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ON THE SEMIRING $L^+(C_0(X))$

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ABSTRACT. Let X and Y be locally compact Hausdorff spaces. Let $C_0(X)$ (resp. $C_0(Y)$) denote the Banach space of all continuous functions on X vanishing at infinity on X (resp. Y) and $L^+(C_0(X))$ (resp. $L^+(C_0(Y))$) the semiring of positive operators on $C_0(X)$ (resp. $C_0(Y)$). We prove that if there exists a semiring isomorphism φ from $L^+(C_0(X))$ onto $L^+(C_0(Y))$, then X and Y are homeomorphic. If X and Y are assumed to be compact then the same conclusion holds under the milder condition that φ is an affine isomorphism and $\varphi(I_{C(X)})$ is order bounded.

1. INTRODUCTION

Let X be a locally compact Hausdorff space. Denote by $C_0(X)$ the Banach space of continuous functions on X vanishing at infinity under the supremum norm. When X is compact, $C(X)$ denotes the Banach space of all continuous functions on X. Let $C_0(X)^+$ denote the cone of all nonnegative functions in $C_0(X)$. If we let $L(C_0(X))$ be the Banach space of all bounded linear operators on $C_0(X)$ into itself, the partial ordering on $C(X)$ induces a partial ordering on $L(C_0(X))$ by $T \geq 0$ if $Tf \geq 0$ whenever $f \geq 0$. We shall write $L^+(C_0(X))$ for the positive cone under this ordering. The set $L^+(C_0(X))$ is a semiring under the usual addition and composition of linear operators. Such semirings of positive operators have been studied by some authors like Horne [2] and Tam [5] in the finite dimensional cases. They proved, among others, that if K is any proper generating closed cone in $\mathbf{R}^n$, then the set of linear operators on $\mathbf{R}^n$ which maps K into itself (in other words the set of operators which is positive relative to the partial ordering defined by K), together with its semiring structures, determine K (see [2, Theorem 1.8] and [5, Theorem 4.4]). In this article we investigate along this line in the infinite dimensional case. However we shall restrict our discussion to continuous function spaces. We shall prove

Theorem 1.1. *Let X and Y be locally compact Hausdorff spaces. Suppose that $L^+(C_0(X))$ and $L^+(C_0(Y))$ are isomorphic as semirings, then X and Y are homeomorphic.*

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To prove this theorem we look for invariants of the semiring which are related explicitly to the underlying topological spaces. The objects that suit our purpose are the minimal left facial ideals, i.e., the subsets of $L^+(C_0(X))$ (or $L^+(C_0(Y))$ which are minimal left ideals in the respective semirings and are at the same time faces of respective sets. By a left ideal we mean of course a subset I of $L^+(C_0(X))$ which is closed under addition and satisfies $S \circ T \in I$ whenever $S \in L^+(C_0(X))$ and $T \in I$. It is called minimal if it does not properly contain any other proper ideal. Clearly every minimal ideal is a principal ideal. We shall also use the fact that a convex subset $I \subseteq L^+(C_0(X))$ is a face of $L^+(C_0(X))$ if and only if it is hereditary, i.e., whenever $S \in L^+(C_0(X))$ and $S \leq T$ for some $T \in I$, then $S \in I$.

If X and Y are compact instead of only locally compact, Theorem 1.1 can be improved. Following Buck [1] we shall call an operator T on $L^+(C(X))$ order bounded if $-rI_{C(X)} \leq T \leq rI_{C(X)}$ for some $r \geq 0$. An affine map between two convex sets is one which preserves convex combinations. We have

Theorem 1.2. *Suppose there exists an affine isomorphism φ from $L^+(C(X))$ onto $L^+(C(Y))$ for which $\varphi(I_{C(X)})$ is order bounded. Then X and Y are homeomorphic.*

Theorem 1.2 improves Theorem 1.1 in this case because if φ is a semiring isomorphism, then $\varphi(I_{C(X)}) = I_{C(Y)}$.

We note in passing that the space $L(C(X))$ is in general not positively generated. A counterexample can be found in [3]. Although his example is constructed for a mapping T from $C(X)$ into $C(Y)$, an example with the same domain and range can easily be deduced by considering the disjoint union Z of X and Y. For then every element in $C(Z)$ can be identified as an ordered pair $(f, g) \in C(X) \times C(Y)$. Define $S: C(Z) \to C(Z)$ by $S(f, g) = (0, Tf)$; then S is not less than any positive operator on $C(Z)$.

2. Proof of Theorem 1.1

We shall break the proof into some lemmas. The first one is a general fact about minimal ideals. Let $\mu \in C_0(x)^*$ and $h \in C_0(X)$. Denote by $\mu \otimes h$ the rank one operator $\mu \otimes h(f) = \mu(f)h$ for all $f \in C_0(x)$. Then $\mu \otimes h$ is a positive operator if and only if $\mu \geq 0$ and $h \geq 0$. In this case, the left ideal in $L^+(C_0(X))$ generated by $\mu \otimes h$ is the set $\{\mu \otimes g : g \in C_0(X)^+\}$. Note that this is indeed a minimal ideal.

Lemma 2.1. *Every minimal left ideal in $L^+(C_0(X))$ is generated by a rank one operator.*

Proof. This follows from the fact that the composition of a rank one operator with any operator is of rank less than or equal to one. □

Lemma 2.2. *The ideal $\{\mu \otimes g : g \in C_0(X)^+\}$ is a face of $L^+(C_0(X))$ if and only if $\mu = \delta_x$, the Dirac measure at x, for some $x \in X$.*

Proof. $(\Rightarrow)$. Suppose that the support of μ, $\operatorname{supp}\mu$, consists of at least two distinct points, say x and y. There exist disjoint compact neighbourhoods U_x of x and U_y of y such that $\mu(U_x) \geq 0$ and $\mu(U_y) \geq 0$. Take $h \in C_0(X)$ for which $h \geq 0$, $h = 1$ on U_x and $h = 0$ on U_y. Then, for any nonzero $g \in C_0(X)^+$, $(h\mu) \otimes g \leq \mu \otimes g$. However $h\mu \otimes g \notin \{\mu \otimes f : f \in C_0(X)^+\}$.

$(\Leftarrow)$. Suppose that $0 \leq T \leq \delta_x \otimes g$. Then, for any $y \in X$, $0 \leq T^*(\delta y) \leq g(y)\delta_x$. It follows that $T^*(\delta_y) = h(y)\delta_x$ for some $h(y) \geq 0$. All we need to show is that h is a continuous function on X. Toward this end just take any $f \in C_0(X)$ for which $f(x) = 1$. Then $Tf(y) = h(y)$. Hence $h = Tf \in C_0(X)$. □

The next lemma follows immediately.

Lemma 2.3. *A subset of $L^+(C_0(X))$ is a minimal left facial ideal if and only if it is of the form $\{\delta_x \otimes g : g \in C_0(X)^+\}$ for some $x \in X$.*

We shall denote by I_x the left ideal as given by the above lemma. Now let φ be a semiring isomorphism between $L^+(C_0(X))$ and $L^+(C_0(Y))$. As observed in [5, Lemma 4.2], φ preserves scalar multiplications. Therefore φ maps minimal left facial ideals of $L^+(C_0(X))$ into minimal left facial ideals of $L^+(C_0(Y))$. In this way it induces a one-to-one correspondence from X onto Y given by $\tau(x) = y$ if $\varphi(I_x) = I_y$. We have to prove that τ is a homeomorphism.

Lemma 2.4. *The τ defined above is continuous.*

Proof. Let U be an open subset of Y. We shall show that $\tau^{-1}(U)$ is an open subset of X. Let $x \in \tau^{-1}U$ and let $g \in C_0(Y)^+$ be such that $g(\tau(x)) = 1$ and g vanishes outside U. Let $f \in C_0(X)^+$ be given by $\varphi(\delta_x \otimes f) = \delta_{\tau(x)} \otimes g$. Then $\varphi(f(x)\delta x \otimes f) = \varphi((\delta_x \otimes f)\circ(\delta_x \otimes f)) = ((\delta_{\tau(x)} \otimes g)\circ(\delta_{\tau(x)} \otimes g)) = \delta_{\tau(x)} \otimes g$. It follows that $f(x) \neq 0$. For $t \notin \tau^{-1}(U)$ let $\varphi(\delta_t \otimes h) = \delta_{\tau(t)} \otimes g$. Then, from $\varphi(f(t)\delta_x \otimes h) = \varphi((\delta_t \otimes h)\circ(\delta_x \otimes f)) = (\delta_{\tau(t)} \otimes g)\circ(\delta_{\tau(x)} \otimes g) = g(\tau(t))\delta_{\tau(x)} \otimes g = 0$, we have $f(t) = 0$. Hence $r^{-1}(U)$ is open. □

A similar argument shows that τ^{-1} is also continuous, and this completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

In this section, X and Y will be compact Hausdorff spaces. Let φ be an affine isomorphism from $L^+(C(X))$ onto $L^+(C(Y))$ such that $\varphi(I_{C(X)})$ is order bounded. We shall need the following characterisation of order bounded operators due to Buck [1, p. 101].

Lemma 3.1. *An operator $T \in L(C(X))$ is order bounded if and only if it is a multiplication operator, i.e., there exists an $h \in C(X)$ such that $Tf = hf$ for all $f \in C(X)$.*

For $h \in C(X)$ we shall let M_h denote the multiplication operator defined by h. It is obvious that $M_h \geq 0$ if and only if $h \geq 0$. We also observe that φ preserves the orderings in $C(X)^+$ and $C(Y)^+$. For if S and T are two positive operators on $C(X)$ such that $S-T \geq 0$, then $\varphi(S-T) \geq 0$ and, from $\varphi(S) = \varphi(S-T)+\varphi(T)$, we conclude that $\varphi(S)-\varphi(T) \geq 0$. This shows that φ maps order bounded operators (or, equivalently, multiplication operators) into order bounded operators. In this way φ defines an affine isomorphism p from $C(X)^+$ onto $C(Y)^+$ by $\varphi(M_h) = M_{p(h)}$. This p can be extended (uniquely) to an order isomorphism from $C(X)$ onto $C(Y)$. Therefore X and Y are homeomorphic by [4].

The reason why the proof in this section is not valid for locally compact spaces is that we then have that order bounded operators are multiplication by bounded continuous functions, from which we can only conclude that the Stone-Čech compactifications of X and Y are homeomorphic. We do not know, however, whether Theorem 1.2 holds also for locally compact spaces nor whether the requirement that $\varphi(I_{C(X)})$ is order bounded can be done without.

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