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A CHARACTERIZATION OF THE DUAL OF THE CLASSICAL LORENTZ SEQUENCE SPACE $d(w, q)$

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ABSTRACT. A new proof is given that regularity of w implies that the dual of the classical Lorentz sequence space $d(w, q)$ is the nonclassical $d(w^{-q'/q}, q')$, where $1/q + 1/q' = 1$. It is also shown that regularity is necessary for this equality to hold.

I. INTRODUCTION

In this paper we study the topological dual of the classical Lorentz sequence spaces.

If $0 < q < \infty$, $w = (w_n)_{n=1}^{\infty}$ is a nonincreasing sequence of positive real numbers with $w_1 = 1$, $\sum_{n=1}^{\infty} w_n = \infty$, and $\lim_n w_n = 0$, the classical Lorentz sequence space $d(w, q)$ is defined as

$$d(w, q) = \left\{ x = (x_n)_{n=1}^{\infty} : \|x\|_{w,q} = \left(\sum_{n=1}^{\infty} (x_n^*)^q w_n \right)^{1/q} < \infty \right\},$$

where (x_n^*) is the nonincreasing rearrangement of $(|x_n|)$. The sequence (w_n) is said to be regular if there is a constant C such that $\sum_{i=1}^{n} w_i \leqq Cnw_n$ for every positive integer n.

In [1], Allen showed, using a result of Garling [2], that if (w_n) is regular and $1 < q < \infty$, then the dual of $d(w, q)$ is $d(w^{-q'/q}, q')$, where $q' = q/(q-1)$ and $w^{-q'/q} = (w_n^{-q'/q})$. Here we give a shorter direct proof of this fact as well as a proof that the regularity condition is necessary. Our result is the following:

Theorem 1. *Let* $1 < q < \infty$, $w = (w_n)$ *be a nonincreasing sequence of positive real numbers with* $w_1 = 1$, $\sum_{n=1}^{\infty} w_n = \infty$, *and* $\lim_n w_n = 0$. *A necessary and sufficient condition for the topological dual of* $d(w, q)$ *to be* $d(w^{-q'/q}, q')$ *is that* w *is regular.*

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II. Proof of Theorem 1

We will use the following lemma:

Lemma 2. *Let $w = (w_n)$ and $b = (b_n)$ be nonnegative, nonincreasing sequences, and assume that w is regular. If $1 < q' < \infty$, n is a positive integer and $j(i)$ is a permutation of $1, \dots, 4n-1$, then*

$$\sum_{i=2n}^{4n-1} (b_i/w_i)^{q'} w_{j(i)} \leq C \sum_{i=n}^{2n-1} b_i^{q'} w_i^{1-q'}, \tag{2.1}$$

where C is independent of n.

From now to the end of the paper, C will be a constant whose value may change from line to line.

From the regularity of (w_i) and the fact that w_i is nonincreasing, we have

$$nw_n \leq \sum_{i=1}^{2n} w_i \leq 2nCw_{2n} \leq 2C \sum_{i=n}^{2n-1} w_i,$$

and from this it follows that $w_n \leq Cw_{4n}$. Using the monotonicity of b and w and these inequalities, we see that the left side of (2.1) is bounded by

$$\left(\frac{b_{2n}}{w_{4n}}\right)^{q'} \sum_{i=2n}^{4n-1} w_{j(i)} \leq C \left(\frac{b_{2n}}{w_n}\right)^{q'} \sum_{i=1}^{2n} w_i \leq C \sum_{i=n}^{2n-1} \left(\frac{b_{2n}}{w_n}\right)^{q'} w_i.$$

By the monotonicity of b and w, the last term is bounded by the right side of (2.1); this proves the lemma.

To prove the sufficiency part of Theorem 1, assume that w is regular. Since Hölder's inequality implies that $d(w^{-q'/q}, q') \subset d(w, q)^*$, it is sufficient to prove that $d(w, q)^* \subset d(w^{-q'/q}, q')$. To do this suppose that $\phi \in d(w, q)^*$ and $\phi(e_i) = b_i$, where e_i denotes the vector $(0, \dots, 0, 1, 0 \dots)$ with the 1 as the ith coordinate. We may assume without loss of generality that (b_i) is a nonincreasing sequence.

Fix n and let $d_k = 1$ for $1 \leq k \leq n$ and $d_k = 0$ for $k > n$. Then with $a_k = d_k(b_k/w_k)^{1/(q-1)}$, we have

$$\sum_{i=1}^{n} b_i^{q'} w_i^{-q'/q} = \sum_{i=1}^{n} a_i b_i \leq \|\phi\| \|a\| = \|\phi\| \left(\sum_{i=1}^{n} (d_k(b_k/w_k)^{q'})_i^* w_i \right)^{1/q}.$$

Therefore, to prove that (b_i) belongs to $d(w^{-q'/q}, q')$ we need to prove only that there exists a constant C independent of n such that

$$\sum_{i=1}^{n} (d_k(b_k/w_k)^{q'})_i^* w_i \leq C \sum_{i=1}^{n} b_i^{q'} w_i^{-q'/q}. \tag{2.2}$$

To prove (2.2), let $j(i)$ be the permutation of $1, \dots, n$ such that

$$\sum_{i=1}^{n} (d_k(b_k/w_k)^{q'})_i^* w_i = \sum_{i=1}^{n} (b_i/w_i)^{q'} w_{j(i)}, \tag{2.3}$$

and let $j(i) = i$ for $i > n$. For $n = 1$, inequality (2.2) is trivial; for $n > 1$ let L be the greatest integer such that $2^L \leq n$. Then the right side of (2.3) is bounded by

$$(b_1/w_1)^{q'} w_{j(1)} + \sum_{k=1}^{L} \sum_{i=2^k}^{2^{k+1}-1} (b_i/w_i)^{q'} w_{j(i)}.$$

Applying Lemma 2 to the second term shows that it is bounded by the right side of (2.2). This completes the sufficiency proof.

To prove the necessity, let Γ_n be the set of all z with $z_i^* = w_i$ for $1 \leq i \leq 2n$ and $z_i = 0$ for $i > 2n$. Let

$$x^n = \frac{1}{(2n)!} \sum_{z \in \Gamma_n} z = \frac{1}{2n} \left(\sum_{i=1}^{2n} w_i \right) \left(\sum_{i=1}^{2n} e_i \right).$$

Since by hypothesis $d(w^{-q'/q}, q') = d(w, q)^*$, $d(w^{-q'/q}, q')$ is a locally convex space. From this and the fact that x^n is a convex combination of elements with norm $(\sum_{i=1}^{2n} w_i)^{1/q'}$, we have

$$\frac{1}{2n} \left(\sum_{i=1}^{2n} w_i \right) \left(\sum_{i=1}^{2n} w_i^{-q'/q} \right)^{1/q'} = \|x^n\|_{w^{-q'/q}, q'} \leq C \left(\sum_{i=1}^{2n} w_i \right)^{1/q'}.$$

Since w_i is nonincreasing,

$$\sum_{i=1}^{2n} w_i^{-q'/q} \geq n w_n^{-q'/q},$$

and we get $(\sum_{i=1}^{2n} w_i)^{1/q} \leq 2C(nw_n)^{1/q}$, which implies the regularity.

References

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