STABILITY IN INTERPOLATION OF FAMILIES OF BANACH SPACES

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Abstract. Let \( D \) be a simply connected domain in the complex plane whose boundary \( \Gamma \) is a rectifiable simple closed curve. Let \( \{A(y)/y \in \Gamma\} \) and \( \{B(y)/y \in \Gamma\} \) be interpolation families of Banach spaces. Let \( T \) be a linear operator mapping \( A(y) \) continuously into \( B(y) \). For \( z \in D \) let \( T_z \) be the restriction of \( T \) to the interpolation space \( A_z \). Then \( \{z \in D/\text{cod}(T_z) = d < \infty \text{ and dim Ker}(T_z) = 0\} \) and \( \{z \in D/\text{dim Ker}(T_z) = d < \infty \text{ and } T_z \text{ is onto } B_z\} \) are open sets.

1. Introduction

We begin with a brief review of the theory of interpolation for analytic families of Banach spaces. Throughout the paper, \( D \) will denote a simply connected domain in the complex plane whose boundary \( \Gamma \) is a rectifiable simple closed curve.

Definition 1.1. An interpolation family of Banach spaces on \( \Gamma \) is a collection \( \{A(y)|y \in \Gamma\} \), of Banach spaces which satisfies:

1. Every \( A(y) \) is continuously embedded in a common Banach space \( \mathcal{H} \) which is called a containing space for \( \{A(y)|y \in \Gamma\} \).
2. For every \( a \in \bigcap_{y \in \Gamma} A(y) \), \( \|a\|_{A(y)} \) is a measurable function on \( \Gamma \) with respect to \( dP_z \), where \( z \in D \) and \( dP_z \) is a harmonic measure on \( \Gamma \) with respect to \( z \).
3. Let \( \mathcal{A} = \{a \in \bigcap_{y \in \Gamma} A(y) | \int_{\Gamma} \log^+ \|a\|_{A(y)} dP_z(y) < \infty \} \), where \( \log^+ x = \max(\log x, 0) \). \( \mathcal{A} \) is a linear space which is called the log-intersection space for \( \{A(y)|y \in \Gamma\} \). We assume that \( \mathcal{A} \) is dense in each \( A(y) \) and that there exists a measurable function \( \psi \) on \( \Gamma \) which satisfies \( \int_{\Gamma} \log^+ \psi(y) dP_z < \infty \) such that \( \|a\|_\psi \leq \psi(y)\|a\|_{A(y)} \) for all \( a \in \mathcal{A} \) and \( y \in \Gamma \).

We shall denote the spaces \( A(y) \) by \( A_y \) and their norms by \( \| \cdot \|_y \).
We denote by $N^+(D)$ the class of all functions $f(z)$ analytic on $D$ such that $f(\phi(w))$ belongs to the class $N^+ = N^+(A)$, where $A = \{w \mid |w| < 1\}$ and $\phi$ is a conformal map from $D$ to $\Delta$, see [7]. Thus $N^+(D)$ is closed under pointwise addition and multiplication and each $f(z) \in N^+(D)$ possesses a.e. non-tangential limits on $\Gamma$. If these non-tangential limits are essentially bounded on $\Gamma$, then $f \in H^\infty(D)$, the space of bounded analytic functions on $D$. A function $f \in N^+(D)$ is termed an outer function in $N^+(D)$, if $f(\phi(w))$ is an outer function in $N^+(\Delta)$.

Let $\mathcal{S}(A(\cdot), \Gamma) = \mathcal{S}(\mathcal{A}) = \mathcal{S}$ be the space of all functions of the form

$g(z) = \sum_{j=1}^{n} \varphi_j(z) a_j$, where $a_j \in \mathcal{A}$ and $\varphi_j \in N^+(D)$, and such that

$$\|g(\cdot)\|_{\mathcal{S}} = \text{ess sup}_{\gamma \in \Gamma} \|g(\gamma)\| < \infty.$$ 

Note that, in general, $(\mathcal{S}, \| \cdot \|_{\mathcal{S}})$ is not complete.

**Definition 1.2.** For $z \in D$ and $a \in \mathcal{A}$, define

$$\|a\|_{A(z)} = \inf \{\|g\|_{\mathcal{S}} \mid g \in \mathcal{S}, g(z) = a\}.$$ 

The interpolation space $\{A_{\gamma}\}{z} = A\{z\}$ is defined to be the completion in $\mathcal{U}$ of $(\mathcal{A}, \| \cdot \|_{A(z)})$.

In most applications the family of dual spaces $\{A^*_\gamma\}$ is itself an interpolation family and $(A\{z\})^* = \{A^*_\gamma\}{z}$. In this paper we shall assume that the above duality result holds.

We consider in this paper linear operators $T$ mapping $\mathcal{A}$ into $\cap_{\gamma \in \Gamma} B(\gamma)$, with $\|Ta\|_{B(\gamma)} \leq M(\gamma) \|a\|_{A(\gamma)}$ for all $a \in \mathcal{A}$ and $\gamma \in \Gamma$, where $\{A(\gamma)\}_{\gamma \in \Gamma}$ and $\{B(\gamma)\}_{\gamma \in \Gamma}$ are interpolation families on $\Gamma$ and

$$\int_{\Gamma} |\log M(\gamma)| dP_\gamma(\gamma) < \infty.$$ 

We will denote the restriction of $T$ to $A\{z\}$ by $T_z$. If $\ker T_z = \{0\}$, denote by $T_z^{-1}$ the inverse of $T_z$, i.e., $T_z^{-1}$ is defined on $T_z A\{z\}$. If $T_z A\{z\}$ has finite codimension, by the open mapping theorem, $T_z A\{z\}$ is closed and $T_z^{-1}$ is a bounded linear operator.

The question of the stability of Fredholm property when one changes the parameters which determine the interpolation space has been considered by several authors [1, 4, 9, 12] in the context of interpolation of two spaces and by Vignati and Vignati [11] in the case of families of Banach spaces. We consider families of Banach spaces and prove that, if $\ker T_s = \{0\}$ and $\text{codim} T_s = d < \infty$, then there exists a ball $B$ centered at $s$ such that $z \in B$ implies that $\ker T_z = \{0\}$ and $\text{codim} T_z = d$. This generalizes the result in [11], where the theorem is proved in the case $d = 0$.

**Definition 1.3.** Suppose $U(A)$ is a subset of $\mathcal{S}(\mathcal{A})$. Let

$$U_s = \{a \in A\{s\} \mid \exists f \in U(A) \text{ such that } f(s) = a\}.$$
Define
\[ \|a\|_{U_s(A)} = \inf \{ \|f\|_s \mid f(s) = a, \ f \in U(A) \} . \]
Clearly \( \|a\|_{A(s)} \leq \|a\|_{U_s(A)} \) for all \( a \in U_s \).

Let \( U_s(A) \) be a completion in \( A(s) \) of \( \{U_s, \|\cdot\|_U\} \). Clearly \( U_s(A) \subset A(s) \). If also \( \|a\|_{U_s(A)} < k\|a\|_{A(s)} \), for some \( k \) and all \( a \in U_s \) with \( a \neq 0 \), we call \( U(A) \) an \( s \)-Calderón subset (with constant \( k \)). If \( U(A) \) is an \( s \)-Calderón subset and a linear subspace of \( \mathcal{G}(s) \), we will say that it is an \( s \)-Calderón subspace.

Clearly \( U(A) \) is an \( s \)-Calderón subspace for some \( k \) if and only if \( U_s(A) \) is a closed subspace of \( A(s) \).

An important tool in handling finite dimensional subspaces is Auerbach’s lemma:

**Lemma 1.4.** Let \( M \) be a Banach space with dimension \( d < \infty \). Then there exist \( \{e_1, \ldots, e_d\} \subset M \) and \( \{f_1, \ldots, f_d\} \subset M^* \) such that

1. \( \|e_i\|_M = 1 \) and \( \|f_i\|_{M^*} = 1, \ i = 1, \ldots, d \).
2. \( f_i(e_j) = \delta_{ij} \).

For a proof see [8].

Given a finite set of vectors \( \{e_1, \ldots, e_d\} \) in a Banach space \( A \), we define
\[ b(e_1, \ldots, e_d) = \min \left\{ \left\| \sum_{i=1}^d c_i e_i \right\|_A \mid \max \{ |c_i| \} = 1 \right\} . \]
Clearly \( \{e_1, \ldots, e_d\} \) are independent if and only if \( b(e_1, \ldots, e_d) > 0 \). Given \( \sum_{i=1}^d c_i e_i \), let \( m = \max \{ |c_i| \} \). Then \( \| \sum_{i=1}^d c_i e_i \|_A = m \| \sum_{i=1}^d c_i e_i / m \|_A \geq m b(e_1, \ldots, e_d) \), so that \( |c_i| \leq \| \sum_{i=1}^d c_i e_i \|_A / b(e_1, \ldots, e_d) \).

**Lemma 1.5.** Let \( M \) be a Banach space with dimension \( d < \infty \). Then \( \{e_1, \ldots, e_d\} \) can be found so that \( \|e_i\|_M = 1 \), and \( b(e_1, \ldots, e_d) = 1 \).

For a proof, see [4].

**Lemma 1.6.** Let \( E \) be a closed subspace of \( A(s) \) with finite codimension \( d \). Then there exists \( \{v_i, i = 1, \ldots, d\} \subset \mathcal{G}(s) \), such that \( A(s) = E \oplus \text{span}\{v_i(s)\} \).

**Proof.** Since \( E \) has finite codimension \( d \), \( A(s) = E \oplus M \) with \( \dim M = d \). Assume that \( \{e_i, 1 \leq i \leq d\} \) forms a basis for \( M \) with \( \|e_i\|_{A(s)} = 1 \) and \( b(e_1, \ldots, e_d) = 1 \). Let
\[ \rho = \rho(e_1, \ldots, e_d; E) = \inf \left\{ \left\| \sum_{i=1}^d c_i e_i - x \right\|_{A(s)} \mid \max \{ |c_i| \} = 1, \ x \in E \right\} . \]
Clearly \( \rho \leq 1 \). If \( \rho > 0 \) then \( E \cap M = \{0\} \) and \( \{e_i\} \) are independent. Since \( E \) is a closed subspace and \( M \) has finite dimension, then \( E \cap M = \{0\} \) implies \( \rho > 0 \).
Since \( \{v(s), v \in \mathcal{S}(A)\} \) is dense in \( A\{s\} \), we can find \( \{v_i, i = 1, \ldots, d\} \subset \mathcal{S}(A) \) with \( \|v_i(s) - e_i\|_{A\{s\}} < \rho/2d \). Now suppose \( \max\{|c_i|\} = 1 \) and let \( x \in E \); we have
\[
\left\| x - \sum_{i=1}^{d} c_i v_i(s) \right\|_{A\{s\}} \geq \left\| x - \sum_{i=1}^{d} c_i e_i \right\|_{A\{s\}} - \left\| \sum_{i=1}^{d} c_i (v_i(s) - e_i) \right\|_{A\{s\}} > \rho/2.
\]
Taking infimum we get \( \rho(v_1(s), \ldots, v_d(s); E) \geq \rho/2 > 0 \). Hence \( \{v_i(s)\} \) are linearly independent and \( A\{s\} = E \oplus \text{span}\{v_i(s)\} \). \( \square \)

The function \( \rho \) introduced in the last proof is, of course, a generalization of the function \( b(e_1, \ldots, e_d) \). Another application for the function \( \rho \) is the following lemma.

**Lemma 1.7.** Let \( E \) be a closed subspace of a Banach space \( B \), and let \( M \) be a finite dimensional subspace of \( B \). Then \( E + M \) is closed.

**Proof.** We may assume \( E \cap M = \{0\} \). Let \( \{e_1, \ldots, e_d\} \) be a basis for \( M \). Then \( \rho = \rho(e_1, \ldots, e_d; E) > 0 \). Let \( x_n \in E + M \), and \( x_n \to x_0 \in B \).

\[
x_n = y_n + \sum_{i=1}^{d} c^n_i e_i.
\]

Let \( M_n = \max\{|c^n_i|; i = 1, \ldots, d\} \).

Given \( \varepsilon > 0 \), let \( N \) be so that, for all \( n \geq N \), \( \|x_n\| \leq (1 + \varepsilon)\|x_0\| \). Then \( \rho \leq \|\sum_{i=1}^{d} c^n_i e_i + y_n\|/M_n \leq (1 + \varepsilon)\|x_0\|/M_n \), and \( M_n \leq (1 + \varepsilon)\|x_0\|/\rho \). Thus, \( \limsup M_n \leq \|x_0\|/\rho \) and \( \{\sum_{i=1}^{d} c^n_i e_i\} \) is compact. Denote \( \sum_{i=1}^{d} c^n_i e_i = z_n \). Let \( z_{n_k} \to z \in M \); then \( y_{n_k} = x_{n_k} - z_{n_k} \to x_0 - z \) and, since \( E \) is closed, \( x_0 - z \in E \). \( \square \)

We will also need the following well-known result due to Szegö [10].

**Lemma 1.8.** Let \( f(\gamma) \) be a positive \( dP_z \) measurable function on \( \Gamma \) such that
\[
\int_{\Gamma} |\log f(\gamma)|dP_z(\gamma) < \infty
\]
for some (and thus every) \( z \in D \). Then there exists a non-vanishing outer function \( G(z) \) in \( N^+(D) \) whose a.e. non-tangential limits \( G(\gamma) = \lim_{z \to \gamma} G(z) \) satisfy \( |G(\gamma)|f(\gamma) = 1 \) for a.e. \( \gamma \in \Gamma \).

2. Stability

An invariant form of Schwarz's lemma states that if \( f(z) \) is analytic and \( |f(z)| < 1 \) on \( \Delta \), then for any \( z, \zeta \in \Delta \),
\[
\left| \frac{f(z) - f(\zeta)}{1 - f(z)f(\zeta)} \right| \leq \frac{|z - \zeta|}{|1 - \overline{\zeta} z|}.
\]
This implies that if \( z_0, \zeta_0, z \in \Delta \) and if we denote
\[
m_{z_0}(z) = \frac{z - z_0}{1 - \overline{z_0} z}e^{i\alpha},
\]
then
\[
\left| \frac{m_{z_0}(z) - m_{z_0}(\zeta)}{1 - m_{z_0}(z)m_{z_0}(\zeta)} \right| = \left| \frac{z - \zeta}{1 - \overline{z}\zeta} \right| .
\]
This, in turn, implies that if \( z, \zeta \in D \) and \( d \) is a conformal map of \( D \) onto \( \Delta \), then
\[
q(z, \zeta) = \left| \frac{d(z) - d(\zeta)}{1 - d(z)d(\zeta)} \right|
\]
does not depend on \( d \).

The invariant form of Schwarz's lemma also implies the following lemma.

**Lemma 2.1.** Let \( \phi \) be analytic on \( D \), \(|\phi(z)| \leq M \). If \( s, z \in D \), then
\[
|\phi(z)| \geq M \frac{\phi(s) - M q(z, s)}{M - |\phi(s)| q(z, s)} .
\]
For a proof see [9]. Of course, since \( q(z, s) < 1 \), the inequality above implies \(|\phi(z)| \geq |\phi(s)| - q(z, s)M \).

**Lemma 2.2.** If \( f \in \mathcal{F}(A) \) and \( z, s \in D \), then
\[
(2.1) \quad \|f(z)\|_{A(z)} \geq \|f(s)\|_{A(s)} - q(z, s)\|f\|_{\mathcal{F}} .
\]

**Proof.** Let us fix \( s \in D \). Since \( f(s) \in A\{s\} \), it is possible to find \( \psi \in A^*\{s\} \) such that \(|\psi|_{A^*\{s\}} = 1 \) and \( \psi(f(s)) = \|f(s)\|_{A(s)} \).

Given \( \epsilon > 0 \), we can find \( G \in \mathcal{F}(A^*) \) such that \( \|G(s) - \psi|^{|A^*\{s\}} < \epsilon \) and \( \|G\|_{\mathcal{F}(A^*)} < 1 + \epsilon \). Thus \( \phi(z) = G(z)(f(z)) \) is analytic in \( D \), \( \phi \in H^\infty(D) \), and
\[
|\phi(z)| \leq (1 + \epsilon)\|f(z)\|_{A(z)} \leq (1 + \epsilon)\|f\|_{\mathcal{F}} .
\]
By Lemma 2.1, we have
\[
\|f(z)\|_{A(z)} \geq \frac{|\phi(z)|}{1 + \epsilon} \geq \frac{|\phi(s)|}{1 + \epsilon} - q(z, s)\|f\|_{\mathcal{F}} 
\]
\[
\geq \frac{1 - \epsilon}{1 + \epsilon} \|f(s)\|_{A(s)} - q(z, s)\|f\|_{\mathcal{F}} .
\]
Letting \( \epsilon \to 0 \), we get the theorem. \( \square \)

**Theorem 2.3.** Let \( \{v_1, \ldots, v_d\} \subset \mathcal{F}(A) \), and assume that \( \{v_i(s)\} \) are independent. Denote \( b_z = b(v_1(z), \ldots, v_d(z)) \). Then there exists \( \delta > 0 \) so that, for all \(|z - s| < \delta \), \( b_z > b_s/2 \). Consequently, if we denote by \( U \) the span of \( \{v_i\} \) in \( \mathcal{F}(A) \) then, for these \( z \), \( \dim(U_{\{z\}}) = d \).

**Proof.** Without loss of generality, we can assume that \( \|v_i\|_{\mathcal{F}} = 1 \). As we consider \( v_i(z) \) in \( A\{z\} \), \( b_z \) will be controlled by inequality (2.1), which will prove the theorem:
\[
\left\| \sum_{i=1}^d c_i v_i(z) \right\|_{A\{z\}} \geq \left\| \sum_{i=1}^d c_i v_i(s) \right\|_{A\{s\}} - q(z, s) \left\| \sum_{i=1}^d c_i v_i \right\|_{\mathcal{F}} .
\]
Since \( \{v_i(s)\} \) are independent, we have \( b_s > 0 \). Suppose \( \max\{|c_i|\} = 1 \); then

\[
 b_s \leq \left\| \sum_{i=1}^{d} c_i v_i(s) \right\|_{A(s)} \quad \text{and} \quad \left\| \sum_{i=1}^{d} c_i v_i \right\| \leq \sum_{i=1}^{d} |c_i| \cdot \|v_i\|_A \leq d .
\]

Hence,

\[
 \left\| \sum_{i=1}^{d} c_i v_i(z) \right\|_{A(z)} \geq b_s - dq(z,s).
\]

We can choose \( \delta \) small enough such that \( |z - s| < \delta \) implies \( b_s - dq(z,s) > b_z/2 \).

It follows that \( b_z > b_s/2 \). Therefore, if \( |z - s| < \delta \), \( \{v_i(z)\} \) are independent and \( \dim(U_{(z)}) = d \). \( \Box \)

**Theorem 2.4.** Let \( \{v_1, \ldots, v_d\} \subset \mathscr{F}(\mathcal{A}) \), and assume that \( \{v_i(s)\} \) are independent. Denote by \( U \) the span of \( \{v_i\} \) in \( \mathscr{F}(\mathcal{A}) \). Then \( \exists \delta > 0 \), such that \( |z - s| < \delta \) implies \( U \) is a \( z \)-Calderón space with a uniform constant \( k \).

**Proof.** Without loss of generality, we can assume that \( \|v_i(s)\|_{A(s)} = 1 \). Let \( m = \max\{\|v_i\|_A\} \) and \( b_z = b(v_1(z), \ldots, v_d(z)) \). Then, as we have seen before, if \( b_z > 0 \), we have

\[
|c_i| \leq \frac{\|\sum_{i=1}^{d} c_i v_i(z)\|_{A(z)}}{b_z}
\]

so that

\[
\left\| \sum_{i=1}^{d} c_i v_i \right\| \leq \frac{md}{b_z} \left\| \sum_{i=1}^{d} c_i v_i(z) \right\|_{A(z)} .
\]

Since \( b_s > 0 \), by Theorem 2.3, \( \exists \delta > 0 \) such that \( |z - s| < \delta \) implies that \( b_z > b_s/2 \) and \( U \) is a \( z \)-Calderón subspace with constant \( 2md/b_s \). \( \Box \)

**Theorem 2.5.** Let \( U(A), V(A) \) be two \( z \)-Calderón subspaces with constant \( k \), for all \( z \) in a ball \( S \) which is contained in \( D \). Then the function

\[
\chi_z = \inf\{d(f(z), U(z)) : f \in V(A) \text{ and } \|f(z)\|_{A(z)} = 1\}
\]

is continuous in \( S \).

**Proof.** Given \( 0 < \varepsilon < 1 \) and \( s \in S \), there are \( v \in V(A), u \in U(A) \) so that

\[
1 = \|v(s)\|_{A(s)} \leq \|v\|_A \leq k \|v(s)\|_{A(s)} = k
\]

\[
\|u(s) - v(s)\|_{A(s)} < \chi_s + \varepsilon/2
\]

with

\[
\|u\|_A \leq k \|u(s)\|_{A(s)} .
\]

Therefore

\[
\|u\|_A \leq k(\|u(s) - v(s)\|_{A(s)} + \|v(s)\|_{A(s)}) < k(\chi_s + 2) .
\]
Clearly we have
\[ \chi_z \leq \frac{\|u(z) - v(z)\|_{A(z)}}{\|v(z)\|_{A(z)}}. \]

By Lemma 2.2, we have
\[ \|v(z)\|_{A(z)} \geq \|v(s)\|_{A(s)} - q(z, s)\|v\|_\mathcal{F} = 1 - q(z, s)\|v\|_\mathcal{F}. \]

Hence we can choose \( \eta > 0 \) small enough such that \( |z - s| < \eta \) implies both \( \|v(z)\|_{z}^{-1} < 1 + \epsilon \) and \( z \in S \). Again, invoking Lemma 2.2, we have
\[ \|u(z) - v(z)\|_{A(z)} \leq \|u(s) - v(s)\|_{A(s)} + q(z, s)\|u - v\|_\mathcal{F}. \]
\[ \lessdot \leq \|u(s) - v(s)\|_{A(s)} + q(z, s)(\|u\|_\mathcal{F} + \|v\|_\mathcal{F}). \]
\[ < (\chi_s + \epsilon/2) + q(z, s)(\chi_s + 3). \]

Therefore, if necessary, choosing a smaller \( \eta > 0 \), we have that, for \( |z - s| < \eta \),
\[ \chi_z \leq \frac{\|u(z) - v(z)\|_{A(z)}}{\|v(z)\|_{A(z)}} \leq (1 + \epsilon)(\chi_s + \epsilon). \]

Similarly \( \chi_s \leq (1 + \epsilon)(\chi_s + \epsilon) \) in \( |z - s| < \eta_1 \), and hence \( \chi_z \) is continuous in \( S \).

Note that when \( V(A) \) is finite dimensional, as it is in the proof below, \( \chi_z \) is equivalent to the function \( \rho \) defined in the proof of Lemma 1.6.

**Theorem 2.6.** Let \( U \) be a \( z \)-Calderón subspace with constant \( k \), for all \( z \) in a ball \( S \) contained in \( D \). Assume that, for an \( s \in S \), \( U_{\{s\}} \) has codimension \( d < \infty \). Then there exists \( \delta > 0 \), so that, for all \( z \) with \( |z - s| < \delta \), we have codim \( U_{\{z\}} = d \).

**Proof.** By Lemma 1.6, there exist \( \{v_i\}_{i = 1, \ldots, d} \subset \mathcal{F}(\mathcal{A}) \) so that \( \{v_i(s)\} \) forms a basis for \( M_s \) and \( A\{s\} = U_{\{s\}} \oplus M_s \). We can assume \( \|v_i\|_\mathcal{F} = 1 \). Denote by \( M_z \) the space spanned by \( \{v_i(z)\} \). Since \( A\{s\} = U_{\{s\}} \oplus M_s \) and \( U_{\{s\}}, M_s \) are closed, there exists a projection \( P \) of \( A\{s\} \) onto \( U_{\{s\}} \) with \( \ker P = M_s \), and \( \|P\| < \infty \).

Since \( U \) is a \( z \)-Calderón subspace, \( U_{\{z\}} \) is a closed subspace of \( A\{z\} \). By Lemma 1.7, since \( M_z \) is finite-dimensional, \( U_{\{z\}} + M_z \) is closed. Hence, if \( A\{z\} \neq U_{\{z\}} + M_z \) then there exists \( e \in \mathcal{F}(\mathcal{A}) \), with \( \|e\|_{A\{z\}} = 1 \) and \( 1 \leq \|e\|_{\mathcal{F}(\mathcal{A})} < 2 \), such that \( \text{dist}(e(z), U_{\{z\}} + M_z) > 2/3 \).

Consider \( e(s) \in A\{s\} \). We have \( e(s) = y_s + z_s \), where \( y_s = Pe(s) \in U_{\{s\}} \), \( z_s = (I - P)e(s) = \sum_{i=1}^d c_i v_i(s) \in M_s \). We have
\[ \|y_s\|_{A\{s\}} \leq \|P\| \|e(s)\|_{A\{s\}} < 2\|P\| \]
\[ \|z_s\|_{A\{s\}} \leq (\|P\| + 1)\|e(s)\|_{A\{s\}} < 2(\|P\| + 1). \]

Since \( U \) is \( s \)-Calderón, there exists \( f \in U(A) \) such that \( \|f(s) - y_s\|_{A\{s\}} \leq \min\{1/3, \|y_s\|_{A\{s\}}\} \) and
\[ \text{(2.2)} \|f\|_\mathcal{F} \leq k\|f(s)\|_{A\{s\}} \leq 2k\|y_s\|_{A\{s\}} < 4k\|P\|. \]
On the other hand we have, denoting \( b = b(v_1(s), \ldots, v_d(s)) \),
\[
|c_i| \leq \frac{\|\sum_{i=1}^d c_i v_i(s)\|_{A(s)}}{b} = \frac{\|z\|_{A(s)}}{b} < 2 \frac{\|P\| + 1}{b}.
\]

It follows that
\[
(2.3) \quad \left| \sum_{i=1}^d c_i v_i \right|_{\mathcal{G}} \leq \sum_{i=1}^d |c_i| \|v_i\|_{\mathcal{G}} \leq 2d \frac{\|P\| + 1}{b}.
\]

Let \( g = f + \sum_{i=1}^d c_i v_i \). Clearly \( \|g(s) - e(s)\|_{A(s)} < 1/3 \) and
\[
\|g - e\|_{\mathcal{G}} \leq \|f\|_{\mathcal{G}} + \left| \sum_{i=1}^d c_i v_i \right|_{\mathcal{G}} + \|e\|_{\mathcal{G}} \leq 4k\|P\| + \frac{2d(\|P\| + 1)}{b} + 2.
\]

Choose \( \delta \) such that \( |z - s| < \delta \) implies
\[
q(z, s) \left[ 4k\|P\| + \frac{2d(\|P\| + 1)}{b} + 2 \right] < 1/3.
\]

Then, by Lemma 2.2,
\[
\|e(z) - g(z)\|_{A(z)} \leq \|e(s) - g(s)\|_{A(s)} + q(z, s)\|e - g\|_{\mathcal{G}} \leq 1/3 + q(z, s) \left[ 4k\|P\| + \frac{2d(\|P\| + 1)}{b} + 2 \right] < 2/3
\]

which contradicts \( \text{dist}(e(z), U, z) + M_z > 2/3 \). Therefore \( U(z) + M_z = A(z) \).

Since \( A(s) = U(s) \oplus M_s \), \( U(s) \) is closed and \( M_s \) has finite dimension, we have \( \chi_s > 0 \). By Theorem 2.5, there exists \( \delta > 0 \) such that \( |z - s| < \delta \) implies \( \chi_z > 0 \), which implies \( U(z) \cap M_z = \{0\} \). Hence the codimension of \( U(z) \) is \( d \). □

**Theorem 2.7.** If \( \ker T_s = \{0\} \) and \( \text{codim}(T_s A(s)) = d < \infty \), then there exists \( \delta > 0 \) so that, for all \( |z - s| < \delta \), \( \ker T_z = \{0\} \) and \( \text{codim} T_z A(z) = d \).

**Proof.** The control over \( \ker T_z \) is achieved by considering the following function:
\[
r(T_z) = \inf\{\|Ta\|_{B(z)} | \|a\|_{A(z)} = 1\}.
\]

Clearly if \( r(T_z) > 0 \), \( \ker T_z = \{0\} \) and the range of \( T_z \) is closed. Conversely, if the range of \( T_z \) is closed and \( \ker T_z = \{0\} \) then, by the open mapping theorem, \( r(T_z) = \|T_z^{-1}\|^{-1} > 0 \). Since \( \text{codim} T_z = d < \infty \), the range of \( T_z \) is closed, and, since \( \ker T_z = \{0\} \), we have \( r(T_z) > 0 \). Theorem 2.3 in [11] says that \( r(T_z) > r(T_s)/2 > 0 \) for all \( z \) which satisfy \( |z - s| < \delta \), some \( \delta > 0 \).

Since \( \|T\|_{A(y) \rightarrow B(y)} \leq M(y) \) with \( \int_{\gamma} -\log M(y) dP_z(y) < \infty \), by Lemma 1.8, there exists an outer function \( G(\cdot) \) such that
\[
(2.4) \quad |G(y)M(y)| = 1 \quad \text{for a.e. } \gamma \in \Gamma.
\]
We claim that $G(\cdot)T\mathcal{F}(\mathcal{A})$ is a $z$-Calderón subspace of $\mathcal{F}(\mathcal{B})$ with constant $c$, for all $z$ such that $|z - s| < \delta$, where
\[
c = 4\|T_{z}^{-1}\| \sup_{\eta} \left\{ \exp \left( \int \log M(\gamma) \, dP_{\eta}(\gamma) \right) \right\} |\eta - s| < \delta.
\]

In fact, let $y \in G(\cdot)T\mathcal{F}(\mathcal{A})_{z}$. Then $y \in T_{z}\mathcal{A}$, and so there exists $x \in \mathcal{A}$ such that $Tx = y$ and $\|x\|_{A(z)} \leq \|T_{z}^{-1}\| \|y\|_{B(z)}$. We choose $g \in \mathcal{F}(\mathcal{A})$ such that $g(z) = x$ and
\[
(2.5) \quad \|g\|_{\mathcal{F}(\mathcal{A})} \leq 2\|x\|_{A(z)} \leq 2\|T_{z}^{-1}\| \|y\|_{B(z)}.
\]
Then $g(\zeta)/G(z) \in \mathcal{F}(\mathcal{A})$ and $H(\zeta) = G(\zeta)T(g(\zeta)/G(z)) \in G(\cdot)T\mathcal{F}(\mathcal{A})$. Clearly $H(z) = y$, and, by (2.4), (2.5),
\[
\|H\|_{\mathcal{F}(\mathcal{B})} \leq \frac{2\|T_{z}^{-1}\| \|y\|_{B(z)}}{|G(z)|} \leq c\|y\|_{B(z)}.
\]
Therefore $G(\cdot)T\mathcal{F}(\mathcal{A})$ is a $z$-Calderón subspace with constant $c$ for all $z$ such that $|z - s| < \delta$.

It is also clear that
\[
(G(\cdot)T\mathcal{F}(A))_{\{z\}} = T_{z}A(z).
\]
Thus, by Theorem 2.6,
\[
\ker T_{z} = \{0\}, \text{ and the theorem is proved. } \Box
\]

Corollary 2.8. If $\dim \ker T_{z} = d < \infty$ and $T_{z}$ is surjective, then there exists $\delta > 0$, so that for all $|z - s| < \delta$, $\dim \ker T_{z} = d < \infty$ and $T_{z}$ is surjective.

Proof. Consider $T^{*} : B^{*}(\gamma) \rightarrow A^{*}(\gamma)$. Clearly $\dim \ker T_{z}^{*} = \dim \ker T_{z} = d < \infty$, $\dim \ker T_{z}^{*} = \codim T_{z} = 0$ and $\|T^{*}\|_{B^{*}(\gamma) \rightarrow A^{*}(\gamma)} = \|T\|_{A(\gamma) \rightarrow B(\gamma)}$. By assumption, $\{A_{\gamma}^{*}\}$, $\{B_{\gamma}^{*}\}$ are interpolation families and $(A(z))^{*} = \{A_{\gamma}^{*}\}_{\{z\}}$, $(B(z))^{*} = \{B_{\gamma}^{*}\}_{\{z\}}$.

By Theorem 2.7, there exists $\delta > 0$ so that $|z - s| < \delta$ implies $\ker T_{z}^{*} = \{0\}$ and $\codim T_{z}^{*} = d$. Therefore, for all these $z$ we have $\dim \ker T_{z} = d$ and $T_{z}$ is surjective. $\Box$

References


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