A CONTINUUM HAVING ITS HYPERSPACES NOT LOCALLY CONTRACTIBLE AT THE TOP

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Abstract. For a continuum $X$, let $C(X)$ (resp. $2^X$) be the spaces of all nonempty subcontinua (resp. closed subsets) of $X$. In this paper we answer a question of Dilks by showing an example of a continuum $X$ such that if $\mathcal{H} = C(X)$ or $2^X$, then $\mathcal{H}$ does not have nonempty open subsets which are contractible in $\mathcal{H}$. In particular, $\mathcal{H}$ is not locally contractible at any of its points.

Introduction

A continuum is a compact, connected, nondegenerate metric space. The hyperspaces of a continuum $X$ are the spaces $2^X$ consisting of all nonempty closed subsets of $X$, and $C(X)$, consisting of the connected elements in $2^X$, each with the Hausdorff metric. If $U$ is a subset of a topological space $Y$, we say that $U$ is contractible in $Y$ if there exists a continuous function $F: U \times I \to Y$ and there exists a point $y \in Y$ such that $F(u, 0) = u$ and $F(u, 1) = y$ for every $u \in U$.

In [4] the contractibility of hyperspaces is discussed in detail. More recent results may be found in [1], [2], [5–9]. In [3, Problem 111], appears the following question by Dilks: Is $C(X)$ or $2^X$ locally contractible at the point $X$? In this paper we give an example of a continuum $X$ such that if $\mathcal{H} = C(X)$ or $2^X$, then $\mathcal{H}$ does not have nonempty open subsets which are contractible in $\mathcal{H}$. In particular, $\mathcal{H}$ is not locally contractible at any of its points.

1. An auxiliary construction

Let $Q = [-1, 1] \times [-1, 1] \times \ldots$ with the metric $d(x, y) = \sum |x_n - y_n|/(2^n)$, where $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$. Let $J = [-1, 1]$, and let $\mathbb{N} = \{1, 2, \ldots\}$. For $n \in \mathbb{N}$, define the projection $\Pi_n: Q \to J^n$ by $\Pi_n(x) = (x_1, \ldots, x_n)$. Given $x, y \in Q$, define the segment joining $x$ and $y$ by $\overline{xy} = \{tx + (1 - t)y: t \in [0, 1]\}$.

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Given \( n \in \mathbb{N} \), \( A \subset J^n \times \{0\} \times \ldots \), and \( p, q \in A \) define, for \( m \in \mathbb{N} \), 
\[
A_m = \Pi_n(A) \times \{1/m\} \times \{0\} \times \ldots , \quad p_m = (\Pi_n(p), 1/m, 0, \ldots) \in A_m, \quad q_m = (\Pi_n(q), 1/m, 0, \ldots),
\]
and
\[
L_m = \begin{cases} \frac{p_mp_{m+1}}{q_mq_{m+1}} & \text{if } m \text{ is even,} \\ q_mq_{m+1} & \text{if } m \text{ is odd.} \end{cases}
\]
Then we define \( P(n, A, p, q) = A \cup (\bigcup \{A_m \cup L_m : m \in \mathbb{N}\}) \) and \( N(n, A, p, q) = \{(x_1, \ldots, x_n, -x_{n+1}, x_{n+2}, \ldots) \in Q : x = (x_1, x_2, \ldots) \in P(n, A, p, q)\} \).

Since \( A = \Pi_n(A) \times \{0\} \times \ldots \), we have that \( P(n, A, p, q) \subset \Pi_n(A) \times [0, 1] \times \{0\} \times \ldots \). Then \( \Pi_n(P(n, A, p, q)) = \Pi_n(A) = \Pi_n(N(n, A, p, q)) \).

Notice that \( A = \text{Cl}_Q(\bigcup \{A_m \cup L_m : m \in \mathbb{N}\}) \). So if \( A \) is a continuum, then \( P(n, A, p, q) \) and \( N(n, A, p, q) \) are continua. Notice also that \( P(n, A, p, q) \cap (J^n \times \{0\} \times \ldots) = A = N(n, A, p, q) \cap (J^n \times \{0\} \times \ldots) \).

For \( n \in \mathbb{N} \) and \( u \in J \), define \( Q(n, u) = J^n \times \{u\} \times J \times \ldots \). If \( m \in \mathbb{N} \) and \( u \in (1/(m+1), 1/m) \), then \( Q(n, u) \cap P(n, A, p, q) = Q(n, u) \cap L_m \) is a point which separates \( P(n, A, p, q) \).

2. The Example \( X \)

Let \( a = (-1, 0, \ldots) \) and \( b = (1, 0, \ldots) \). For \( n \in \mathbb{N} \), put \( a_n = (-1 + 1/2^n, 0, \ldots) \) and \( b_n = (1 - 1/2^n, 0, \ldots) \). Define
\[
A^* = a_1b_1; \quad B_1 = P(1, A^*, a_1, b_1); \quad C_1 = P(2, B_1 \cup a_2b_1, a_2, b_1), \quad D_1 = N(2, B_1 \cup a_1b_2, a_1, b_2).
\]
In general, for \( n \geq 2 \), define \( B_n = P(2n - 1, C_{n-1} \cup D_{n-1}, a_n, b_n); \quad C_n = P(2n, B_n \cup a_{n+1}b_n, a_{n+1}, b_n) \) and \( D_n = N(2n, B_n \cup a_nb_{n+1}, a_n, b_{n+1}) \).

Then define \( X = \text{Cl}_Q(\bigcup \{B_n : n \in \mathbb{N}\}) \). Since \( A^* \) is a continuum, then \( B_1, C_1, \) and \( D_1 \) are continua. It follows that each \( B_n \) is a continuum. Furthermore \( B_1 \subset C_1 \cap D_1 \subset B_2 \subset C_2 \cap D_2 \subset \ldots \). So \( X \) is a continuum. We will prove some properties of \( X \):

(A) \( \Pi_{2n+1}(X) = \Pi_{2n+1}(C_n \cup D_n \cup \overline{ab}) \) for all \( n \in \mathbb{N} \).

Since \( \overline{a_nb_n} \subset B_n \) and \( \overline{a_nb_n} \to \overline{ab} \), we have that \( \overline{ab} \subset X \). Then \( C_n \cup D_n \cup \overline{ab} \subset X \). If \( m > n+1 \), then \( \Pi_{2m-1}(B_m) = \Pi_{2m-1}(C_{m-1} \cup D_{m-1}) \). This implies that
\[
\Pi_{2m-2}(B_m) = \Pi_{2m-2}(C_{m-1}) \cup \Pi_{2m-2}(D_{m-1}) \\
= \Pi_{2m-2}(B_{m-1} \cup a_mb_{m-1}) \cup \Pi_{2m-2}(B_{m-1} \cup a_{m-1}b_m) \\
\subset \Pi_{2m-2}(B_{m-1} \cup \overline{ab}).
\]
So \( \Pi_{2m-3}(B_m) \subset \Pi_{2m-3}(B_{m-1} \cup \overline{ab}) \). Then \( \Pi_{2n+1}(B_m) \subset \Pi_{2n+1}(B_{m-1} \cup \overline{ab}) \).

It follows that
\[
\Pi_{2n+1}(B_m) \subset \Pi_{2n+1}(B_{n+1} \cup \overline{ab}).
\]
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But thus

\[ C_{n-1} \cup D_{n-1} \times J^{2n-1} \times \{0\} \times \ldots \]

\[ B_n \subset J^{2n-1} \times [0, 1] \times \{0\} \times \ldots \]

\[ C_n \subset J^{2n-1} \times [0, 1] \times [0, 1] \times \{0\} \times \ldots \]

\[ D_n \subset J^{2n-1} \times [0, 1] \times [-1, 0] \times \{0\} \times \ldots \]

Since \( B_1 \subset \cdots \subset B_n \subset C_n \cup D_n \) and \( \Pi_{2n+1}(C_n \cup D_n \cup \overline{ab}) \) is compact, we conclude that \( \Pi_{2n+1}(X) = \Pi_{2n+1}(C_n \cup D_n \cup \overline{ab}) \).

Notice that \( \Pi_{2n}(X) \subset \Pi_{2n}(C_n) \cup \Pi_{2n}(D_n) \cup \Pi_{2n}(\overline{ab}) \subset \Pi_{2n}(B_n \cup a_{n+1}b_n) \cup \Pi_{2n}(B_n \cup a_nb_{n+1}) \cup \Pi_{2n}(\overline{ab}) \subset \Pi_{2n}(B_n \cup \overline{ab}) \). Since \( C_n \cup D_n \cup \overline{ab} \subset J^{2n+1} \times \{0\} \times \ldots \), we have that \( C_n \cup D_n \cup \overline{ab} = \Pi_{2n+1}(X) \times \{0\} \times \ldots \). Then \( r_1 \) is a retraction. Similarly, \( r_2 \) is a retraction.

\( B_n \cap \overline{ab} = \overline{a_nb_n} \) for every \( n \in \mathbb{N} \).
\[ B_1 \cap \overline{ab} = P(1, A^*, a, b) \cap (J \times \{0\} \times \ldots) = A^* = \overline{a_1b_1}. \]

Suppose that \[ B_n \cap \overline{ab} = \overline{a_nb_n}. \] Then \[ B_{n+1} \cap \overline{ab} = P(2n+1, C_n \cup D_n, a_{n-1}, b_{n-1}) \cap (J^{2n-1} \times \{0\} \times \ldots) \cap (J \times \{0\} \times \ldots) = (C_n \cup D_n) \cap (J^{2n} \times \{0\} \times \ldots) \cap (J \times \{0\} \times \ldots) = \left((B_n \cup a_{n+1}b_{n+1}) \cup (B_n \cup a_n b_{n+1})\right) \cap \overline{ab} = a_{n+1}b_{n+1}. \]

(D) Assume that \( n \in \mathbb{N}, Y = C_n \cup D_n \cup \overline{ab} \) and \( \mathcal{H}_0 = C(Y) \) or \( 2^Y \). Then there do not exist \( B \in \mathcal{H}_0, W \) open in \( \mathcal{H}_0 \) and \( F: W \times I \to \mathcal{H}_0 \) continuous such that \( B \subseteq B_n, B \subseteq W \), for each \( A \in W \), \( F(A, 0) = A, F(A, 1) = Y \) and, for every \( A \in W \) and \( s \leq t \), \( F(A, s) \subseteq F(A, t) \).

Suppose that there exist such \( B, W, \) and \( F \). Since \( F(B, 0) \subseteq B_n \) and \( F(B, 1) = Y \not\subseteq B_n \), there exists \( t_0 = \max\{t \in [0, 1]: F(B, t) \subseteq B_n\} \) and \( 0 \leq t_0 < 1 \). By (C), \( a_{n+1}, b_{n+1} \not\in B_n \) and, since \( B_n \subseteq J^{2n} \times \{0\} \times \ldots \), we have that \( B_n \cap \left((\Pi_{2n}(a_{n+1})) \times J \times \ldots\right) = \emptyset \). So there exists \( \delta > 0 \) such that \( t_0 + \delta/2 < 1; \mathcal{B} = \{A \in \mathcal{H}_0: H(A, B) < \delta\} \subseteq W \) (\( H \) is the Hausdorff metric for \( 2^Q \)) and, if \( A \in \mathcal{B} \) and \( t \in (t_0 - \delta, t_0 + \delta) \cap [0, 1] \), then \( F(A, t) \cap \left((\Pi_{2n}(a_{n+1})) \times J \times \ldots\right) = \emptyset \).

We will prove that \( F(B, t_0 + \delta/2) \subseteq C_n \). Recall that
\[
C_n = P(2n, B_n \cup a_{n+1}b_n, a_{n+1}, b_n)
= B_n \cup a_{n+1}b_n \cup \left(\bigcup\{A_m \cup L_m: m \in \mathbb{N}\}\right)
\]
where \( A_m = (\Pi_{2n}(B_n \cup a_{n+1}b_n)) \times \{1/m\} \times \{0\} \times \ldots; \)
\[
L_m = \begin{cases} \overline{p_m p_{m+1}} & \text{if } m \text{ is even} \\ \overline{q_m q_{m+1}} & \text{if } m \text{ is odd} \end{cases}
\]
\[
p_m = (\Pi_{2n}(a_{n+1}), 1/m, 0, \ldots),
q_m = (\Pi_{2n}(b_n), 1/m, 0, \ldots).
\]

Since \( B \subseteq B_n \subseteq J^{2n} \times \{0\} \times \ldots \), we have that \( B = \Pi_{2n}(B) \times \{0\} \times \ldots \). For each \( m \in \mathbb{N}, \) put \( E_m = \Pi_{2n}(B) \times \{1/m\} \times \{0\} \times \ldots \subseteq A_m \). Then \( E_m \in \mathcal{H}_0 \) and \( E_m \to B \), so there exists \( M \in \mathbb{N} \) such that \( H(B, E_m) < \delta \) for all \( m \geq M \).

Let \( m \) be an even number with \( m \geq M \). We will show that
\[
T = F(E_m, t_0 + \delta/2) \subseteq C_n.
\]
Choose \( u \in \left(1/(m+1), 1/m\right) \). Then \( Q(2n, u) \cap C_n = Q(2n, u) \cap \overline{p_m p_{m+1}} \). Since \( \{F(E_m, s) \in 2^Y: 0 \leq s \leq t_0 + \delta/2\} \) is an order arc in \( 2^Y \) from \( F(E_m, 0) \) to \( T \), then [4, Theorem 1.8] each component of \( T \) intersects \( E_m \subseteq \Pi_{2n}(B) \times \{1/m\} \times \{0\} \times \ldots \). Thus \( S = T \cup (\Pi_{2n}(B_n) \times \{1/m\} \times \{0\} \times \ldots) \) is connected. Since \( \overline{p_m p_{m+1}} \subseteq (\Pi_{2n}(a_{n+1})) \times J \times \ldots, B_n \subseteq J^{2n} \times \{0\} \times \ldots \), and \( a_{n+1} \not\in B_n \), we have that \( S \cap \overline{p_m p_{m+1}} = \emptyset \). On the other hand, \( S \subseteq Y = C_n \cup D_n \cup \overline{ab} \) and \( D_n \cup \overline{ab} \subseteq J^{2n} \times [-1, 0] \times \{0\} \times \ldots \), so \( S \cap Q(2n, u) = S \cap Q(2n, u) \cap C_n = S \cap Q(2n, u) \cap \overline{p_m p_{m+1}} = \emptyset \). Furthermore \( S \) is connected and \( E_m \subseteq S \cap J^{2n} \times \left[u, 1\right] \times J \times \ldots \).
Then \( S \subset (J^{2n} \times (u, 1] \times J \times \ldots) \cap Y \subset C_n \). Thus \( F(E_m, t_0 + \delta/2) \subset C_n \) for all \( m \geq M \) with \( m \) even. Hence \( F(B, t_0 + \delta/2) \subset C_n \).

In a similar way it may be proved that
\[
F(B, t_0 + \delta/2) \subset D_n.
\]

Then \( F(B, t_0 + \delta/2) \subset D_n \). Then \( F(B, t_0 + \delta/2) \subset C_n \cap D_n = C_n \cap (J^{2n} \times \{0\} \times \ldots) \cap D_n = (B_n \cup a_{n+1} b_n) \cap (B_n \cup a_n b_{n+1}) = B_n \). Therefore \( F(B, t_0 + \delta/2) \subset B_n \).

This contradicts the choice of \( t_0 \) and proves (D).

(E)
\[
C(X) = \text{Cl}_{c(x)} \left( \bigcup \{C(B_n) : n \in \mathbb{N}\} \right),
\]

and
\[
2^X = \text{Cl}_{2^X} \left( \bigcup \{2^{B_n} : n \in \mathbb{N}\} \right).
\]

For \( n \in \mathbb{N} \), take the natural retraction \( f: \overline{ab} \to \overline{a_n b_n} \) and define \( r: X \to B_n \) by
\[
r(x) = \begin{cases} 
 f(r_2(x)) & \text{if } r_2(x) \in \overline{ab}, \\
r_2(x) & \text{if } r_2(x) \in B_n,
\end{cases}
\]

with \( r_2 \) as in (B). Then \( r \) is a retraction such that \( D(x, r(x)) \leq 1/(2^{n-1}) \) for every \( x \in X \). Hence, for each \( B \in 2^X \), \( H(B, r(B)) \leq 1/(2^{n-1}) \).

(F)

If \( \mathcal{H} = C(X) \) or \( 2^X \), then \( \mathcal{H} \) does not have nonempty open subsets which are contractible in \( \mathcal{H} \).

Suppose that there exist an open nonempty subset \( U \) of \( \mathcal{H} \), a continuous function \( G: U \times I \to \mathcal{H} \), and \( A_0 \in \mathcal{H} \) such that \( G(A, 0) = A \) and \( G(A, 1) = A_0 \) for every \( A \in U \). Then the function \( K: U \times I \to \mathcal{H} \) given by \( K(A, t) = \bigcup \{G(A, s) : 0 \leq s \leq t\} \) is continuous [4, Lemma 16.3]. Let \( \alpha: [1/2, 1] \to \mathcal{H} \) be a continuous function such that \( \alpha(1/2) = A_0 \), \( \alpha(1) = X \) and \( \alpha(s) \subset \alpha(t) \) if \( s \leq t \). Choose \( n \in \mathbb{N} \) such that \( U \cap 2^{B_n} \neq \emptyset \). Let \( Y = C_n \cup D_n \cup \overline{ab} \), \( \mathcal{H}_0 = 2^X \cap \mathcal{H} \), and \( W = U \cap \mathcal{H}_0 \). Fix \( B \in W \cap 2^{B_n} \). Define \( F: W \times I \to \mathcal{H}_0 \) by:
\[
F(A, t) = \begin{cases} 
 r_1(K(A, 2t)) & \text{if } 0 \leq t \leq 1/2, \\
r_1(K(A, 1) \cup \alpha(t)) & \text{if } 1/2 \leq t \leq 1,
\end{cases}
\]

where \( r_1 \) is the retraction defined in (B) and \( r_1(Z) \) means the image of \( Z \) under \( r \). Properties of \( B \), \( W \), and \( F \) contradict property (D), and the contradiction completes the proof of (F).

Added in proof. The question answered in this paper has also been answered by Hisao Kato, On local contractibility at \( X \) in hyperspaces \( C(X) \) and \( 2^X \), Houston J. Math. 15 (1989), 363–370.
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