THE HYPERSPACES OF SUBCONTINUA OF THE PSEUDO-ARC
AND OF SOLENOIDS OF PSEUDO-ARCS
ARE CANTOR MANIFOLDS

PAWEL KRUPSKI

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ABSTRACT. New proofs of the above facts are based on specific homogeneity
properties of the pseudo-arc and of solenoids of pseudo-arcs.

The reader is referred to [5] for hyperspace theory. It is known that if \( X \)
the pseudo-arc or a solenoid of pseudo-arcs (see [7] for the definition), then
the hyperspace \( C(X) \) of all nonvoid subcontinua of \( X \) is 2-dimensional. It is
proved in [6] that if \( X \) is the pseudo-arc, then \( C(X) \) is also a Cantor manifold, i.e., no 0-dimensional subset separates \( C(X) \). In [2] a general theorem is
presented that \( C(X) \) has this property for an arbitrary metric, nondegenerate
continuum \( X \). Our proof of the theorem in the title is an application of the
following result [3].

Lemma 1. If \( X \) is an \( n \)-dimensional, locally compact, connected, homogeneous,
metric space, then no \((n - 2)\)-dimensional subset separates \( X (n \geq 1) \).

Lemma 2. If a dense, connected subset of a metric separable space \( X \) is separated
by no \( n \)-dimensional subset, then the space \( X \) has the same property. \( \Box \)

(1) Let \( X \) be the pseudo-arc. To show that \( C(X) \) is a Cantor manifold it
suffices to observe, by Lemmas 1 and 2, that the subspace \( Y \subset C(X) \) of all
nondegenerate, proper subcontinua of \( X \) is connected, locally compact, homogeneous (see [1]), as well as 2-dimensional and dense.

(2) Let \( X \) be a solenoid of pseudo-arcs with the continuous decomposition
\( D \) into pseudo-arcs such that \( X/D \) is a solenoid \( S \). The set \( D \) as a subspace
of \( C(X) \) is homeomorphic to \( S \). As in (1) the open subspace \( Y \) of \( C(X) \)
is connected and dense. The set \( Y \setminus D \) is dense in \( Y \) and is the union of two
disjoint, open, connected, 2-dimensional subsets \( M = \{ y \in Y : d \neq y \subset d \in D \} \).
and \( N = \{ y \in Y : d \neq y \supset d \in D \} \). It follows from [4] and from properties of solenoids of pseudo-arcs [7] that for every pair \( y_1, y_2 \in M(y_1, y_2 \in N) \) there exists a homeomorphism \( h : X \to X \) such that \( h(y_1) = h(y_2) \). The induced homeomorphism \( \hat{h} : C(X) \to C(X) \) satisfies \( \hat{h}(M) = M, \hat{h}(N) = N \) and \( \hat{h}(y_1) = y_2 \), so both \( M \) and \( N \) are homogeneous and, by Lemma 1, no 0-dimensional subset separates neither \( M \) nor \( N \). Suppose a 0-dimensional subset \( C \) separates \( Y \). Without loss of generality we may assume that \( C \) is a closed subset of \( Y \). It means \( Y \setminus C = A \cup B \), where \( A, B \) are nonvoid, disjoint and open subsets of \( C(X) \). In view of the above properties of \( M \) and \( N \) we may assume \( M \subseteq A \) and \( N \subseteq B \). Thus \( C \subseteq D \). If there is \( d \in D \setminus C \), then some order arc \( \alpha \subseteq C(X) \) passing through \( d \) joins \( M \) and \( N \), which is impossible, since \( \alpha \cap D = \{d\} \) and \( C \) separates \( Y \) between \( M \) and \( N \). Therefore \( C = D \), hence \( C \) is not 0-dimensional, a contradiction.

Remark. A similar proof works for \( X \) being a solenoid. However in this case \( C(X) \) is the cone over \( X \), which is evidently a Cantor manifold.

References