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A REGULAR TOPOLOGICAL SPACE HAVING NO CLOSED SUBSETS OF CARDINALITY $\aleph_2$

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ABSTRACT. Using $\diamondsuit_{\lambda^+}$, we construct a regular topological space in which all closed sets are of cardinality either $< \lambda$ or $\geq 2^{\lambda^+}$. In particular (answering a question of Juhász) there is always a regular space in which no closed set has cardinality $\aleph_2$.

1. INTRODUCTION

A topological space X is said to *omit* a cardinal κ, if $|X| > \kappa$ but no closed subset of X has cardinality κ.

For example, the space $\beta\omega$ (the Stone-Čech compactification of the countable discrete set ω) omits all cardinals κ with $\aleph_0 \leq \kappa < 2^{2^{\aleph_0}}$. (This is in some sense "best possible," since for no κ can there be a Hausdorff space omitting all cardinals in $[\kappa, 2^{2^\kappa}]$.)

[Hu] showed that there is always a Hausdorff space omitting $\aleph_2$. [J, §6] uses [HJ] to show that if $2^\kappa = \kappa^+$, then there is a zerodimensional (hence regular) Hausdorff space omitting κ, and he asks whether one could prove the existence of a regular space omitting $\aleph_2$ in ZFC alone, i.e. without assumptions on cardinal arithmetic. (See [J] for related results and references.)

We will show that

$$ZFC + \diamondsuit_S(\lambda^+) \vdash \text{"there is a regular space omitting all } \kappa \in [\lambda, 2^{\lambda^+})\text{"}$$

and derive as a corollary

$$ZFC \vdash \text{"there is a regular space omitting } \aleph_2\text{."}$$

Here is a sketch of the construction: Our space X will be (essentially) the set ${}^{\lambda^+}\lambda^+$. For every subset of X of size λ we will mark 2^{λ^+} many points as limit points of this set. We then construct a sequence of λ^+ many sets that will serve as a subbasis for a topology. Using $\diamondsuit$ we can "guess" subsets of size λ

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and make sure that no such set will be separated from its predetermined limit points.

The proof is due to the third author.

2. The main theorem and its corollaries

Theorem (Shelah). *Assume $\diamondsuit_S(\lambda^+)$, where S is stationary in λ^+. Then there is a regular topological space X of size 2^{λ^+} in which every set of cardinality less than λ is closed, but $|cl(A)| = 2^{\lambda^+}$, for all A of size λ. ($cl(A) =$ the closure of the set A.)*

Corollary. *The following is true in ZFC:*

There exists a regular space of power $> \aleph_2$, in which there are no closed sets of power $\aleph_1$ or $\aleph_2$.

Proof of the corollary. If $2^{2^{\aleph_0}} > \aleph_2$, then $\beta\omega$ will work, since all infinite closed sets have size $2^{2^{\aleph_0}}$. Otherwise we have $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$, so by [G], $\diamondsuit_S$ holds with $S = \{\delta < \aleph_2 : cf(\delta) = \aleph_0\}$, so we can apply the theorem for $\lambda = \aleph_1$.

Second corollary. *If $cf(\kappa) = \aleph_0$, κ a strong limit $> \aleph_0$, then there is a regular space of size $> \kappa^+$ with no closed subsets of size κ^+.*

Proof of the second corollary. If $2^\kappa = \kappa^+$, then by [S] $\diamondsuit_S(\kappa^+)$ holds for $S = \{\delta < \kappa^+ : cf(\delta) = \aleph_1\}$, so we can apply the theorem with $\lambda = \kappa$ to get a space where all closed sets are of size $< \kappa$ or $\geq 2^{\kappa^+} > \kappa^+$.

If $2^\kappa > \kappa^+$, then it is well known that there is a compact space omitting all cardinals in $[\kappa, 2^\kappa)$: The space $\beta\kappa - \kappa$ is a compact F-space (of size 2^{2^κ}), so every closed subspace Y is also a compact F-space and hence satisfies $|Y| < \kappa$ or $|Y| = |Y|^{\aleph_0} \geq \kappa^{\aleph_0} = 2^\kappa$ (see [vD, §6–7]). (Alternatively, we can consider $X = {}^\omega\kappa$, the product of countably many discrete spaces of size κ, and prove directly that it has no closed sets of size κ^+.)

Proof of the theorem. Let T_α = the set of increasing α-sequences in λ^+. Our space X will be T_{λ^+}.

Notation. For $\nu \in T_\delta$, $\alpha < \delta$, let $\nu \restriction \alpha \in T_\alpha$ be the restriction of ν to α.

Idea of the proof. A subbase of the topology will be given by a sequence

$$\{\mathscr{U}_\beta : \beta \in S\} \cup \{\mathscr{V}_\beta : \beta \in S\}$$

where $\mathscr{U}_\beta = X - \mathscr{V}_\beta$, and both $\mathscr{U}_\beta$ and $\mathscr{V}_\beta$ have the form $\bigcup [\eta_i]$, where $\eta_i \in T_{\alpha_i}$, $\alpha_i < \lambda^+$, and $[\eta] = \{f \in T_{\lambda^+} : f \supseteq \eta\}$. The construction will be done in stages, where at each stage δ we "promise" $[\eta] \subseteq \mathscr{U}_\beta$ or $[\eta] \subseteq \mathscr{V}_\beta$ for certain η's and β's. Formally, this promise is represented by the sets $\mathscr{U}_\beta^\delta$ and $\mathscr{V}_\beta^\delta$. We must be careful not to make contradictory promises.

At each stage we take care of some approximations $\{f_i \restriction \delta : i < \lambda\}$ to sets $\{f_i : i < \lambda\}$ of size λ and make sure that in the end their limit points include

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a certain predetermined set of size 2^{λ^+}. Every initial segment $g \restriction \delta$ of such a limit point can be computed from $\langle f_i \restriction \delta : i < \lambda \rangle$. Hence, using $\diamondsuit$, we can ensure that all sets of size λ have been considered. We also take care of certain approximations to sets of size $< \lambda$, and make sure that they will turn out to be closed. Again, by $\diamondsuit$ we ensure that all possible sets have been considered.

3. Construction of a subbasis for X

We will construct a sequence $\langle \mathscr{U}_\beta, \mathscr{V}_\beta : \beta \in S \rangle$ such that the following will hold:

(A) $\mathscr{U}_\beta \cup \mathscr{V}_\beta = X$.

(B) $\mathscr{U}_\beta \cap \mathscr{V}_\beta = \varnothing$.

(C) For every set $\{f_i : i \leq \gamma\}$ $(\gamma < \lambda)$ of different functions $(f_i \in T_{\lambda^+})$ there is a β such that $\{f_i : i < \gamma\}$ and f_γ are separated by $\langle \mathscr{U}_\beta, \mathscr{V}_\beta \rangle$.

Clearly, (A)–(C) imply that T_{λ^+}, with the topology generated by the subbase $\{\mathscr{U}_\beta, \mathscr{V}_\beta : \beta \in S\}$ is regular: By (A) and (B) the subbasis sets and hence the basis sets are clopen. By (C), small sets are closed. In particular, X is a T_1 space with a clopen base, so it is (completely) regular. We will also ensure:

(D) If $|A| = \lambda$, then $|cl(A)| \geq 2^{\lambda^+}$.

Definition. For $\beta < \lambda^+$, ${}^{\lambda+1}T_\beta$ is the set of all sequences

$$\vec{\eta} = \langle \eta_i : i \leq \lambda \rangle$$

where each η_i is in T_β.

Definition. Let $c_\beta : \lambda^+ \to {}^{\lambda+1}T_\beta$ be a function such that for all $\vec{\eta} \in {}^{\lambda+1}T_\beta$ the set $c_\beta^{-1}(\vec{\eta})$ is unbounded in λ^+. Elements of $c_\beta^{-1}(\vec{\eta})$ are called "codes" of $\vec{\eta}$.

It is possible to find such functions, since $\diamondsuit_S(\lambda^+)$ implies $2^\lambda = \lambda^+$.

We will assume $\forall \delta \in S : \ \delta > \lambda$.

Definition. Let (by $\diamondsuit_S$) $\langle X_\delta : \delta \in S \rangle$ be a sequence such that $\forall \delta \in S \ X_\delta \subseteq \delta$, and for every $X \subseteq \lambda^+$ the set

$$\{\delta \in S : X_\delta = X \cap \delta\}$$

is stationary.

Claim. There is a sequence

$$\langle s^\delta : \delta \in S \rangle, \qquad s^\delta = \langle s_i^\delta : i \leq \sigma_\delta \rangle, \quad s_i^\delta \in T_\delta, \quad \sigma_\delta < \lambda$$

such that for all $\gamma < \lambda$, for all $\vec{f} = \langle f_i : i \leq \gamma \rangle \in {}^{\gamma+1}T_{\lambda^+}$:

$$\{\delta \in S : \langle f_i \restriction \delta : i \leq \gamma \rangle = s^\delta\} \text{ is stationary.}$$

Proof of the claim. Every such sequence $\vec{f}$ can be "coded" by a set $A_{\vec{f}} \subseteq \lambda \times \lambda^+$, e.g. by

$$A_{\vec{f}} = \bigcup_{i \leq \gamma} \{i\} \times \operatorname{range}(f_i).$$

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(Note that γ and $\vec{f}$ can be computed from $A_{\vec{f}}$.)

Let $F : \lambda^+ \to \lambda \times \lambda^+$ be a bijection. Then the set

$$C_1 = \{\delta : F \restriction \delta \text{ is a bijection between } \delta \text{ and } \lambda \times \delta\}$$

is closed unbounded.

For $\delta \in S$ let

$$s^\delta = \langle \eta_i : i \leq \gamma \rangle$$

if

$$\delta \in C_1 \text{ and } F(X_\delta) = \bigcup_{i \leq \gamma} \{i\} \times \text{range}(\eta_i)$$

for some sequence $\langle \eta_i : i \leq \gamma \rangle \in {}^{\gamma+1}T_\delta$, $\gamma < \lambda$. If $F(X_\delta)$ cannot be written as above, let $s^\delta =$ any sequence in ${}^1T_\delta$.

To show that this sequence is as required, consider any sequence $\vec{f} = \langle f_i : i \leq \gamma \rangle \in {}^{\gamma+1}T_{\lambda^+}$, for any $\gamma < \lambda$. Let $X = A_{\vec{f}}$. The set

$$C_2 = \bigcap_{i \leq \gamma} \{\delta < \lambda^+ : \text{range}(f_i \restriction \delta) = \text{range}(f_i) \cap \delta\}$$

is closed unbounded. Hence it is enough to check that if $\delta \in S \cap C_1 \cap C_2$ and $X_\delta = X \cap \delta$, then $s^\delta = \langle f_i \restriction \delta : i \leq \gamma \rangle$. But this follows easily from

$$\begin{aligned} F(X_\delta) &= F(X \cap \delta) = F(X) \cap (\lambda \times \delta) \\ &= \bigcup_{i \leq \gamma} \{i\} \times (\text{range}(f_i) \cap \delta) = \bigcup_{i \leq \gamma} \{i\} \times \text{range}(f_i \restriction \delta). \end{aligned}$$

Definition. For $\delta \in S$, let $(M_\delta, \in) < (H(\kappa), \in)$ (where $\kappa = (2^{2^\lambda})^+$), $|M_\delta| = \lambda$, $\delta + 1 \subseteq M_\delta$, $X_\delta \in M_\delta$, $\langle c_\beta : \beta < \lambda^+ \rangle \in M_\delta$, $s^\delta \in M_\delta$.

Fact.

$$\forall \nu \in T_{\lambda^+} \quad \{\delta : \nu \restriction \delta \in M_\delta\} \textit{ is stationary.}$$

Proof. Let $X = \text{range}(\nu)$. $C = \{\delta : \text{range}(\nu \restriction \delta) = \text{range}(\nu) \cap \delta\}$ is closed unbounded. If $X_\delta = X \cap \delta$ and $\delta \in C$, then $\text{range}(\nu \restriction \delta) \in M_\delta$ and hence $\nu \restriction \delta$ ($=$ the increasing enumeration of its range) is in M_δ.

Definition. Let $G : \lambda^+ \to \lambda^+$ be an increasing function such that $\beta < \delta \to G(\delta) \notin M_\beta$.

Definition. $\vec{f} = \langle f_i : i \leq \lambda \rangle$, ($f_i \in T_{\lambda^+}$), is called a "candidate for convergence" iff

(i) All f_i are distinct, for $i < \lambda$.

(ii) $f_\lambda(0) = G(\gamma_0)$, where γ_0 is such that all $f_i \restriction \gamma_0$ are distinct.

(iii) $\forall 0 \neq \beta < \lambda^+$, $f_\lambda(\beta)$ codes $\vec{f} \Uparrow \beta$, i.e. $c_\beta(f_\lambda(\beta)) = \vec{f} \Uparrow \beta$, where

$$\vec{f} \Uparrow \beta = \langle f_i \restriction \beta : i \leq \lambda \rangle$$

(iv) $\forall \beta < \lambda^+ \forall i < \lambda \; f_i(\beta) < f_\lambda(\beta)$.

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We will satisfy (D) by demanding

(D'): If $\vec{f}$ is a candidate for convergence, then

$$f_\lambda \in cl(\{f_i : i < \lambda\}).$$

Fact. *For any $\langle f_i : i < \lambda \rangle$ there are 2^{λ^+} many possibilities to choose f_λ such that $\langle f_i : i \leq \lambda \rangle$ becomes a candidate for convergence.* (*Hence* (D') *guarantees* (D).)

Proof. For every $\beta < \lambda$, $\beta \neq 0$, we have λ^+ possible choices for $f_\lambda(\beta)$, since the only requirements are $f_\lambda(\beta) \in c_\beta^{-1}(\vec{f} \Uparrow \beta)$ and $f_\lambda(\beta) > f_\lambda(\gamma)$ for all $\gamma < \beta$, and $f_\lambda(\beta) > f_i(\beta)$ for all $i < \lambda$.

Definition. For $\delta \in S$, $\vec{\eta} \in {}^{\lambda+1}T_\delta$ is called "δ-candidate for convergence," iff

(1) All η_i are distinct, for $i < \lambda$.

(2) $\eta_\lambda(0) = G(\gamma_0)$, where γ_0 is such that all $\eta_i \restriction \gamma_0$ are distinct.

(3) $\forall 0 \neq \beta < \lambda^+$, $\eta_\lambda(\beta)$ codes $\vec{\eta} \Uparrow \beta$, i.e. $c_\beta(\eta_\lambda(\beta)) = \vec{\eta} \Uparrow \beta$, where

$$\vec{\eta} \Uparrow \beta = \langle \eta_i \restriction \beta : i \leq \lambda \rangle.$$

(4) $\forall \beta < \lambda^+ \forall i < \lambda\ \eta_i(\beta) < \eta_\lambda(\beta)$.

(5) $\vec{\eta} \in M_\delta$, or equivalently, $\eta_\lambda \in M_\delta$.

Fact. $\vec{\eta} \in {}^{\lambda+1}T_\delta$ *is a δ-candidate for convergence, iff $\vec{\eta} \in M_\delta$ and for some convergence candidate $\vec{f} \in {}^{\lambda+1}T_{\lambda^+}$, $\vec{\eta} = \vec{f} \Uparrow \delta$.*

Proof. "$\rightarrow$" is clear. To prove the other direction, note that $\eta_\lambda(0) = G(\gamma_0) \in M_\delta$ implies $\gamma_0 < \delta$, so all $(f_i \restriction \delta)$'s are distinct.)

Remark. Conversely, $\vec{f}$ is a candidate for convergence, iff for all δ such that $f_\lambda \restriction \delta \in M_\delta$ we have that $\vec{f} \Uparrow \delta$ is a δ-candidate for convergence. (Note that by construction of the M_δ's, $\{\delta : f_\lambda \restriction \delta \in M_\delta\}$ is stationary.)

Hence if $\vec{\eta}$ is a δ-candidate for convergence, then for all $\alpha \in S$ such that $\eta_\lambda \restriction \alpha \in M_\alpha$, $\vec{\eta} \Uparrow \alpha$ is an α-candidate for convergence (because then $\vec{\eta} = \vec{f} \Uparrow \delta$ for some convergence candidate $\vec{f}$, and $\vec{\eta} \Uparrow \alpha = \vec{f} \Uparrow \alpha$).

For any $\nu \in T_\delta$ there is at most one δ-candidate for convergence (call it $\vec{\eta}^\nu$) such that $\eta_\lambda = \nu$, since all the η_i's can be computed from ν.

Whenever $\vec{\eta}^\nu$ is defined, let

$$B_\nu = \{(\alpha, \beta) \in S \times S : \nu \restriction \alpha \in M_\alpha, \beta \leq \alpha < \delta\}.$$

Eventually, the sets $\mathscr{U}_\beta$ and $\mathscr{V}_\beta$ ($\subseteq T_{\lambda^+}$) will be defined by

$$f \in \mathscr{U}_\beta \leftrightarrow \exists \delta : f \restriction \delta \in \mathscr{U}_\beta^\delta,$$

$$f \in \mathscr{V}_\beta \leftrightarrow \exists \delta : f \restriction \delta \in \mathscr{V}_\beta^\delta,$$

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where the sets $\mathscr{U}_\beta^\delta$ and $\mathscr{V}_\beta^\delta$ ($\beta, \delta \in S$, $\beta \le \delta$) will be constructed by induction on δ satisfying the following conditions (for all $\delta, \alpha, \beta \in S$):

(a) $T_\delta \cap M_\delta = \mathscr{U}_\beta^\delta \cup \mathscr{V}_\beta^\delta$, for $\beta \le \delta$.

(b) If $\eta \in T_\delta$, $\delta \ge \alpha \ge \beta$, then:

$$\eta \restriction \alpha \in \mathscr{U}_\beta^\alpha \rightarrow \eta \notin \mathscr{V}_\beta^\delta,$$

$$\eta \restriction \alpha \in \mathscr{V}_\beta^\alpha \rightarrow \eta \notin \mathscr{U}_\beta^\delta.$$

In particular, $\mathscr{U}_\beta^\delta \cap \mathscr{V}_\beta^\delta = \varnothing$.

(c) $\mathscr{U}_\delta^\delta$ separates $s_{\sigma_\delta}^\delta$ from $\{s_i^\delta : i < \sigma_\delta\}$.

(d) If $\vec{\eta}$ is a δ-candidate for convergence, $\nu = \eta_\lambda$, then for all finite $F \subseteq B_\nu$

$$\left|\left\{ i < \lambda : \bigwedge_{(\alpha, \beta) \in F} (\nu \restriction \alpha \in \mathscr{U}_\beta^\alpha \leftrightarrow \eta_i \restriction \alpha \in \mathscr{U}_\beta^\alpha \right\}\right| = \lambda.$$

Fact. (a) → (A), (b) → (B), (c) → (C), and (d) → (D').

Proof. "(b) → (B)" is clear.

To show "(a) → (A)," consider any $f \in T_{\lambda^+}$. By construction of the M_δ's there is a δ such that $\nu = f \restriction \delta \in M_\delta$. Hence $\nu \in \mathscr{U}_\beta^\delta$ or $\nu \in \mathscr{V}_\beta^\delta$, so $f \in \mathscr{U}_\beta$ or $f \in \mathscr{V}_\beta$.

A similar argument proves "(c) → (C)."

To prove "(d) → (D')," assume that $f_\lambda \notin cl(\{f_i : i < \lambda\})$ for some candidate for convergence $\vec{f} = \langle f_i : i \le \lambda \rangle$. Then we can find finitely many subbasis sets $\mathscr{U}_{\beta_1}, \dots, \mathscr{U}_{\beta_n}$ such that

$$\forall i < \lambda \, \exists j \le n : \eta_\lambda \in \mathscr{U}_{\beta_j} \leftrightarrow \eta_i \notin \mathscr{U}_{\beta_j}.$$

Find $\alpha, \delta \in S$, $\alpha < \delta$ such that $f_\lambda \restriction \alpha \in M_\alpha$, and $\nu = f_\lambda \restriction \delta \in M_\delta$, and $\beta_1, \dots, \beta_n < \alpha$. Note that also $f_i \restriction \alpha \in M_\alpha$ for all $i < \lambda$ and that for all $\beta \le \alpha$ $f_i \restriction \alpha \in \mathscr{U}_\beta^\alpha \leftrightarrow f_i \in \mathscr{U}_\beta$. Let $F \subseteq B_\nu$ be defined by

$$F = \{\alpha\} \times \{\beta_1, \dots, \beta_n\}.$$

For this F the set considered in (d) is empty, a contradiction.

4. Inductive construction of the sequence $(\mathscr{U}_\beta^\delta, \mathscr{V}_\beta^\delta)$

Assume that we have already constructed $\mathscr{U}_\beta^\alpha, \mathscr{V}_\beta^\alpha$ for all $\beta \le \alpha, \alpha < \delta$. Now we have, for each $\beta \le \delta$ and each $\nu \in M_\delta \cap T_\delta$, to decide whether $\nu \in \mathscr{U}_\beta^\delta$ or $\nu \in \mathscr{V}_\beta^\delta$. We will first deal with the cases $\beta < \delta$.

Let $\prec$ be the transitive closure of the following relation $\prec_0$: $\eta \prec_0 \nu$ iff for some δ-candidate for convergence $\vec{\eta}$ and for some $i < \lambda$ we have $\eta = \eta_i$ and $\nu = \eta_\lambda$. The relation $\prec$ is well founded (by (4)). By $\prec$-induction on

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$\{\nu : \nu \in T_\delta \cap M_\delta\}$ we will decide whether $\nu \in \mathscr{U}^\delta_\beta$ or $\nu \in \mathscr{V}^\delta_\beta$ (so (a) will be satisfied):

Case 1. There is a δ-candidate for convergence $\vec{\eta} = \vec{\eta}^\nu \in M_\delta$ such that $\eta_\lambda = \nu$. Note that we already know (for all $i < \lambda$, all $\beta < \delta$), whether $\eta_i \in \mathscr{U}^\delta_\beta$ or $\eta_i \in \mathscr{V}^\delta_\beta$. Consider the filter base

$$\{A_F : F \subseteq B_\nu \text{ finite}\},$$

where

$$A_F = \left\{ i < \lambda : \bigwedge_{(\alpha, \beta) \in F} (\nu \restriction \alpha \in \mathscr{U}^\alpha_\beta \leftrightarrow \eta_i \restriction \alpha \in \mathscr{U}^\alpha_\beta) \right\}.$$

(Notice that $A_{F_1} \cap A_{F_2} \supseteq A_{F_1 \cup F_2}$.) It generates a uniform filter (by induction hypothesis (d)), so it can be extended to a uniform ultrafilter $\mathscr{F}$. Now let, for each $\beta < \delta$,

$$\nu \in \mathscr{U}^\delta_\beta \leftrightarrow \{i < \lambda : \eta_i \in \mathscr{U}^\delta_\beta\} \in \mathscr{F},$$
$$\nu \in \mathscr{V}^\delta_\beta \leftrightarrow \{i < \lambda : \eta_i \in \mathscr{V}^\delta_\beta\} \in \mathscr{F}.$$

(Note that $\forall i : \eta_i \in \mathscr{V}^\delta_\beta \leftrightarrow \eta_i \notin \mathscr{U}^\delta_\beta$.)

Case 2. Not case 1, i.e. there is no such $\vec{\eta}^\nu$. For each $\beta < \delta$, there are two possibilities:

Case 2.1. For some α, $(\alpha, \beta) \in B_\nu$, i.e. $\alpha \geq \beta$, $\nu \restriction \alpha \in M_\alpha$. In this case, we already know whether $\nu \restriction \alpha \in \mathscr{U}^\alpha_\beta$ or $\nu \restriction \alpha \in \mathscr{V}^\alpha_\beta$ is true, so decide $\nu \in \mathscr{U}^\delta_\beta$ or $\nu \in \mathscr{V}^\delta_\beta$ accordingly. Note that if there are $\alpha, \alpha' < \beta$ such that $\nu \restriction \alpha \in M_\alpha$, $\nu \restriction \alpha' \in M_{\alpha'}$, then $\mathscr{U}^\alpha_\beta$ and $\mathscr{U}^{\alpha'}_\beta$ "agree."

Case 2.2. Otherwise (i.e. the previous promises do not impose any restrictions), let $\nu \in \mathscr{U}^\delta_\beta$.

Remark. The "not Case 1" is not really necessary in Case 2.1: Assume $\nu = \eta_\lambda$, $\vec{\eta}$ a δ-candidate for convergence, $\nu \restriction \alpha \in M_\alpha$, $\beta \leq \alpha < \delta$. W.l.o.g. $\nu \restriction \alpha \in \mathscr{U}^\alpha_\beta$. Then Case 2 would decide "$\nu \in \mathscr{U}^\delta_\beta$". But $\vec{\eta} \restriction\restriction \alpha$ is an α-candidate for convergence. Since

$$\begin{aligned} A_{\{(\alpha, \beta)\}} &= \{i < \lambda : \nu \restriction \alpha \in \mathscr{U}^\alpha_\beta \leftrightarrow \eta_i \restriction \alpha \in \mathscr{U}^\alpha_\beta\} \\ &= \{i < \lambda : \eta_i \restriction \alpha \in \mathscr{U}^\alpha_\beta\} \in \mathscr{F} \end{aligned}$$

also Case 1 decides "$\nu \in \mathscr{U}^\delta_\beta$."

The construction in Case 1 ensures that (d) is satisfied even for $F \subseteq B_\nu \cup (\{\delta\} \times (\delta \cap S))$: Let $F' = F \cap B_\nu$, and $F \cap (\{\delta\} \times (\delta \cap S)) = \{\delta\} \times \{\beta_1, \dots, \beta_n\}$. To see that $|A_F| = \lambda$, note that

$$A_F = A_{F'} \cap \bigcap_{j \leq n} \{i < \lambda : \nu \in \mathscr{U}^\delta_{\beta_i} \leftrightarrow \eta_i \in \mathscr{U}^\delta_{\beta_i}\}$$

is in $\mathscr{F}$.

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The construction in Case 2 (together with the remark about Case 2.1) ensures that (b) is satisfied, and Case 2.2 handles (a).

Finally, we have to decide what $\mathscr{U}_\delta^\delta$ and $\mathscr{V}_\delta^\delta$ will be. We have to satisfy the following conditions: For each δ-candidate for convergence $\vec{\eta} \in M_\delta$, if $\nu = \eta_\lambda$, then for each finite set $F \subseteq B_\nu \cup (\{\delta\} \times (\delta \cap S))$

$$C(F, \vec{\eta}): \quad \left| \left\{ i < \lambda : \bigwedge_{(\alpha,\beta)\in F} (\nu \restriction \alpha \in \mathscr{U}_\beta^\alpha \leftrightarrow \eta_i \restriction \alpha \in \mathscr{U}_\beta^\alpha) \wedge (\eta_i \in \mathscr{U}_\delta^\delta \leftrightarrow \nu \in \mathscr{U}_\delta^\delta) \right\} \right| = \lambda.$$

This will imply that (d) is still true for δ', the successor of δ in S.

Also, for each $\nu \in T_\delta \cap M_\delta$, we have to ensure

$$C(\nu): \qquad \nu \in \mathscr{U}_\delta^\delta \cup \mathscr{V}_\delta^\delta.$$

Enumerate these conditions as $\langle C_i : i < \lambda \rangle$, such that each $C(F, \vec{\eta})$ occurs λ many times, but only after $C(\eta_\lambda)$.

We will construct the sets $\mathscr{U} = \mathscr{U}_\delta^\delta$ and $\mathscr{V} = \mathscr{V}_\delta^\delta$ in λ stages, such that in each stage i we have committed less than λ many elements of $T_\delta \cap M_\delta$. These commitments will be given by sets $\mathscr{U}(i)$, $\mathscr{V}(i)$, for $i < \lambda$. Let

$$\mathscr{U}(0) = \{s_{\sigma_\delta}^\delta\}, \qquad \mathscr{V}(0) = \{s_i^\delta : i < \sigma_\delta\}.$$

Abbreviate $\bigcup_{j<k} \mathscr{U}(j)$ to $\mathscr{U}(<k)$, and similarly for $\mathscr{V}$.

Given the sets $\mathscr{U}(j)$, $\mathscr{V}(j)$, for $j < k$, consider the condition C_k:

Case 1: $C_k = C(\nu)$. If $\nu \in \mathscr{U}(<k) \cup \mathscr{V}(<k)$, then let $\mathscr{U}(k) = \mathscr{U}(<k)$, $\mathscr{V}(k) = \mathscr{V}(<k)$. Otherwise, let $\mathscr{U}(k) = \mathscr{U}(<k) \cup \{\nu\}$, $\mathscr{V}(k) = \mathscr{V}(<k)$.

Case 2: $C_k = C(F, \vec{\eta})$, Let $\nu = \eta_\lambda$. $C(\nu)$ is already satisfied, w.l.o.g. $\nu \in \mathscr{U}(<k)$. (The case $\nu \in \mathscr{V}(<k)$ is handled similarly.) Find an $i < \lambda$ such that

$$(*1) \qquad \eta_i \notin \mathscr{U}(<k) \cup \mathscr{V}(<k)$$

and

$$(*2) \qquad \bigwedge_{(\alpha,\beta)\in F} (\eta_i \restriction \alpha \in \mathscr{U}_\beta^\alpha \leftrightarrow \eta_\lambda \restriction \alpha \in \mathscr{U}_\beta^\alpha)$$

and let $\mathscr{U}(k) = \mathscr{U}(<k) \cup \{\eta_i\}$, $\mathscr{V}(k) = \mathscr{V}(<k)$.

Since the sets $\mathscr{V}(k)$ and $\mathscr{V}(<k)$ differ by at most one element for $k > 0$, we get by induction $|\mathscr{V}(<k)| \leq |k| + |\sigma_\delta| < \lambda$. Similarly, $|\mathscr{U}(<k)| \leq |k| < \lambda$. So $(*1)$ and $(*2)$ can always be satisfied by some i, since the set of i's satisfying $(*2)$ has cardinality λ.

Clearly, all $C(\nu)$'s and $C(\vec{\eta}, F)$'s will be satisfied after λ many steps.

This completes the construction of $\langle \mathscr{U}_\beta^\delta, \mathscr{V}_\beta^\delta : \beta \leq \delta, \beta \in S, \delta \in S \rangle$, and hence of the subbasis of X.

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