

## A CHARACTERIZATION OF THE DUAL OF THE CLASSICAL LORENTZ SEQUENCE SPACE $d(w, q)$

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**ABSTRACT.** A new proof is given that regularity of  $w$  implies that the dual of the classical Lorentz sequence space  $d(w, q)$  is the nonclassical  $d(w^{-q'/q}, q')$ , where  $1/q + 1/q' = 1$ . It is also shown that regularity is necessary for this equality to hold.

### I. INTRODUCTION

In this paper we study the topological dual of the classical Lorentz sequence spaces.

If  $0 < q < \infty$ ,  $w = (w_n)_{n=1}^{\infty}$  is a nonincreasing sequence of positive real numbers with  $w_1 = 1$ ,  $\sum_{n=1}^{\infty} w_n = \infty$ , and  $\lim_n w_n = 0$ , the classical Lorentz sequence space  $d(w, q)$  is defined as

$$d(w, q) = \left\{ x = (x_n)_{n=1}^{\infty} : \|x\|_{w, q} = \left( \sum_{n=1}^{\infty} (x_n^*)^q w_n \right)^{1/q} < \infty \right\},$$

where  $(x_n^*)$  is the nonincreasing rearrangement of  $(|x_n|)$ . The sequence  $(w_n)$  is said to be regular if there is a constant  $C$  such that  $\sum_{i=1}^n w_i \leq Cnw_n$  for every positive integer  $n$ .

In [1], Allen showed, using a result of Garling [2], that if  $(w_n)$  is regular and  $1 < q < \infty$ , then the dual of  $d(w, q)$  is  $d(w^{-q'/q}, q')$ , where  $q' = q/(q-1)$  and  $w^{-q'/q} = (w_n^{-q'/q})$ . Here we give a shorter direct proof of this fact as well as a proof that the regularity condition is necessary. Our result is the following:

**Theorem 1.** *Let  $1 < q < \infty$ ,  $w = (w_n)$  be a nonincreasing sequence of positive real numbers with  $w_1 = 1$ ,  $\sum_{n=1}^{\infty} w_n = \infty$ , and  $\lim_n w_n = 0$ . A necessary and sufficient condition for the topological dual of  $d(w, q)$  to be  $d(w^{-q'/q}, q')$  is that  $w$  is regular.*

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II. PROOF OF THEOREM 1

We will use the following lemma:

**Lemma 2.** *Let  $w = (w_n)$  and  $b = (b_n)$  be nonnegative, nonincreasing sequences, and assume that  $w$  is regular. If  $1 < q' < \infty$ ,  $n$  is a positive integer and  $j(i)$  is a permutation of  $1, \dots, 4n - 1$ , then*

$$(2.1) \quad \sum_{i=2n}^{4n-1} (b_i/w_i)^{q'} w_{j(i)} \leq C \sum_{i=n}^{2n-1} b_i^{q'} w_i^{1-q'}$$

where  $C$  is independent of  $n$ .

From now to the end of the paper,  $C$  will be a constant whose value may change from line to line.

From the regularity of  $(w_i)$  and the fact that  $w_i$  is nonincreasing, we have

$$nw_n \leq \sum_{i=1}^{2n} w_i \leq 2n C w_{2n} \leq 2C \sum_{i=n}^{2n-1} w_i$$

and from this it follows that  $w_n \leq C w_{4n}$ . Using the monotonicity of  $b$  and  $w$  and these inequalities, we see that the left side of (2.1) is bounded by

$$\left(\frac{b_{2n}}{w_{4n}}\right)^{q'} \sum_{i=2n}^{4n-1} w_{j(i)} \leq C \left(\frac{b_{2n}}{w_n}\right)^{q'} \sum_{i=1}^{2n} w_i \leq C \sum_{i=n}^{2n-1} \left(\frac{b_{2n}}{w_n}\right)^{q'} w_i$$

By the monotonicity of  $b$  and  $w$ , the last term is bounded by the right side of (2.1); this proves the lemma.

To prove the sufficiency part of Theorem 1, assume that  $w$  is regular. Since Hölder’s inequality implies that  $d(w^{-q'/q}, q') \subset d(w, q)^*$ , it is sufficient to prove that  $d(w, q)^* \subset d(w^{-q'/q}, q')$ . To do this suppose that  $\phi \in d(w, q)^*$  and  $\phi(e_i) = b_i$ , where  $e_i$  denotes the vector  $(0, \dots, 0, 1, 0 \dots)$  with the 1 as the  $i$ th coordinate. We may assume without loss of generality that  $(b_i)$  is a nonincreasing sequence.

Fix  $n$  and let  $d_k = 1$  for  $1 \leq k \leq n$  and  $d_k = 0$  for  $k > n$ . Then with  $a_k = d_k (b_k/w_k)^{1/(q-1)}$ , we have

$$\sum_{i=1}^n b_i^{q'} w_i^{-q'/q} = \sum_{i=1}^n a_i b_i \leq \|\phi\| \|a\| = \|\phi\| \left( \sum_{i=1}^n (d_k (b_k/w_k)^{q'})_i^* w_i \right)^{1/q}$$

Therefore, to prove that  $(b_i)$  belongs to  $d(w^{-q'/q}, q')$  we need to prove only that there exists a constant  $C$  independent of  $n$  such that

$$(2.2) \quad \sum_{i=1}^n (d_k (b_k/w_k)^{q'})_i^* w_i \leq C \sum_{i=1}^n b_i^{q'} w_i^{-q'/q}$$

To prove (2.2), let  $j(i)$  be the permutation of  $1, \dots, n$  such that

$$(2.3) \quad \sum_{i=1}^n (d_k (b_k/w_k)^{q'})_i^* w_i = \sum_{i=1}^n (b_i/w_i)^{q'} w_{j(i)}$$

and let  $j(i) = i$  for  $i > n$ . For  $n = 1$ , inequality (2.2) is trivial; for  $n > 1$  let  $L$  be the greatest integer such that  $2^L \leq n$ . Then the right side of (2.3) is bounded by

$$(b_1/w_1)^{q'} w_{j(1)} + \sum_{k=1}^L \sum_{i=2^k}^{2^{k+1}-1} (b_i/w_i)^{q'} w_{j(i)}.$$

Applying Lemma 2 to the second term shows that it is bounded by the right side of (2.2). This completes the sufficiency proof.

To prove the necessity, let  $\Gamma_n$  be the set of all  $z$  with  $z_i^* = w_i$  for  $1 \leq i \leq 2n$  and  $z_i = 0$  for  $i > 2n$ . Let

$$x^n = \frac{1}{(2n)!} \sum_{z \in \Gamma_n} z = \frac{1}{2n} \left( \sum_{i=1}^{2n} w_i \right) \left( \sum_{i=1}^{2n} e_i \right).$$

Since by hypothesis  $d(w^{-q'/q}, q') = d(w, q)^*$ ,  $d(w^{-q'/q}, q')$  is a locally convex space. From this and the fact that  $x^n$  is a convex combination of elements with norm  $(\sum_{i=1}^{2n} w_i)^{1/q'}$ , we have

$$\frac{1}{2n} \left( \sum_{i=1}^{2n} w_i \right) \left( \sum_{i=1}^{2n} w_i^{-q'/q} \right)^{1/q'} = \|x^n\|_{w^{-q'/q}, q'} \leq C \left( \sum_{i=1}^{2n} w_i \right)^{1/q'}.$$

Since  $w_i$  is nonincreasing,

$$\sum_{i=1}^{2n} w_i^{-q'/q} \geq n w_n^{-q'/q},$$

and we get  $(\sum_{i=1}^{2n} w_i)^{1/q} \leq 2C(nw_n)^{1/q}$ , which implies the regularity.

### REFERENCES

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