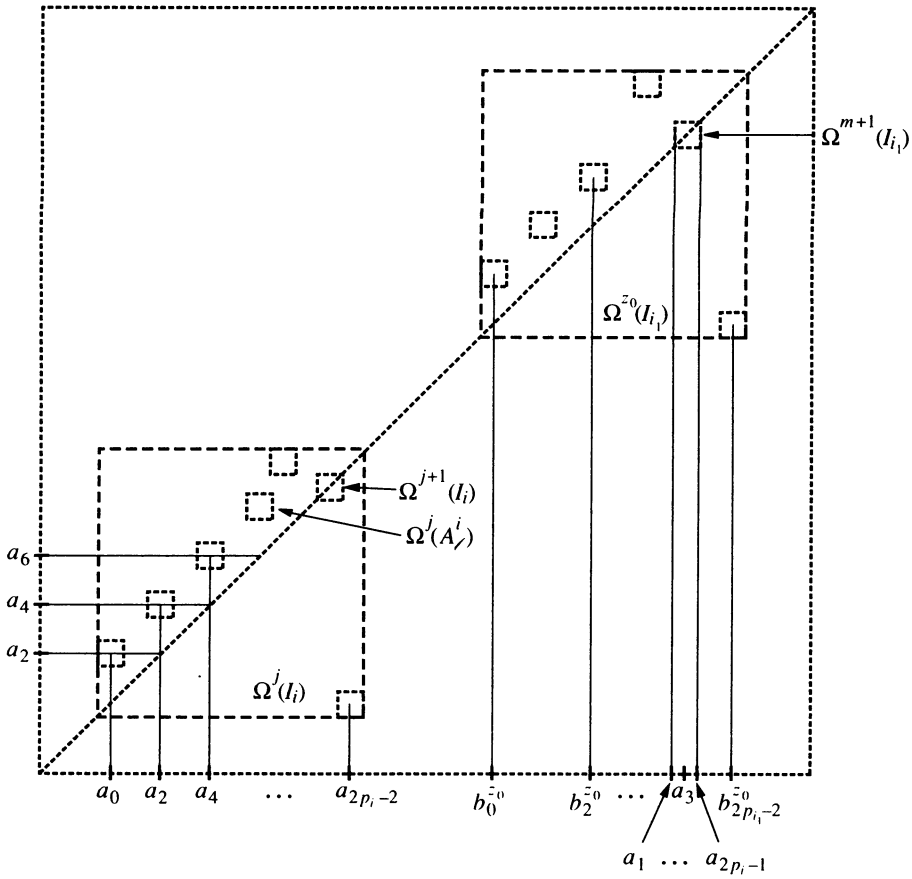

FIGURE 3

rules between $\{a_k\}_{k=0}^{2p_i-1}$ and $\{b_l^{z_0}\}_{l=0}^{2p_{i_1}-1}$ was investigated in the first part of this proof showing that this is impossible.

We shall write $U_i \backsim V_j$ if there is a periodic orbit $\{a_k\}_{k=0}^{2p_i-1}$ of φ such that $\{a_{2l}\}_{l=0}^{p_i-1} \subset U_i$ and $\{a_{2l-1}\}_{l=1}^{p_i} \subset V_j$. Lemmas 1, 2, and 3 imply that for every $1 \le i \le n+1$ there exists $1 \le j \le n$ such that $U_i \backsim V_j$.

Lemma 4. *There exist indices i, j, k such that $1 \le i \le j < k \le n+1$, with $U_i \backsim V_j$ and $U_k \backsim V_j$.*

Proof of Lemma 4. Since the number of the U_j's is greater than the number of the V_j's, there exist i, j and k such that

$$U_i \backsim V_j, \qquad U_k \backsim V_j.$$

If the desired condition: $(i \le j < k)$, does not hold for these i, j and k, we can assume that $i < k \le j$. (The situation when $j < i, k$ is similar). Then we claim there is no $1 \le l \le n+1$ such that $V_i \backsim U_l$. For if $V_i \backsim U_l$ then,

since $\varphi(V_j) \cap U_i \neq \varnothing$ and $\varphi(V_j) \cap U_k \neq \varnothing$, we would get $\varphi(V_j) \supset V_i$. Thus $\varphi^2(V_j) \cap U_l \neq \varnothing$ which contradicts Lemma 0. Therefore deleting both U_i and V_i the remaining U_j's and V_j's still have the property that for each j there is a k with $U_j \backsim U_k$.

We continue with this process of checking the condition and deleting until we have either the condition fulfilled or have deleted down to two U's and only one V. Then the desired condition will hold.

Let i, j, k be as in Lemma 4. Then, since $i \leq j < k$, we have both $\varphi(V_j) \cap U_i \neq \varnothing$ and $\varphi(V_j) \cap U_k \neq \varnothing$ and $\varphi(V_j) \supset V_j$. This implies $\varphi^2(V_j) \cap U_i \neq \varnothing$ and φ in turn contradicts Lemma 0. This completes the proof of the theorem.

Acknowledgment

The author would like to thank professor Miklós Laczkovich for his useful advice.

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