ON THE CONVERGENCE IN $\mathcal{S}'$

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Abstract. We prove the following assertion: Let $T_j, j \in \mathbb{N}$, be a sequence in $\mathcal{S}'$ such that $T_j \ast \phi$ converges to $0$ in $\mathcal{S}'$ as $j \to \infty$, for any $\phi \in \mathcal{D}$.

Then $T_j \to 0$ in $\mathcal{S}'$ as $j \to \infty$.

The following problem was presented by A. Kamiński: Let $T_j$ be a sequence in $\mathcal{S}'$ such that $T_j \ast \phi$ converges to $0$ in $\mathcal{S}'$ as $j \to \infty$ for any $\phi \in \mathcal{S}$. Does $T_j \to 0$ in $\mathcal{S}'$ as $j \to \infty$? K. Keller gave a positive answer to this question in [3] by using an original method based on a theorem of Grothendieck. But the assertion follows directly from the equality of the linear hull of $\mathcal{S} \ast \mathcal{S}$, $\text{lin}(\mathcal{S} \ast \mathcal{S})$, and of $\mathcal{S}$ proved by J. Voigt [9] and even earlier by H. Petzeltová and P. Vrbová [4]. Note that $\text{lin}(\mathcal{D} \ast \mathcal{D}) = \mathcal{D}$ is known from the following papers: L. A. Rubel, W. A. Squires, and B. A. Taylor [5], and J. Dixmier and P. Malliavin [1].

The aim of this paper is to prove the following theorem: 

Theorem 1. Let $T_j, j \in \mathbb{N}$, be a sequence from $\mathcal{S}'$ such that $T_j \ast \phi$ converges to $0$ in $\mathcal{S}'$ as $j \to \infty$ for any $\phi \in \mathcal{S}$. Then $T_j \to 0, j \to \infty$, in $\mathcal{S}'$.

The well-known result of Schwartz should be mentioned: “If $T \in \mathcal{D}'$ and $T \ast \phi \in \mathcal{S}'$, then $T \in \mathcal{S}'$.”

In the proof we shall use Keller's method [3], and since it is not known whether $\text{lin}(\mathcal{D} \ast \mathcal{S})$ is equal to $\mathcal{S}$, this method is essential in our formulation of the problem.

As usual, $\mathbb{N}$ is the set of strictly positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{Z} = \mathbb{N}_0 \cup (-\mathbb{N})$. We denote by $\mathcal{D}$, $\mathcal{S}$, and $\mathcal{D}_K$, where $K$ is a compact subset of the Euclidean space $\mathbb{R}^n$ ($K \subseteq \mathbb{R}^n$), the well-known Schwartz testing-function spaces.

As in [8, Chapter I, §5], $\mathcal{S}_k$, $k \in \mathbb{N}_0$, is the completion of $\mathcal{S}$ in the norm

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Recall that $\mathcal{S} = \bigcap_{k \in \mathbb{N}_0} \mathcal{S}_k$ with the projective topology obtained from the injections $\mathcal{S} \to \mathcal{S}_k$, $k \in \mathbb{N}_0$, $\mathcal{S}' = \bigcup_{k \in \mathbb{N}_0} \mathcal{S}'_k$ with the inductive topology obtained from the injections $\mathcal{S}'_k \to \mathcal{S}'$, $k \in \mathbb{N}_0$, and if a set is bounded in $\mathcal{S}'$ then it belongs to some $\mathcal{S}'_k$ and is bounded there. The norm in $\mathcal{S}'_k$ is denoted by $\| \|_{-k}$, $k \in \mathbb{N}_0$.

For all the details we refer to [6] and [8].

Let us recall the Grothendieck theorem: "Let $E$ be a locally convex Hausdorff space, $F$ and $F_i$, $i \in \mathbb{N}$, Fréchet spaces. Let $u$ be a continuous mapping $F \to E$ and $u_i$ continuous mappings $F_i \to E$, $i \in \mathbb{N}$. If $u(F) \subset \bigcup_{i \in \mathbb{N}} u_i(F_i)$, then there exists some index $i_0$ such that $u(F) \subset u_{i_0}(F_{i_0})$.

For the proof of Theorem 1 we need the following lemma:

**Lemma.** With the assumptions on $T_j$ as in Theorem 1, there exists $k \in \mathbb{N}$ such that, for every $K \in \mathbb{R}^n$, there exists a neighborhood $U_K$ of 0 in $\mathcal{S}_K$ and $B_K > 0$ such that $\phi \in U_K$ implies

\[(1) \quad \| (T_j * \phi)(x) \| \leq B_K (1 + |x|^k), \quad \text{for all } x \in \mathbb{R}^n \text{ and } j \in \mathbb{N}.

**Proof of the lemma.** First, we shall show that for every $\phi \in \mathcal{D}$ there are $k_\phi \in \mathbb{N}$ and $C_\phi > 0$ such that

\[(2) \quad \| (T_j * \phi)(x) \| \leq C_\phi (1 + |x|)^{k_\phi}, \quad x \in \mathbb{R}^n, \quad j \in \mathbb{N}.

From [5, Theorem 3] it follows that, for a given $\phi \in \mathcal{D}$, there are $\psi_1, \ldots, \psi_N$, $\theta_1, \ldots, \theta_N$ from $\mathcal{D}$ such that

$$\phi = \psi_1 * \theta_1 + \cdots + \psi_N * \theta_N.$$  

Sequences $T_j * \psi_j$, $j \in \mathbb{N}$, $k = 1, \ldots, N$, are bounded in $\mathcal{S}'$ so [2, Theorem 13, p. 41] implies that there are sequences of continuous functions $H_{j,k}$, $j \in \mathbb{N}$, $k = 1, \ldots, N$, and nonnegative integers $\beta_1, \ldots, \beta_N$ and $m_1, \ldots, m_N$ such that

$$T_j * \psi_j = (1 + |x|^2)^{m_k} H_{j,k}^{(\beta_k)}$$

and

$$\sup_{x \in \mathbb{R}^n} |H_{j,k}(x)| < \infty, \quad k = 1, \ldots, N.$$  

Thus, we get

$$T_j * \phi = (1 + |x|^2)^{m_1} H_{j,1}^{(\beta_1)} * \theta_1^{(\beta_1)} + \cdots + (1 + |x|^2)^{m_N} H_{j,N}^{(\beta_N)} * \theta_N^{(\beta_N)}, \quad j \in \mathbb{N},$$

from which (2) follows.
(2) implies that, for some $B_\phi > 0$,
\[ \ln(||(T_j * \phi)(x)|| + 1) \leq B_\phi \ln(2 + |x|), \quad x \in \mathbb{R}^n, \ j \in \mathbb{N}. \]

Let $Z$ be a fixed closed ball in $\mathbb{R}^n$, and let $\mathcal{S}_b^N$ be the space of all bounded sequences of bounded continuous functions defined on $\mathbb{R}^n$ supplied with the norm $\gamma(\{\alpha_j\}) = \sup_{x \in \mathbb{R}^n} |\alpha_j(x)|$.

We define the mapping $h$ from $\mathcal{S}_Z$ into $\mathcal{S}_b^N$ by $h(\phi) = \{\alpha_j\}$, where
\[ \alpha_j(x) = \ln((T_j * \phi)(x)| + 1)/\ln(2 + |x|), \quad x \in \mathbb{R}^n, \ j \in \mathbb{N}. \]

Let us prove that the mapping $\gamma \circ h$ is bounded on some neighborhood of zero in $\mathcal{S}_Z$. We have $\mathcal{S}_Z = \bigcup_{R > 0} \mathcal{S}_{Z,R}$, where
\[ \mathcal{S}_{Z,R} = \{\phi \in \mathcal{S}_Z; \gamma(h(\phi)) \leq R\}, \quad R > 0. \]

Let $\phi$ belong to the closure of $\mathcal{S}_{Z,R}$ with respect to $\mathcal{S}_Z$. Since $\mathcal{S}_Z$ is metrizable, there is a sequence $\phi_\nu, \nu \in \mathbb{N}$, from $\mathcal{S}_{Z,R}$ which converges to $\phi$ in $\mathcal{S}_Z$.

Fix $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$. For any $\nu \in \mathbb{N}$, we have
\[ (\ln((T_j * \phi_\nu)(x)| + 1)/\ln(2 + |x|)) \leq R. \]

From $T_j * \phi_\nu(x) \to T_j * \phi(x)$, $\nu \to \infty$, it follows that
\[ (\ln((T_j * \phi)(x)| + 1)/\ln(2 + |x|)) \leq R. \]

Since $j$ and $x$ were arbitrarily chosen, we get that $\phi \in \mathcal{S}_{Z,R}$, and thus that $\mathcal{S}_{Z,R}$ is closed.

Because $\mathcal{S}_Z$ is of second category, there is a neighborhood of zero in $\mathcal{S}_Z$, denoted by $\mathcal{U}_Z$, and a $D > 0$ such that
\[ \gamma(h(\phi)) \leq D, \quad \phi = \mathcal{U}_Z. \]

So, for any $\phi \in \mathcal{U}_Z$, $j \in \mathbb{N}$, and $x \in \mathbb{R}^n$,
\[ \ln((T_j * \phi)(x)| + 1) \leq D \ln(2 + |x|); \]
i.e., for a set $\mathcal{U}_Z$ there are $k \in \mathbb{N}$ and $B_Z > 0$ such that $\phi \in \mathcal{U}_Z$ implies that
\[ ||(T_j * \phi)(x)|| \leq B_Z (1 + |x|)^k, \quad \text{for all } x \in \mathbb{R}^n, \ j \in \mathbb{N}. \]

Since $\mathcal{U}_Z$ is absorbing in $\mathcal{S}_Z$, we get that, for any $\phi \in \mathcal{S}_Z$, there exists $B_\phi > 0$ such that

(3) \[ ||(T_j * \phi)(x)|| \leq B_\phi (1 + |x|)^k, \quad x \in \mathbb{R}^n, \ j \in \mathbb{N}. \]

Let $K \subset \mathbb{R}^n$ be an arbitrary compact set. Any $\phi$ from $\mathcal{S}_K$ can be written in the form
\[ \phi(x) = \sum_{i=1}^p \phi_i(x - \tau_i), \quad x \in \mathbb{R}^n, \]
where $\phi_i$, $i = 1, \ldots, p$, are functions from $\mathcal{D}_Z$. This implies that
\[
|(T_j \ast \phi)(x)| \leq \sum_{i=1}^{p} |(T_j \ast \phi)(x - \tau_i)|, \quad x \in \mathbb{R}^n, \ j \in \mathbb{N}.
\]

Since
\[
|(T_j \ast \phi)(x - \tau_i)| \leq B_i (1 + |x - \tau_i|)^k \leq B_i (1 + |\tau_i|)^k (1 + |x|)^k, \quad x \in \mathbb{R}^n, \ j \in \mathbb{N},
\]
where $B_i$ are suitable constants, $i = 1, \ldots, p$, we get that for any $\phi \in \mathcal{D}$ there exists a $B_\phi > 0$ such that (3) holds. Again, by using the fact that $\mathcal{D}_K$ is of second category, it follows that there exist $B_K > 0$ and a neighborhood of zero $\mathcal{V}_K$ such that
\[
|(T_j \ast \phi)(x)| \leq B_K (1 + |x|)^k, \quad x \in \mathbb{R}^n, \ j \in \mathbb{N}, \ \phi \in \mathcal{V}_K.
\]
Since $K$ is arbitrary, the proof is complete.

Proof of Theorem 1. It is enough to prove that $T_j$, $j \in \mathbb{N}$, is a bounded sequence in $\mathcal{S}'$. Namely, for any $\phi$, $\psi \in \mathcal{S}$ we have
\[
\langle T_j \ast \phi, \psi \rangle = \langle T_j, \phi \ast \psi \rangle, \quad j \in \mathbb{N} \quad (\phi(x) = \phi(-x), \ x \in \mathbb{R}^n),
\]
and since the linear hull of $\mathcal{S} \ast \mathcal{D}$ is equal to $\mathcal{S}$, we get that for any $\phi \in \mathcal{S}$, $\langle T_j, \phi \rangle \to 0$, $j \to \infty$. Now, by the boundedness of $T_j$ in $\mathcal{S}'$ and [7, Theorem 33.2 and Corollary 1, p. 356] the assertion will follow.

As in [3, p. 88], we put
\[
E_k = \left\{ a \equiv \{a_{jq} \}, \ j \in \mathbb{N}, \ q \in \mathbb{Z}^n, \ a_{jq} \in \mathbb{C} ; \right. \\
\|a\|_k = \sup_{j, q} |a_{jq}|((1 + |q|) \ln(1 + j))^{-k} \leq \infty \right\}, \quad k > 0,
\]
and
\[
E = \lim \inf_{k \to \infty} E_k.
\]

Let $e \in \mathcal{D}$ have the properties $e(x) = 1$, $x \in I^n$, $I = [-1, 1]$, $\text{supp } e \subset J^n$, $J = [-3/2, 3/2]$, and let $e_q(x) = e(x + q)$, $x \in \mathbb{R}^n$, $q \in \mathbb{Z}^n$. We shall prove that the mapping $u$ defined on $\mathcal{S}$ by
\[
(4) \quad \mathcal{S} \ni \psi \to u(\psi) = \{a_{jq} \} = \{ \langle T_j(t)e(t + q), \psi(t) \rangle \}, \quad j \in \mathbb{N}, \ q \in \mathbb{Z}^n,
\]
is continuous from $\mathcal{S}$ to $E$. Then Grothendieck's theorem implies that
\[
u(\mathcal{S}) \subset E_s \text{ for some } s > 0.
\]

By Keller's construction [3, p. 89], in which the definition of the norm in $E_k$ is essential, this implies that $\{T_j, j \in \mathbb{N}\}$ is a bounded subset of $\mathcal{S}'$.

Let us prove that
\[
(5) \quad \{ T_j(\cdot)e(\cdot + q)/(1 + |q|)^k, \ j \in \mathbb{N}, \ q \in \mathbb{Z}^n \},
\]
where $k \in \mathbb{N}$, from the lemma, is bounded in $\mathcal{S}'$. 
Let $K = \text{supp} \hat{e}$ and $\mathcal{U}_K$ be a neighborhood of zero in $\mathcal{D}_K$ for which (1) holds. Since the sets of the form

$$\mathcal{V}(m, \varepsilon) = \left\{ \phi \in \mathcal{D}_K, \sup_{x \in K} \sum_{|\alpha| \leq m} |\phi^{(\alpha)}(x)| < \varepsilon \right\}, \quad m \in \mathbb{N}_0, \varepsilon > 0,$$

form a basis of neighborhoods of zero in $\mathcal{D}_K$, we have that for some $m \in \mathbb{N}_0$ and $\varepsilon > 0$, $\mathcal{V}(m, \varepsilon) \subset \mathcal{U}_K$. Fix $\psi \in \mathcal{S}$. Let $a \in \mathbb{R}^n$, $\alpha \in \mathbb{N}_0^n$. We have

$$\sup_{i \in \mathcal{K}} |(\tilde{\psi}(t - a)\hat{e}(t))^{(\alpha)}| \leq 2|\alpha| \sup_{i \in \mathcal{K}} |\tilde{\psi}^{(i)}(t - a)| \sup_{i \in \mathcal{K}} |\hat{e}^{(i)}(t)|,$$

where $i \leq \alpha$ means that the components of $i$ are $\leq$ the corresponding components of $\alpha$. For any $i \leq \alpha$, $\tilde{\psi}^{(i)}$ and $\hat{e}^{(i)}$ are bounded on $\mathbb{R}^n$. This implies that there is a $C_{\psi, \alpha}$ such that

$$\sup_{i \in \mathcal{K}} |(\psi(t - a)e(t))^{(\alpha)}| \leq C_{\psi, \alpha}.$$

So, for $M = Ae^{-1} \max\{C_{\psi, \alpha}; |\alpha| \leq m\}$, where $A$ is the number of $\alpha$, $|\alpha| \leq m$, all the functions

$$\mathbb{R}^n \ni t \to \frac{1}{M} \tilde{\psi}(t - a)\hat{e}(t), \quad a \in \mathbb{R}^n,$$

are from $\mathcal{V}(m, \varepsilon)$ and, for any $a \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $j \in \mathbb{N}$, we have

$$\left| \frac{1}{(1 + |x|)^k} \left( T_j(t) * \left( \frac{1}{M} \tilde{\psi}(t - a)\hat{e}(t) \right) \right)(x) \right| = \frac{1}{M(1 + |x|)^k} |T_j(t), \tilde{\psi}(x - t - a)\hat{e}(x - t)| < B_K.$$

Take $x = a = -q \in \mathbb{Z}^n$. The boundedness of (5) in $\mathcal{S}'$ follows from

$$\left| \frac{1}{(1 + |q|)^k} \langle T_j(t), \tilde{\psi}(-t)\hat{e}(-q - t) \rangle \right| = \frac{1}{(1 + |q|)^k} |\langle T_j(t)\hat{e}(t + q), \tilde{\psi}(t) \rangle| < MB_K.$$

So we have that for some $k_0$ (i.e., in some $\mathcal{S}'_{k_0}$) and some $C > 0$,

$$\|T_j(\cdot)e(\cdot + q)/(1 + |q|)^k\|_{-k_0} < C, \quad j \in \mathbb{N}, \quad q \in \mathbb{Z}^n.$$

This implies for the mapping $u$ defined by (4) that, for every $j \in \mathbb{N}$ and $q \in \mathbb{Z}^n$,

$$|a_{jq}| \leq C(1 + |q|)^k \|\psi\|_{k_0}.$$

It follows that $u$ is continuous from $\mathcal{S}$ into $E$. This completes the proof.
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