Approximation of fixed points of asymptotically nonexpansive mappings

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Abstract. Let $T$ be an asymptotically nonexpansive self-mapping of a non-empty closed, bounded, and starshaped (with respect to zero) subset of a smooth reflexive Banach space possessing a duality mapping that is weakly sequentially continuous at zero. Then, if $\text{id} - T$ is demiclosed and $T$ satisfies a strengthened regularity condition, the iteration process $z_{n+1} := \mu_{n+1} T^n(z_n)$ converges strongly to some fixed point of $T$, provided $(\mu_n)$ has certain properties.

0. Introduction

Motivated by the papers of B. Halpern [4] and P. Vijayaraju [7], we deal with the convergence of almost fixed points $x_n = \mu_n T^n(x_n)$ of an asymptotically nonexpansive mapping $T$. This class of functions was introduced by K. Goebel and W. A. Kirk [3] in 1972. In §1, convergence of the sequence $(x_n)$ to some fixed point of $T$ is shown, assuming that $T$ is uniformly asymptotically regular and $\text{id} - T$ is demiclosed. These assumptions have been made by P. Vijayaraju [7] to ensure the existence of a fixed point of $T$.

By strengthening the regularity condition on $T$ in §2, we establish the convergence of the explicit iteration scheme $z_{n+1} := \mu_{n+1} T^n(z_n)$ to some fixed point of $T$. This iteration method is similar to one that was introduced by B. Halpern [4].

1. Convergence of the sequence of approximate fixed points

We shall begin by recalling some definitions needed in the sequel.

Definition 1.1. Let $(E, || \cdot ||)$ be a normed space, $\emptyset \neq A \subseteq E$; $\mu : \mathbb{R}^+ \to \mathbb{R}^+$; $J : E \to E^*$; and $x_0 \in E$.

1. $(E, || \cdot ||)$ is called smooth: $\iff || \cdot ||$ is Gateaux-differentiable on $\partial B(0, 1)$.

2. $\mu$ is said to be a gauge function: $\iff \mu$ is continuous and strictly increasing with $\mu(0) = 0$ and $\lim_{x \to \infty} \mu(x) = \infty$.

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The set-valued duality mapping associated with \( \mu \) is given by 
\[ J^\mu_E(x) := \{ u \in E^* | u(x) = \|u\| \|x\| \text{ and } \|u\| = \mu(\|x\|) \} \]
and \( J \) is called a duality mapping with respect to \( \mu \): \( \iff J(x) \in J^\mu_E(x) \) for all \( x \in E \). For abbreviation, we set \( J^\mu_E := J^\mu_E \).

(4) \( J \) is weakly sequentially continuous at \( x_0 \): \( \iff \) for all \( (x_n) \subseteq E \) with \( x_n \rightarrow x_0 \), it follows that \( (J(x_n)) \rightarrow J(x_0) \).

Remark. It is well known that \( J^\mu_E \) is single-valued if and only if \( (E, \| \cdot \|) \) is smooth (see, e.g., [1]). In this case, we regard \( J^\mu_E \) as a mapping from \( E \) to \( E^* \). In all our proofs we assume, without loss of generality, that \( J^\mu_E \) is normalized, i.e., \( \mu = \text{id} \). Furthermore, in the sequel all normed spaces are assumed to be real Banach spaces.

Definition 1.2. Let \( (E, \| \cdot \|) \) be a normed space; \( \emptyset \neq A \subseteq E \); \( T: A \rightarrow A \); \( (k_n) \subseteq [1, \infty) \).

1. \( T \) is said to be asymptotically nonexpansive with sequence \( (k_n) \): \( \iff \) \( \lim(k_n) = 1 \) and \( \|T^n(x) - T^n(y)\| \leq k_n \|x - y\| \) for all \( n \in \mathbb{N} \) and all \( x, y \in A \).
2. \( T \) is called demiclosed: \( \iff \) for all \( (x_n) \subseteq A \) and all \( x, y \in E \) with \( (x_n) \rightarrow x \) and \( \lim(T(x_n)) = y \), it follows that \( x \in A \) and \( T(x) = y \).
3. \( T \) is called uniformly asymptotically regular: \( \iff \) for each \( \epsilon > 0 \) there is \( n_0 \in \mathbb{N} \), such that \( \|T^n(x) - T^{n+1}(x)\| \leq \epsilon \) for all \( n \geq n_0 \) and all \( x \in A \).

Lemma 1.3. Let \( (E, \| \cdot \|) \) be a smooth normed space; \( \emptyset \neq A \subseteq E \); \( T: A \rightarrow A \) asymptotically nonexpansive with sequence \( (k_n) \subseteq [1, \infty)^\mathbb{N} \); \( x, y \in A \); \( \lambda \in (0, 1) \); \( n \in \mathbb{N} \); \( x = \lambda T^n(x) \); and \( y = T^n(y) \). Then

\[ J_E(y - x)(x) \geq -\frac{k_n - 1}{\lambda - 1} \|x - y\|^2. \]

Proof. Set \( \alpha := \frac{1}{\lambda} - 1 > 0 \), \( z := \alpha x \), and \( w := y - x \). Then

\[ \|z - w\| = \|(1 + \alpha)x - y\| = \|\frac{1}{\lambda}x - y\| = \|T^n(x) - T^n(y)\| \leq k_n \|x - y\| = k_n \|w\|. \]

Hence

\[ J_E(y - x)(x) = \frac{1}{\alpha} J_E(w)(z) = \frac{1}{\alpha} J_E(w)(z - w) + \frac{1}{\alpha} J_E(w)(w) \geq -\frac{1}{\alpha} \|w\| \|z - w\| + \frac{1}{\alpha} \|w\|^2 \geq \frac{1}{\alpha} (1 - k_n) \|w\|^2 \]

\[ = ((1 - k_n) / ((\lambda^2) - 1)) \|x - y\|^2. \]

The following lemma can be verified by an easy calculation.

Lemma 1.4. Let \( \lambda \in \left( \frac{1}{2}, 1 \right) \). Then

1. \( \lambda^2 / (2\lambda - 1) > 1 \), and
2. \( (\lambda k - \lambda) / (k - \lambda) \leq 1 - \lambda \) for all \( k \in [1, \lambda^2 / (2\lambda - 1)] \).
Remark. Part (1) of Lemma 1.4 shows that the assumption \( k_n \leq \lambda_n^2/(2\lambda_n - 1) \) and \( k_n \in [1, \infty) \) of our next lemma makes sense.

Lemma 1.5. Let \((E, \| \cdot \|)\) be a smooth normed space possessing a duality mapping \( J : E \to E^* \) that is weakly sequentially continuous at 0; \( \emptyset \neq A \subset E \) bounded; \( T : A \to A \) asymptotically nonexpansive with sequence \( (k_n) \in [1, \infty)^N ; \) \((x_n) \in A^N ; \) \( x \in A ; \) \((x_{n_k}) \) a subsequence of \((x_n)\) with \((x_{n_k}) \to x ; \)
\((\lambda_n) \in (\frac{1}{2}, 1)^N \) with \( \lim(\lambda_n) = 1 ; \) \( x = T(x) ; \) \( x_n = (\lambda_n/k_n)T^n(x_n) \) for all \( n \in \mathbb{N} ; \) and \( k_n \leq \lambda_n^2/(2\lambda_n - 1) \) for all \( n \in \mathbb{N} \). Then

1. \( \lim(x_{n_k}) = x \), and
2. \( J(y - x)(x) \geq 0 \) for all \( y \in \text{Fix}(T) \).

Proof. (1) Since \((\lambda_n/k_n) \in (0, 1)^N \), it follows from Lemma 1.3 that
\[
J(x - x_n)(x_n) \geq -\frac{k_n - 1}{k_n/\lambda_n - 1} \| x_n - x \|^2 \quad \text{for all } n \in \mathbb{N}.
\]
We may choose \( M > 0 \) such that \( \| x_n - z \|^2 \leq M \) for all \( n \in \mathbb{N} \) and all \( z \in A \) because of the boundedness of \( A \). Obviously, \( (k_n - 1)/(k_n/\lambda_n - 1) \geq 0 \) and from Lemma 1.4 we know, that
\[
\frac{k_n - 1}{k_n/\lambda_n - 1} = \frac{\lambda_n k_n - \lambda_n}{k_n - \lambda_n} \leq 1 - \lambda_n.
\]
Hence \( J(x - x_n)(x_n) \geq -(1 - \lambda_n)M \), and therefore, for all \( n \in \mathbb{N} \),
\[
\| x - x_n \|^2 = J(x - x_n)(x) - J(x - x_n)(x_n) \leq J(x - x_n)(x) + (1 - \lambda_n)M,
\]
where

(a) \( \lim(1 - \lambda_n) = 0 \),
(b) \((x_{n_k} - x) \to 0 \), hence \( \lim J(x - x_{n_k})(x) = 0 \).

Thus we conclude that \( \lim \| x - x_{n_k} \| = 0 \).

(2) \( y \in \text{Fix}(T) \). As already shown in (1), for \( y = T(y) \) instead of \( x \), it follows that\( (*) \)
\[
J(y - x_n)(x_n) \geq -(1 - \lambda_n)M \quad \text{for all } n \in \mathbb{N}.
\]
Since \((E, \| \cdot \|)\) is smooth, \( J \) is strong-weak\(^{(\ast)}\) continuous (see e.g., [2]), and so from \( \lim(y - x_{n_k}) = y - x \), we conclude that \( J(y - x_{n_k}) \rightharpoonup J(y - x) \). Hence \( \lim J(y - x_{n_k})(x_{n_k}) = J(y - x)(x) \). This, together with \((*)\) and \( \lim(1 - \lambda_n) = 0 \), yields \( J(y - x)(x) \geq 0 \). \( \Box \)

Remark. Lemmas 1.3 and 1.5 are similar to some related results for nonexpansive mappings; namely, [6, Lemmas 4, 5]. The following theorem was demonstrated by P. Vijayaraju in the course of the proof of [7, Theorems 2.1, 2.2].
Theorem 1.6. Let \((E, \| \cdot \|)\) be a Banach space; \(\emptyset \neq A \subset E\) closed, bounded, and starshaped with respect to 0; \(T: A \to A\) asymptotically nonexpansive with sequence \((k_n) \in [1, \infty)^\mathbb{N}\) and uniformly asymptotically regular; \((\lambda_n) \in [0, 1)^\mathbb{N}\); and \(\lim(\lambda_n) = 1\). Then

1. for each \(n \in \mathbb{N}\) there is exactly one \(x_n \in A\) such that \(x_n = (\lambda_n/k_n)T^n(x_n)\), and
2. \(\lim \|x_n - T(x_n)\| = 0\).

Now we are able to prove the main result of this section.

Theorem 1.7. Let \((E, \| \cdot \|)\) be a smooth reflexive Banach space possessing a duality mapping \(J: E \to E^*\) that is weakly sequentially continuous at 0; \(\emptyset \neq A \subset E\) closed, bounded, and starshaped with respect to 0; \(T: A \to A\) asymptotically nonexpansive with sequence \((k_n) \in [1, \infty)^\mathbb{N}\) and uniformly asymptotically regular; \(\text{id} - T\) demiclosed; \((\lambda_n) \in (\frac{1}{2}, 1)^\mathbb{N}\); \(\lim(\lambda_n) = 1\); \(k_n \leq \lambda_n^2/(2\lambda_n - 1)\) for all \(n \in \mathbb{N}\). Then

1. for each \(n \in \mathbb{N}\) there is exactly one \(x_n \in A\) such that \(x_n = (\lambda_n/k_n)T^n(x_n)\), and
2. \((x_n)\) converges strongly to some fixed point of \(T\).

Proof. Part (1) follows from Theorem 1.6, so that it remains to prove part (2). Since \((E, \| \cdot \|)\) is reflexive and \(A\) is bounded, there exists \(z \in E\) and a subsequence \((x_{\mu_n})\) of \((x_n)\) such that \((x_{\mu_n}) \longrightarrow z\) (Pettis' theorem). From Theorem 1.6, \(\lim \|x_n - T(x_n)\| = 0\), so from the demiclosedness of \(\text{id} - T\), \(z \in A\) and \(T(z) = z\).

Applying Lemma 1.5, we get

\(*\) \quad \(J(y - z)(z) \geq 0\) for all \(y \in \text{Fix}(T)\).

Fix \(\varphi: \mathbb{N} \to \mathbb{N}\) and injective.

Repeating the above argument, we find a subsequence \((x_{\varphi_{\mu_n}})\) of \((x_{\mu_n})\) converging weakly to some \(x \in A\) such that \(x = T(x)\) and

\(**\) \quad \(J(y - x)(x) \geq 0\) for all \(y \in \text{Fix}(T)\).

Substituting \(x\) into (*) and \(z\) into (**), it follows that \(J(x - z)(z) \geq 0\) and \(J(z - x)(x) \geq 0\). Hence \(0 \leq J(x - z)(-x) + J(x - z)(z) = -\|x - z\|^2 \leq 0\), so that \(x = z\), and therefore \((x_{\varphi_{\mu_n}}) \longrightarrow z\). This shows that \((x_n) \longrightarrow z\), and we now apply Lemma 1.5 (with \(\varphi := \text{id}\)) to obtain \(\lim \|x_n - z\| = 0\). \(\square\)

Remark. For \((k_n) \in (1, \infty)^\mathbb{N}\) with \(\lim(k_n) = 1\), the conditions \((\lambda_n) \in (\frac{1}{2}, 1)^\mathbb{N}\), \(\lim(\lambda_n) = 1\), and \(k_n \leq \lambda_n^2/(2\lambda_n - 1)\) for all \(n \in \mathbb{N}\) are fulfilled, for example, if one chooses \(\lambda_n := k_n - \sqrt{k_n^2 - k_n}\) for all \(n \in \mathbb{N}\).

2. An explicit iteration scheme

Now we give an explicit iteration scheme for the approximation of a fixed point of \(T\). In analogy to B. Halpern's work [4], we use the fact that the
implicit iteration process $x_n = (\lambda_n/k_n)T^n(x_n)$ already converges to some fixed point of $T$ to show that the same holds for the explicit iteration procedure $z_{n+1} := (\lambda_{n+1}/k_{n+1})T^n(z_n)$.

In addition to several technical assumptions on the values $\lambda_n$ and $k_n$, we require that $T$ fulfills a certain strengthened regularity condition (assumption (b) of Theorem 2.2). Although parts of the proof of Theorem 2.2 are parallel to those of the proof of [4, Theorem 3], they are included for the sake of completeness.

**Definition 2.1.** Let $(\varepsilon_n) \in (0, \infty)^N$ and $(\mu_n) \in (0, 1)^N$. $((\varepsilon_n), (\mu_n))$ is called admissible if

1. $(\varepsilon_n)$ is decreasing;
2. $(\mu_n)$ is strictly increasing with $\lim(\mu_n) = 1$;
3. there exists $(\beta_n) \in \mathbb{N}^N$ such that
   i. $(\beta_n)$ is increasing,
   ii. $\lim(\beta_n(1 - \mu_n)) = \infty$,
   iii. $\lim((1 - \mu_n + \beta_n)/(1 - \mu_n)) = 1$, and
   iv. $\lim((\varepsilon_n\beta_n\mu_n\beta_n+\beta_n)/(1 - \mu_n)) = 0$.

**Remark.** (a) Parts (2) and (3) (i)-(iii) are due to B. Halpern [4]. (b) $\lim(\beta_n) = \infty$ because of (3)(ii) and (2), so that it follows from (3)(iv) that $\lim(\varepsilon_n) = 0$.

**Theorem 2.2.** Let $(E, \| \cdot \|)$ be a normed space; $\emptyset \neq A \subset E$ bounded and starshaped with respect to $0$; $T: A \to A$ asymptotically nonexpansive with sequence $(k_n) \in [1, \infty)^N$; $(\lambda_n) \in (0, 1)^N$ with $\lim(\lambda_n) = 1$; $(x_n) \in A^N$; $\mu_n := \lambda_n/k_n$ for all $n \in \mathbb{N}$; $z_0 \in A$; $x_n = (\lambda_n/k_n)T^n(x_n)$ for all $n \in \mathbb{N}$, and $z_{n+1} := (\lambda_{n+1}/k_{n+1})T^n(z_n)$ for all $n \in \mathbb{N}_0$; $(\varepsilon_n) \in (0, \infty)^N$ such that $((\varepsilon_n), (\mu_n))$ is admissible;

(a) $((1 - \mu_n)/(1 - \lambda_n))$ bounded;
(b) $\|T^n(x) - T^{n+1}(x)\| \leq \varepsilon_n$ for all $n \in \mathbb{N}$ and all $x \in A$.

Assume further that $q := \lim(x_n)$ exists. Then $\lim(z_n) = q$.

**Proof.** Observe that $(z_n) \in A^N$ is well defined, because $T(A) \subset A$, $(\mu_n) \in (0, 1)^N$, and $A$ is starshaped with respect to 0. Since $A$ is bounded and $T$ is Lipschitzian, there is a $d > 0$ such that $\|z\| \leq d$ for all $z \in A$ and there is a $c > 0$ such that $\|T(w)\| \leq c$ for all $w \in A$. Additionally, (a) implies the existence of a number $\kappa > 0$, such that $(1 - \mu_n)/(1 - \lambda_n) \leq \kappa$ for all $n \in \mathbb{N}$, and we may choose $(\beta_n)$ according to Definition 2.1. For $n > p \geq i$, we have

$\|z_{p+1} - x_i\| \leq \|\mu_{p+1}T^p(z_p) - \mu_iT^i(z_p)\| + \|\mu_iT^i(z_p) - \mu_{i+1}T^{i+1}(x_i)\|$,

where

$\|\mu_iT^i(z_p) - \mu_iT^i(x_i)\| \leq \mu_i k_i \|z_p - x_i\| = \lambda_i \|z_p - x_i\|$,
and

\[
\|\mu_{p+1} T^p(z_p) - \mu_i T^i(z_p)\| \\
\leq \|\mu_{p+1} T^p(z_p) - \mu_{p+1} T^i(z_p)\| + \|\mu_{p+1} T^i(z_p) - \mu_i T^i(z_p)\| \\
\leq \mu_{p+1} \|T^p(z_p) - T^i(z_p)\| + |\mu_{p+1} - \mu_i|c.
\]

Since \( n > p \geq i \), it follows that \(|\mu_{p+1} - \mu_i| \leq \mu_n - \mu_i \). Additionally,

\[
\|T^p(z_p) - T^i(z_p)\| \leq \sum_{j=i}^{p-1} \|T^{j+1}(z_p) - T^j(z_p)\| \\
\leq \sum_{j=i}^{p-1} e_j \leq \sum_{j=i}^{p-1} e_i = (p-i)e_i \leq (n-i)e_i.
\]

Combining the above inequalities, we get

(1) \(|z_{p+1} - x_i| \leq \lambda_i |z_p - x_i| + (\mu_n - \mu_i)c + \mu_n(n-i)e_i \quad \text{for all } n > p \geq i,

from which it follows by induction that

\[
|z_{m+j} - x_i| \leq \lambda_i^j |z_m - x_i| + (\mu_n - \mu_i)c \sum_{\nu=0}^{j-1} \lambda_i^\nu + \mu_n(n-i)e_i \sum_{\nu=0}^{j-1} \lambda_i^\nu,
\]

for all \( n > m \geq i \) and \( j \leq n - m \). Since \( \sum_{\nu=0}^{j-1} \lambda_i^\nu \leq 1/(1 - \lambda_i) \), we obtain, defining \( j = n - m \),

(2) \(|z_n - x_i| \leq \lambda_i^{n-m} |z_m - x_i| + (\mu_n - \mu_i)(1 - \lambda_i)^{-1}c + \mu_n(n-i)e_i(1-\lambda_i)^{-1}

for all \( n > m \geq i \).

Set \( m := i \) and \( n := i + \beta_i > m \) to get

\[
|z_{i+\beta_i} - x_i| \leq 2d \lambda_i^{\beta_i} + (\mu_{i+\beta_i} - \mu_i)(1 - \lambda_i)^{-1}c + \mu_{i+\beta_i} \beta_i e_i(1 - \lambda_i)^{-1},
\]

for all \( i \in \mathbb{N} \). Observe that

(i) \(|\mu_{i+\beta_i} - \mu_i| |1 - \lambda_i|^{-1} = |1 - (1 - \mu_{i+\beta_i})(1 - \mu_i)|/(1 - \lambda_i)|
\leq \kappa|1 - (1 - \mu_{i+\beta_i})(1 - \mu_i)|^{-1},

from which we conclude that \( \lim((\mu_{i+\beta_i} - \mu_i)/(1 - \lambda_i)) = 0 \), taking into account the admissibility of \(((\epsilon_n), (\mu_n))\). Also,

(ii) \( \log(\lambda_i^{\beta_i}) \leq \beta_i(\lambda_i - 1) \leq -\beta_i/(1 - \mu_i)1/\kappa \). Since \(((\epsilon_n), (\mu_n))\) is admissible, it follows that \( \lim(\log(\lambda_i^{\beta_i})) = -\infty \), and thus \( \lim(\lambda_i^{\beta_i}) = 0 \). In addition,

(iii) \( 0 \leq \mu_{i+\beta_i} \beta_i e_i(1 - \lambda_i)^{-1} \leq \kappa \mu_{i+\beta_i} \beta_i e_i(1 - \mu_i)^{-1} \), from which it follows that \( \lim((\mu_{i+\beta_i} \beta_i e_i)/(1 - \lambda_i)) = 0 \), again using the admissibility of \(((\epsilon_n), (\mu_n))\).

Hence

(3) \( \lim |z_{i+\beta_i} - x_i| = 0. \)
As already noticed in the proof of [4, Theorem 3], there exists \( j : \mathbb{N} \to \mathbb{N} \) and \( n_0 \in \mathbb{N} \) such that \( \lim (j_n) = \infty \) and \( j_n + \beta_{j_n} < n \leq j_n + 1 + \beta_{j_n+1} \) for all \( n \geq n_0 \). Now let \( n \geq n_0 \). With \( n > m := j_n + \beta_{j_n} \geq i := j_n + 1 \), it follows from (2) that

\[
\|z_n - x_{j_n+1}\| \leq \|z_{(j_n+\beta_{j_n}) - x_{j_n+1}}\| + (\mu_n - \mu_{j_n+1})(1 - \lambda_{j_n+1})^{-1}c
\]

\[ + \mu_n(n - (j_n + 1))e_{j_n+1}(1 - \lambda_{j_n+1})^{-1} \quad \text{for all} \ n \in \mathbb{N}, \]

where

(i) \( 0 \leq (\mu_n - \mu_{j_n+1})(1 - \lambda_{j_n+1})^{-1} \leq (\mu_{(j_n+1+\beta_{j_n+1})} - \mu_{j_n+1})(1 - \lambda_{j_n+1})^{-1} \), which tends to 0 by 2(i).

(ii) \( 0 \leq \mu_n(n - (j_n + 1))e_{j_n+1}(1 - \lambda_{j_n+1})^{-1} \leq \mu_{(j_n+1+\beta_{j_n+1})}e_{j_n+1}(1 - \lambda_{j_n+1})^{-1} \), which tends to 0 by (2)(iii).

(iii) \( \|z_{(j_n+\beta_{j_n}) - x_{j_n+1}}\| \leq \|z_{(j_n+\beta_{j_n}) - x_{j_n}}\| + \|x_{j_n} - q\| + \|q - x_{j_n+1}\| \), which also tends to 0 because \( \lim \|x_n - q\| = 0 \) and \( \lim \|z_{(j_n+\beta_{j_n}) - x_{j_n}}\| = 0 \) by (3). Hence \( \lim \|z_n - x_{j_n+1}\| = 0 \). Finally, since \( \|z_n - q\| \leq \|z_n - x_{j_n+1}\| + \|x_{j_n+1} - q\| \), it follows that \( \lim \|z_n - q\| = 0 \). □

We now prove the main result of this section.

**Theorem 2.3.** Let \((E, \| \cdot \|)\) be a reflexive smooth Banach space possessing a duality mapping that is weakly sequentially continuous at 0; \( \emptyset \neq A \subset E \) closed, bounded, and starshaped with respect to 0; \( T : A \to A \) asymptotically nonexpansive with sequence \((k_n) \in [1, \infty) \); \( \text{id} - T \) demiclosed; \((\lambda_n) \in \left(\frac{1}{2}, 1\right)^\mathbb{N} \); \( z_0 \in A \); \( z_{n+1} := (\lambda_{n+1}/k_{n+1})T^n(z_n) \) for all \( n \in \mathbb{N}_0 \);

(a) \( \lim(\lambda_n) = 1 \); \( \mu_n := \lambda_n/k_n \) for all \( n \in \mathbb{N} \); \( k_n \leq \lambda_n^2/(2\lambda_n - 1) \) for all \( n \in \mathbb{N} \); \( ((1 - \mu_n)/(1 - \lambda_n)) \) bounded; \((e_n) \in (0, \infty)^\mathbb{N} \), such that \((e_n), (\mu_n)) \) is admissible;

(b) \( \|T^n(x) - T^{n+1}(x)\| \leq e_n \) for all \( n \in \mathbb{N} \) and all \( x \in A \).

Then \((z_n) \) converges strongly to some fixed point of \( T \).

**Proof.** Since \((e_n), (\mu_n)) \) is admissible, it follows that \( \lim(e_n) = 0 \), so that assumption (b) implies that \( T \) is uniformly asymptotically regular. The result follows by a combination of Theorems 1.7, 2.2. □

**Remark.** (1) For a related result concerning the weak convergence of the successive approximations \( x_{n+1} := T(x_n) \) to some fixed point of an asymptotically nonexpansive mapping \( T \) in an Opial space we refer the reader to [5].

(2) Theorem 2.3, of course, is applicable to nonexpansive mappings as well (set \( k_n := 1 \) for all \( n \in \mathbb{N} \)). In this special case, however, there is a much better result: [6, Theorem 10] allows us to drop the assumptions that \( \text{id} - T \) is demiclosed and that \( \|T^n(x) - T^{n+1}(x)\| \leq e_n \) for all \( n \in \mathbb{N} \) and all \( x \in A \), provided use is made of the iteration scheme \( z_{n+1} := \lambda_{n+1}T(z_n) \), \( T \) is nonexpansive, and \((E, \| \cdot \|) \) possesses a duality mapping that is weakly sequentially continuous on the whole of \( E \).
We close this section with a special case of Theorem 2.3, which shows that it is really possible to fulfill the assumptions made on \((\lambda_n), (\mu_n), (e_n),\) and \((k_n).\)

**Theorem 2.4.** Let \((E, \| \cdot \|)\) be a reflexive smooth Banach space possessing a duality mapping that is weakly sequentially continuous at 0; \(\emptyset \neq A \subseteq E\) closed, bounded, and starshaped with respect to 0; \(T : A \rightarrow A\) asymptotically nonexpansive with sequence \((1 + 1/(\sqrt{n + 1} - 1))\); \(\text{id} - T\) demiclosed; \(\mu_n := 1 - (n + 1)^{-1/4}\) for all \(n \in \mathbb{N}\); \(z_0 \in A\); \(z_{n+1} := \mu_{n+1} T^n(z_n)\) for all \(n \in \mathbb{N}_0\); and

\[(*) \quad \|T^n(x) - T^{n+1}(x)\| \leq (n + 1)^{-7/8} \quad \text{for all } n \in \mathbb{N} \text{ and all } x \in A. \]

Then \((z_n)\) converges strongly to some fixed point of \(T\).

**Proof.** For \(n \in \mathbb{N}\) define \(k_n := 1 + 1/(\sqrt{n + 1} - 1), \lambda_n := k_n - \sqrt{k_n^2 - k_n}, e_n := (n + 1)^{-7/8},\) and \(\beta_n := [(n + 1)^{1/2}]_G\) (cf. Definition 2.1), where \([x]_G := \max\{n \in \mathbb{N}_0 | n \leq x\}\). Then easy calculations show that all the assumptions of Theorem 2.3 are fulfilled, especially that \(\mu_n = \lambda_n/k_n\) for all \(n \in \mathbb{N}\) and that \(((e_n), (\mu_n))\) is admissible. □

**Remark.** For \(x \in A\) and \(n, m \in \mathbb{N}\) with \(n > m\), the following holds:

\[
\|T^n(x) - T^m(x)\| \leq \sum_{j=m}^{n-1} \|T^{j+1}(x) - T^j(x)\| \leq \sum_{j=m}^{n-1} (j + 1)^{-7/8}.
\]

Nevertheless, since \(\sum_{j=1}^{\infty} (j + 1)^{-7/8} = \infty\), \((*)\) does not imply that \((T^n(x))\) is a Cauchy sequence (from which it would follow that \((T^n(x))\) converges strongly to some fixed point of \(T\)).

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**References**


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