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ON INTERSECTIONS OF COMPACTA IN EUCLIDEAN SPACE

A. N. DRANISHNIKOV

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ABSTRACT. Let $\mathbf{X}$ be a codimension-three tame compactum in Euclidean space $\mathbf{E}^n$. If $\dim \mathbf{X} \times \mathbf{Y} < n$, then every map $f: \mathbf{Y} \to \mathbf{E}^n$ can be approximated by map g with $\mathbf{X} \cap \operatorname{Im} g = \varnothing$.

1. INTRODUCTION

This paper is a sequel to the recent series of works [M-Ru1-2], [K-L], [K], [Sp1-2] [D-S], [D-R-S1-2], [D2].

Suppose a compactum $\mathbf{X}$ lies in Euclidean space $\mathbf{E}^n$. Under what conditions can every map $f: \mathbf{Y} \to \mathbf{E}^n$ of compactum $\mathbf{Y}$ be approximated by a map whose image does not intersect $\mathbf{X}$? This paper suggests a solution of the problem in terms of the dimension of the product. Denote by $\mathbf{C}(\mathbf{X}, \mathbf{Z})$ the space of all continuous maps of $\mathbf{X}$ into $\mathbf{Z}$.

Theorem 1. *Let $\mathbf{X}$ be a codimension-three tame compactum in Euclidean space $\mathbf{E}^n$. Suppose that $\dim(\mathbf{X} \times \mathbf{Y}) < n$ for some compactum $\mathbf{Y}$. Then the space $\mathbf{C}(\mathbf{Y}, \mathbf{E}^n - \mathbf{X})$ is dense in $\mathbf{C}(\mathbf{Y}, \mathbf{E}^n)$.*

By the tameness of $\mathbf{X}$ we understand the equality $\operatorname{dem} \mathbf{X} = \dim \mathbf{X}$, where dem is Shtanko's embedding dimension [Sh], [E].

This theorem with the additional restriction $\dim \mathbf{X} + \dim \mathbf{Y} \leq n$ was proved in [D-Š] and [D-R-Š]. The restriction $\dim \mathbf{X} + \dim \mathbf{Y} \leq n$ is very strong because some recent results in dimension theory [D1] imply that there exist compacta $\mathbf{X}$ and $\mathbf{Y}$ with $\dim(\mathbf{X} \times \mathbf{Y}) < n$ and $\dim \mathbf{X} + \dim \mathbf{Y} = m$ for given $m < 2n - 3$ (of course, $m > \max\{\dim \mathbf{X}, \dim \mathbf{Y}\}$).

Corollary 1 [D2]. *Let $\mathbf{X}$ and $\mathbf{Y}$ be compacta with $\dim \mathbf{X} + \dim \mathbf{Y} \leq 4/3n - 1$. Suppose that $\dim(\mathbf{X} \times \mathbf{Y}) < n$. Then every pair of maps $f: \mathbf{X} \to \mathbf{E}^n$ and $j: \mathbf{Y} \to \mathbf{E}^n$ has unstable intersection.*

A pair of maps $f: \mathbf{X} \to \mathbf{Z}$ and $g: \mathbf{Y} \to \mathbf{Z}$ of compacta $\mathbf{X}$ and $\mathbf{Y}$ into a metric space $\mathbf{Z}$ is said to have an unstable intersection [D-R-Š2] if, for every

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$\varepsilon > 0$, there exists a pair of maps $f': \mathbf{X} \to \mathbf{Z}$ and $g': \mathbf{Y} \to \mathbf{Z}$ satisfying the conditions that $d(f, f') < \varepsilon$ and $d(g, g') < \varepsilon$ and with the property $\operatorname{Im} f' \cap \operatorname{Im} g' = \varnothing$.

Corollary 1 with the restriction $\dim \mathbf{X} + \dim \mathbf{Y} \leq n$ was proved in [D-R-Š2] and [Sp2]. It was also proved without this restriction in recent work [Sp-T].

Corollary 2 [D2]. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be compacta with* $\dim(\mathbf{X} \times \mathbf{Y}) < n$. *Then there exists a number* m *such that every pair of maps* $f: \mathbf{X} \to \mathbf{E}^{n+m}$ *and* $g: \mathbf{Y} \times \mathbf{I}^m \to \mathbf{E}^{n+m}$ *has an unstable intersection.*

Here $\mathbf{I}^m$ is an m-dimensional cube.

Conjecture. The inequality $\dim \mathbf{X} \times \mathbf{Y} < n$ for a pair of compacta implies the instability of intersections for every pair of maps $f: \mathbf{X} \to \mathbf{E}^n$ and $g: \mathbf{Y} \to \mathbf{E}^n$.

The positive answer to the questions in any of the following problems implies the conjecture.

Problem 1. Let $\mathbf{X}$ and $\mathbf{Y}$ be compacta with $\dim(\mathbf{X} \times \mathbf{Y}) < n$. Can a given map $f: \mathbf{X} \to \mathbf{E}^n$ be approximated by f' with the property $\dim(\operatorname{Im} f' \times \mathbf{Y}) < n$?

Problem 2. Let $\mathbf{X}$ be a compactum with $\dim \mathbf{X} \leq n - 2$, and let G be an Abelian group. Can an arbitrary map $f: \mathbf{X} \to \mathbf{E}^n$ be approximated by f' with the property $c\text{-}\dim_G \operatorname{Im} f' \leq c\text{-}\dim_G \mathbf{X}$?

Recall that the cohomological dimension of a compactum $\mathbf{Z}$ with respect to G as coefficient, $c\text{-}\dim_G \mathbf{Z}$, is the largest number m for which there exists a closed subset $A \subset Z$ with $\check{H}^m(Z, A; G) \neq 0$, where $\check{H}^*$ is a Čech cohomology group [Ku].

Problem 3. Let $\mathbf{X}$ and $\mathbf{Y}$ be compacta. Suppose that a pair of maps $f: \mathbf{X} \to \mathbf{E}^n$ and $g: \mathbf{Y} \to \mathbf{E}^n$ has a stable intersection. Does it imply that the pair $f: \mathbf{X} \to \mathbf{E}^n \times \{0\} \hookrightarrow \mathbf{E}^{n+1}$ and $g \times \mathrm{id}_I: \mathbf{Y} \times I \to \mathbf{E}^{n+1}$, where $I = [-1, 1]$, has a stable intersection in $\mathbf{E}^{n+1}$?

As a consequence of the proof of Theorem 1, a generalization of the Alexandrov theorem [A] for an arbitrary group is obtained.

2. The main lemma

A homology group $H_k(\mathbf{X}; \mathbf{Z})$ with the integers as coefficients is denoted by $H_k(\mathbf{X})$.

Lemma 1. *Let* $\mathbf{M}$ *be a simply connected* CW*-complex and* $\mathbf{Y}$ *be a compactum. If* $c\text{-}\dim_{H_k(\mathbf{M})} \mathbf{Y} \leq k$ *for all* k, *then* $c\text{-}\dim_{\pi_k(\mathbf{M})} \mathbf{Y} \leq k$ *for every* k.

We will denote the homotopy group $\pi_k(\mathbf{M})$ by π_k and the homology group $H_k(\mathbf{M})$ by H_k.

Let us consider a Postnikov tower for the space $\mathbf{M}$ [Spa]: $* \leftarrow E_2 \leftarrow E_3 \leftarrow \cdots \leftarrow E_n \leftarrow \cdots$.

Assertion 1. *For each* n *and arbitrary* $\mathbf{G}$ *there exist the equalities* $H_n(\mathbf{E}_n; \mathbf{G}) = H_n(\mathbf{E}_{n+1}; \mathbf{G}) = \dots = H_n(\mathbf{M}; \mathbf{G})$.

Proof. It is easy to see that $\mathbf{E}_n$ is homotopy equivalent to CW-complex $\mathbf{T}$, which is obtained from $\mathbf{M}$ by attaching cells of dimension $\geq n+2$. Hence $H_n(\mathbf{E}_n; \mathbf{G}) = H_n(\mathbf{T}; \mathbf{G}) = H_n(\mathbf{T}^{(n+1)}; \mathbf{G}) = H_n(\mathbf{M}^{(n+1)}; \mathbf{G}) = H_n(\mathbf{M}; \mathbf{G})$. The other equalities can be obtained similarly.

Recall that $\mathbf{Z} = \mathbf{Z}/p\mathbf{Z}$ and $\mathbf{Z}_{p^\infty} = \varinjlim \mathbf{Z}_{p^k}$.

Assertion 2. *Let* π *be an Abelian group with no element of order* p, *and suppose that each element of* π *is divisible by* p. *Then* $\widetilde{H}_*(K(\pi, n); \mathbf{G}) = 0$ *for all* n *if* $\mathbf{G} = \mathbf{Z}_p$ *or* $\mathbf{G} = \mathbf{Z}_{p^\infty}$.

Here $K(\pi, n)$ is Eilenberg-McLane complex.

Proof. It is easy to verify that $\pi \otimes \mathbf{Z}[\frac{1}{p}] = \pi$. Let $\mathbf{X}_{[p]}$ denote a localization of space $\mathbf{X}$ out of the prime number p. Then [Su] $K(\pi, n)_{[p]} = K(\pi_{[p]}, n) = K(\pi \otimes \mathbf{Z}[\frac{1}{p}], n) = K(\pi, n)$. Hence

$$H_m(K(\pi, n)) = H_m(K(\pi, n)_{[p]}) = H_m(K(\pi, n)) \otimes \mathbf{Z}\left[\frac{1}{p}\right].$$

According to the universal coefficients formula we have $0 \to H_m(K(\pi, n)) \otimes \mathbf{Z}[\frac{1}{p}] \otimes \mathbf{G} \to H_m(K(\pi, n); \mathbf{G}) \to (H_{m-1}(K(\pi, n)) \otimes \mathbf{Z}[\frac{1}{p}]) * \mathbf{G} \to 0$. In the case $G = \mathbf{Z}_{p^\infty}$ or $G = \mathbf{Z}_p$, both the right and the left sides of the sequence are zero.

Assertion 3. *Suppose that* H_k *has no* p*-torsion and each element of* H_k *is divisible by* p *for all* $k < n$. *Then* $\widetilde{H}_*(\mathbf{E}_k; \mathbf{G}) = 0$ *for all* $k < n$ *if* $\mathbf{G} = \mathbf{Z}_p$ *or* $\mathbf{G} = \mathbf{Z}_{p^\infty}$.

Proof. We will use induction on k.

If $k = 1$, it is true by virtue of $\mathbf{E}_1 = *$. Assume that the assertion is proved for some $k < n$. Let us consider the spectral sequence of the fibration $\mathbf{E}_{k+1} \to \mathbf{E}_k$ in the Postnikov tower:

$$\mathbf{E}^2_{m,l} = H_m(\mathbf{E}_k; H_l(K(\pi_{k+1}, k+1); \mathbf{G})).$$

The universal coefficient formula gives us the sequence

$$\begin{aligned} 0 &\to H_m(\mathbf{E}_k) \otimes H_l(K(\pi_{k+1}, k+1); \mathbf{G}) \to \mathbf{E}^2_{m,l} \\ &\to H_{m-1}(\mathbf{E}_k) * H_l(K(\pi_{k+1}, k+1); \mathbf{G}) \to 0. \end{aligned}$$

Assertion 2 implies that $\mathbf{E}^2_{m,l} = 0$ if $l > 0$. If $l = 0$ and $m > 0$ then $\mathbf{E}^2_{m,l} = 0$ by the assumption of induction. Since $\mathbf{E}^2_{m,l} = 0$ if $(m, l) \neq (0, 0)$ then $\widetilde{H}_*(\mathbf{E}_{k+1}; \mathbf{G}) = 0$.

Assertion 4. *Let* $f\colon \mathbf{E} \to \mathbf{B}$ *be a fibration whose fibre* $\mathbf{F}$ *is* $(n-1)$*-connected. Suppose that* $\widetilde{H}_*(\mathbf{B}; \mathbf{G}) = 0$. *Then the inclusion* $\mathbf{F} \hookrightarrow \mathbf{E}$ *induces an isomorphism* $H_n(\mathbf{F}; \mathbf{G}) \to H_n(\mathbf{E}; \mathbf{G})$.

Proof. Consider the spectral sequence of f.

Assertion 5. *Let* $f\colon \mathbf{E} \to \mathbf{B}$ *be a fibration with* $(n-1)$*-connected fibre* $\mathbf{F}$. *Suppose that* $\widetilde{H}_*(\mathbf{B}; \mathbf{Z}_{p^\infty}) = 0$. *Then the inclusion* $j\colon \mathbf{F} \hookrightarrow \mathbf{E}$ *induces an isomorphism* $j_*: H_{n+1}(\mathbf{F}; \mathbf{Z}_{p^\infty}) \to H_{n+1}(\mathbf{E}; \mathbf{Z}_{p^\infty})$.

Proof. Consider the spectral sequence of the fibration f. It is enough to show that $\mathbf{E}^2_{1,n} = \mathbf{E}^2_{2,n} = 0$.

Indeed $\mathbf{E}^2_{m,n} = H_m(\mathbf{B}; H_n(\mathbf{F}); \mathbf{Z}_{p^\infty}) = H_m(\mathbf{B}; H_n(\mathbf{F}) \otimes \mathbf{Z}_{p^\infty}) = 0$. The last equality follows from the universal coefficients formula and stipulation

$$\widetilde{H}^*(\mathbf{B}; \mathbf{Z}_{p^\infty}) = 0.$$

Denote by $\mathscr{G}$ Bokshtein's family of groups. It consists of the rationals $\mathbf{Q}, \mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$, p-primary quasi-cyclic groups $\mathbf{Z}_{p^\infty}$, and localizations $\mathbf{Z}_{(p)}$ of the integers in p where p runs on all prime numbers.

Following Bokshtein, define a set $\mathscr{G}(G)$ for an arbitrary Abelian group G:

(1) $\mathbf{Q} \in \mathscr{G}(\mathbf{G}) \Leftrightarrow \mathbf{Q} \otimes \mathbf{G} \neq 0$,
(2) $\mathbf{Z}_{(p)} \in \mathscr{G}(\mathbf{G}) \Leftrightarrow \mathbf{Z}_{p^\infty} \otimes \mathbf{G} \neq 0$,
(3) $\mathbf{Z}_p \in \mathscr{G}(\mathbf{G}) \Leftrightarrow \mathbf{G} \otimes \mathbf{Z}_{p^\infty} = 0$ and $\mathbf{G} \otimes \mathbf{Z}_p \neq 0$,
(4) $\mathbf{Z}_{p^\infty} \in \mathscr{G}(\mathbf{G}) \Leftrightarrow \mathbf{Z}_{p^\infty} \otimes \mathbf{G} = 0$, $\mathbf{Z}_p \otimes \mathbf{G} = 0$, and $\mathbf{Z}_p * \mathbf{G} = 0$.

Bokshtein's Theorem [Ku]. *For arbitrary compactum* $\mathbf{X}$ *and arbitrary Abelian group* $\mathbf{G}$ *there is the equality*

$$c\text{-}\dim_{\mathbf{G}}\mathbf{X} = \max_{\mathbf{H}\in\mathscr{G}(\mathbf{G})} c\text{-}\dim_{\mathbf{H}}\mathbf{X}.$$

Proof of Lemma 1. Denote by δ_k the set $\{\mathbf{G} : c\text{-}\dim_{\mathbf{G}}\mathbf{Y} \leq k\}$. By the conditions of Lemma 1, we have $\mathbf{H}_k \in \delta_k$ and $\delta_k \subset \delta_{k+1}$ for all k. All we need to prove is the inclusion $\pi_k \in \delta_k$ for each k. We do it by induction on k.

Since $\pi_1 = 0$, then $\pi_1 \in \delta_1$. Assume that, for $k < n$, inclusions $\pi_k \in \delta_k$ hold. We must prove the inclusion $\pi_n \in \delta_n$. By virtue of Bokshtein's Theorem, $c\text{-}\dim_{\pi_n}\mathbf{Y} = c\text{-}\dim_{\mathbf{G}}\mathbf{Y}$ for some $\mathbf{G} \in \mathscr{G}(\pi_n)$. Thus, it is sufficient to prove that $\mathscr{G}(\pi_n) \subset \delta_n$. We have to consider four cases: when $\mathbf{G} = \mathbf{Q}$, $\mathbf{Z}_{(p)}$, $\mathbf{Z}_p$, and $\mathbf{Z}_{p^\infty}$.

(∗) Suppose that $\mathbf{G} \in \mathscr{G}(\pi_n)$. Then, if $\mathbf{G} \in \delta_{n-1}$, it follows that $\mathbf{G} \in \delta_{n-1} \subset \delta_n$. Hence we can assume for each $\mathbf{G}$ that $\mathbf{G} \notin \delta_{n-1}$.

Suppose that $\mathbf{G} = \mathbf{Q} \in \mathscr{G}(\pi_n)$. Then $(*)$ and the condition $\mathbf{H}_k \in \delta_k$ imply that $\cup_{i<n}\mathbf{H}_i$ has no element of infinite order. Let us consider a localization $\mathbf{M}_{(0)}$ of $\mathbf{M}$ in 0. Then $\mathbf{H}_k(\mathbf{M}_{(0)}) = \mathbf{H}_k \otimes \mathbf{Q} = 0$ for all $k < n$. Hurewicz's theorem implies $\mathbf{H}_n(\mathbf{M}_{(0)}) = \pi_n(\mathbf{M}_{(0)})$. Since $\mathbf{H}_n \otimes \mathbf{Q} = \mathbf{H}_n(\mathbf{M}_{(0)}) = \pi_n(\mathbf{M}_{(0)}) = \pi_n \otimes \mathbf{Q} \neq 0$, the group $\mathbf{H}_n$ contains an element of infinite order. Hence $\mathbf{Q} \in \mathscr{G}(\mathbf{H}_n)$, by the definition of $\mathscr{G}(\mathbf{H}_n)$. Therefore, $\mathbf{Q} \in \delta_n$.

Suppose that $\mathbf{Z}_{(p)} \in \mathscr{G}(\pi_n)$. There are three possibilities: (1) $\mathbf{Q}$ and some p-torsion group are both $\in \delta_{n-1}$; (2) $\mathbf{Q} \notin \delta_{n-1}$; (3) $\mathbf{Q} \in \delta_{n-1}$ and there is no p-torsion group in δ_{n-1}.

(1) From Bokshtein's inequality [Ku],

$$c\text{-}\dim_{\mathbf{Z}_{(p)}}\mathbf{Y} \le \max\{c\text{-}\dim_{\mathbf{Q}}\mathbf{Y},\quad c\text{-}\dim_{\mathbf{Z}_{p^\infty}}\mathbf{Y}+1\},$$

it follows that $\mathbf{Z}_{(p)} \in \delta_n$.

(2) Consider a localization of $\mathbf{M}$ in prime p, and apply the above argument. By using modulo p-torsions and Hurewicz's theorem [Spa], we will obtain an equality $\mathbf{H}_n \otimes \mathbf{Z}_{(p)} = \pi_n \otimes \mathbf{Z}_{(p)}$. Since $\mathbf{Z}_{(p)} \otimes \mathbf{Z}_{p^\infty} = \mathbf{Z}_{p^\infty}$, this equality together with $\mathbf{Z}_{(p)} \in \mathscr{G}(\pi_n)$ implies $\mathbf{Z}_{(p)} \in \mathscr{G}(\mathbf{H}_n)$. Hence $\mathbf{Z}_{(p)} \in \delta_n$.

(3) In this case all elements of $\mathbf{H}_i$ for $i < n$ are divisible by p. Apply Assertions 3 and 4 to obtain $\mathbf{H}_n(\mathbf{E}_n; \mathbf{Z}_{p^\infty}) = \mathbf{H}_n(\mathbf{K}(\pi_n, n); \mathbf{Z}_{p^\infty})$. The universal coefficient formula implies that $\mathbf{H}_n \otimes \mathbf{Z}_{p^\infty} = \pi_n \otimes \mathbf{Z}_{p^\infty} \neq 0$. Therefore, $\mathbf{Z}_{(p)} \in \mathscr{G}(\mathbf{H}_n)$ and $\mathbf{Z}_{(p)} \in \delta_n$.

Suppose that $\mathbf{G} = \mathbf{Z}_p \in \mathscr{G}(\pi_n)$. There are two possibilities: (1) $\mathbf{Z}_{p^\infty} \in \delta_{n-1}$ and (2) $\mathbf{Z}_{p^\infty} \notin \delta_{n-1}$. In the first case, Bokshtein's inequality

$$c\text{-}\dim_{\mathbf{Z}_p}\mathbf{Y} \le c\text{-}\dim_{\mathbf{Z}_{p^\infty}}\mathbf{Y}+1$$

implies the inclusion $\mathbf{Z}_p \in \delta_n$. In the second case, due to Bokshtein's inequality

$$c\text{-}\dim_{\mathbf{Z}_p}\mathbf{Y} \le c\text{-}\dim_{\mathbf{Z}_{(p)}}\mathbf{Y},$$

we can assume that $\mathbf{Z}_{(p)} \notin \delta_n$. Apply Assertions 1, 3, and 4 to obtain equalities $\mathbf{H}_n(\mathbf{M}; \mathbf{Z}_p) = \mathbf{H}_n(\mathbf{E}_n; \mathbf{Z}_p) = \mathbf{H}_n(\mathbf{K}(\pi_n, n); \mathbf{Z}_p) = \pi_n \otimes \mathbf{Z}_p \neq 0$. From the other side, $\mathbf{H}_n(\mathbf{M}; \mathbf{Z}_p) = \mathbf{H}_n(\mathbf{E}_n; \mathbf{Z}_p) = \mathbf{H}_n \otimes \mathbf{Z}_p$. Hence $\mathbf{H}_n \otimes \mathbf{Z}_p \neq 0$. Since $\mathbf{Z}_{(p)} \notin \delta_n$ and $\mathbf{H}_n \in \delta_n$, $\mathbf{Z}_{(p)} \notin \mathscr{G}(\mathbf{H}_n)$, and hence $\mathbf{H}_n \otimes \mathbf{Z}_{p^\infty} = 0$. Then $\mathbf{Z}_p \in \mathscr{G}(\mathbf{H}_n) \subset \delta_n$.

Suppose at least that $\mathbf{G} = \mathbf{Z}_{p^\infty} \in \mathscr{G}(\pi_n)$. By virtue of Bokshtein's inequalities [Ku],

$$c\text{-}\dim_{\mathbf{Z}_{p^\infty}}\mathbf{Y} \le c\text{-}\dim_{\mathbf{Z}_p}\mathbf{Y} \le c\text{-}\dim_{\mathbf{Z}_{(p)}}\mathbf{Y},$$

we can assume that $\mathbf{Z}_p, \mathbf{Z}_{(p)} \notin \delta_n$.

There are two cases: (1) $\mathbf{Q} \notin \delta_{n-1}$ and (2) $\mathbf{Q} \in \delta_{n-1}$. In the first case we consider the Serre class of Abelian groups $\mathbf{C}$ which consists of torsion groups having no element of order p [Spa]. Hurewicz's modulo-$\mathbf{C}$ theorem implies a modulo-$\mathbf{C}$ equality $\mathbf{H}_n = \pi_n$. Hence $\mathbf{Z}_{p^\infty} \in \mathscr{G}(\mathbf{H}_n) \subset \delta_n$ by the definition of $\mathscr{G}(\mathbf{H}_n)$. In the second case, apply Assertions 3, 4, and 5 to obtain the isomorphism

$$j_*: \mathbf{H}_{n+1}(\mathbf{K}(\pi, n); \mathbf{Z}_{p^\infty}) \to \mathbf{H}_{n+1}(\mathbf{E}_n; \mathbf{Z}_{p^\infty}).$$

Since $\mathbf{H}_n(\mathbf{K}(\pi_n, n)) * \mathbf{Z}_{p^\infty} = \pi_n * \mathbf{Z}_{p^\infty} \neq 0$ we can assume that $\mathbf{H}_{n+1}(\mathbf{E}_n; \mathbf{Z}_{p^\infty}) \neq 0$. The spectral sequence of the fibration $\mathbf{E}_{n+1} \to \mathbf{E}_n$ implies $\mathbf{H}_{n+1}(\mathbf{E}_{n+1}; \mathbf{Z}_{p^\infty}) \neq 0$. Therefore, $\mathbf{H}_{n+1} \otimes \mathbf{Z}_{p^\infty} \neq 0$ or $\mathbf{H}_n * \mathbf{Z}_{p^\infty} \neq 0$. The condition $\mathbf{H}_{n+1} \otimes \mathbf{Z}_{p^\infty} \neq 0$ implies $\mathbf{Z}_{(p)} \in \mathscr{G}(\mathbf{H}_{n+1}) \subset \delta_{n+1}$. By virtue of Bokshtein's inequality $c\text{-}\dim_{\mathbf{Z}_{p^\infty}}\mathbf{Y} \le \max\{c\text{-}\dim_{\mathbf{Q}}\mathbf{Y}, c\text{-}\dim_{\mathbf{Z}_{(p)}}\mathbf{Y} - 1\}$, we can conclude that $\mathbf{Z}_{p^\infty} \in \delta_n$. The condition $\mathbf{H}_n * \mathbf{Z}_{p^\infty} \neq 0$ combined with the fact that $\mathbf{Z}_p$, $\mathbf{Z}_{(p)} \notin \mathcal{G}(\mathbf{H}_n)$ implies that $\mathbf{Z}_{p^\infty} \in \mathcal{G}(\mathbf{H}_n)$.

3. Proof of theorem 1

Notation (Kuratowski). $\mathbf{X}\tau\mathbf{Z} \Leftrightarrow$ for every closed subset $\mathbf{A} \subset \mathbf{X}$ and every map $f: \mathbf{A} \to \mathbf{Z}$ there exists an extension $\overline{f}: \mathbf{X} \to \mathbf{Z}$ of f.

Recall that $\mathbf{X}\tau\mathbf{K}(\mathbf{G}, n)$ is equivalent to $c\text{-}\dim_{\mathbf{G}}\mathbf{X} < n$ [Ku].

Assertion 6. *Let* $\mathbf{Y}$ *and* $\mathbf{Z}$ *be homotopy-equivalent ANE-spaces. Then* $\mathbf{X}\tau\mathbf{Y}$ *if and only if* $\mathbf{X}\tau\mathbf{Z}$.

Proof. Apply the homotopy extension theorem.

Assertion 7. *Let* $f: \mathbf{E} \to \mathbf{B}$ *be a locally trivial fibration with fibre* $\mathbf{F}$, *and assume that* $\mathbf{E}, \mathbf{F} \in ANE$ *and that* $\mathbf{B}$ *is a polyhedron. Suppose that* $\mathbf{X}\tau\mathbf{B}$ *and* $\mathbf{X}\tau\mathbf{F}$ *for some compactum* $\mathbf{X}$. *Then* $\mathbf{X}\tau\mathbf{E}$.

Proof. Consider a map $\varphi: \mathbf{A} \to \mathbf{E}$ where $\mathbf{A}$ is a closed subset of $\mathbf{X}$. By $\mathbf{X}\tau\mathbf{B}$ there exists an extension $\psi: \mathbf{X} \to \mathbf{B}$ of the map $f \circ \varphi$. For each point $x \in \mathbf{B}^{(0)} - f\varphi(\mathbf{A})$ choose a point $x' \in f^{-1}(x)$. Here $\mathbf{B}^{(i)}$ denotes the i-dimensional skeleton of $\mathbf{B}$. By using $\mathbf{X}\tau\mathbf{F}$, extend the lifting of φ to φ' : $\mathbf{A} \cup \psi^{-1}(\psi(\mathbf{A}) \cap \mathbf{B}^{(0)}) \to \mathbf{E}$. Define a map $\varphi_0 : \psi^{-1}(\mathbf{B}^{(0)}) \cup \mathbf{A} \to \mathbf{E}$ by the formula

$$\varphi_0(x) = \begin{cases} \varphi'(x) & \text{if } x \in \psi^{-1}(\psi(\mathbf{A}) \cap \mathbf{B}^{(0)}) \cup \mathbf{A} \\ (\psi(x))' & \text{if } x \in \psi^{-1}(\mathbf{B}^{(0)} - \psi(\mathbf{A})). \end{cases}$$

Assume that there is a map $\varphi_k : \psi^{-1}(\mathbf{B}^{(k)}) \cup \mathbf{A} \to \mathbf{E}$ such that $\varphi_k|_{\mathbf{A}} = \varphi$ and $f \circ \varphi_k = \psi|\ldots$. Consider an arbitrary $(k+1)$-dimensional simplex $\sigma \subset \mathbf{B}^{(k+1)}$. The local triviality of f implies that there exists a fibre-preserving homeomorphism $h_\sigma : f^{-1}(\sigma) \to \sigma \times \mathbf{F}$. Let $\pi_F : \sigma \times \mathbf{F} \to \mathbf{F}$ be the projection. By virtue of $\mathbf{X}\tau\mathbf{F}$ there exists an extension $\eta_\sigma : \psi^{-1}(\sigma) \to \mathbf{B}$ of the map $\xi_\sigma = \pi_F \circ h_\sigma \circ \varphi_k|\ldots : \psi^{-1}(\sigma^{(k)}) \to \mathbf{B}$. The composition of diagonal product $(\psi|_{\psi^{-1}(\sigma)})\Delta\eta_\sigma$ and h_σ^{-1} gives us a map $\beta_\sigma : \psi^{-1}(\sigma) \to f^{-1}(\sigma)$. The union of all β_σ defines an extension $\varphi_{k+1} : \psi^{-1}(\mathbf{B}^{(k+1)}) \cup \mathbf{A} \to \mathbf{E}$ of φ_k. By virtue of the compactness of $\mathbf{X}$ there will be an inclusion $\mathbf{X} \subset \psi^{-1}(\mathbf{B}^{(k)})$ for some k. In that case, φ_k will be a required extension.

Lemma 2. *Let* $\mathbf{X}$ *be a finite-dimensional compactum with the property*

$$c\text{-}\dim_{\pi_k(M)}\mathbf{X} \le k$$

for all k *where* $\mathbf{M}$ *is a simply connected* CW*-complex. Then* $\mathbf{X}\tau\mathbf{M}$.

Proof. Let m be the dimension of $\mathbf{X}$. Consider the Postnikov tower of space $\mathbf{M} : * \leftarrow \mathbf{E}_2 \leftarrow \mathbf{E}_3 \leftarrow \ldots \leftarrow \mathbf{E}_m \leftarrow \ldots$. Let $\mathbf{N}_m$ be a homotopy equivalent to

$\mathbf{E}_m$ CW-complex which is obtained by attaching to $\mathbf{M}$ cells of the dimensions $\geq m+2$. It is easy to see that $\mathbf{X}\tau\mathbf{N}_m$ implies $\mathbf{X}\tau\mathbf{M}$. By Assertion 6, $\mathbf{X}\tau\mathbf{E}_m$ implies $\mathbf{X}\tau\mathbf{N}_m$. Thus, it is sufficient to prove that $\mathbf{X}\tau\mathbf{E}_m$. Applying the Milnor construction of universal bundles [M], we can assume that the Postnikov tower consists of polyhedra with locally trivial fibrations. Apply Assertion 7 and use induction to conclude that $\mathbf{X}\tau\mathbf{E}_m$.

Lemmas 1, 2 imply the following theorem:

Theorem 2. *For finite-dimensional compacta, an inequality* $c\text{-}\dim_{\mathbf{G}}\mathbf{Y} \leq n$ *implies the property* $\mathbf{Y}\tau\mathbf{M}(\mathbf{G}, n)$, *where* $\mathbf{M}(\mathbf{G}, n)$ *is a Moor space.*

Remark. Since $\dim\mathbf{Y} \leq n \Leftrightarrow \mathbf{Y}\tau\mathbf{S}^n$, Theorem 2 is an extension for arbitrary $\mathbf{G}$ of the well-known Alexandroff theorem [A]: for finite-dimensional compacta the inequality $c\text{-}\dim_{\mathbf{Z}}\mathbf{Y} \leq n$ implies $\dim\mathbf{Y} \leq n$.

Proof of Theorem 1. Let $f\colon \mathbf{Y} \to \mathbf{E}^n$ be an arbitrary map and let $\varepsilon > 0$ be given. We can approximate f by g with a polyhedron $\mathbf{K}$ as the image of g. Suppose that $\rho(f, g) < 1/2\varepsilon$. We assume that $\mathbf{K}$ is a subpolyhedron in $\mathbf{E}^n$ with respect to some triangulation τ. Suppose that mesh $\tau < 1/4\varepsilon$. Consider the second barycentric subdivision $\beta^2\tau$ of τ, and denote by $\mathbf{B}_\sigma$ the star of each simplex $\sigma \in \tau$ with respect to $\beta^2\tau$. Then for every pair σ_1, σ_2 we have $\mathbf{B}_{\sigma_1\cap\sigma_2} = \mathbf{B}_{\sigma_1} \cap \mathbf{B}_{\sigma_2}$. It is easy to see that $\mathbf{B}_\sigma$ is homeomorphic to the n-dimensional cell.

By $\mathbf{K}^{(i)}$ we denote the i-skeleton of $\mathbf{K}$ with respect to τ. For every $\sigma \in \mathbf{K}^{(0)}$, choose a point $x_\sigma \in \operatorname{Int}\mathbf{B}_\sigma - \mathbf{X}$. This defines a map $g_0 : g^{-1}(\mathbf{K}^{(0)}) \to \mathbf{E}^n - \mathbf{X}$. Assume that we can define a map $g_i : g^{-1}(\mathbf{K}^{(i)}) \to \mathbf{E}^n - \mathbf{X}$ such that, for every simplex $\sigma \subset \mathbf{K}^{(i)}$, there is the inclusion $g_i(g^{-1}(\sigma)) \subset \operatorname{Int}\mathbf{B}_\sigma$. Let σ be an arbitrary $i+1$-dimensional simplex in $\mathbf{K}$. Denote by $\mathbf{M}$ the set $\operatorname{Int}\mathbf{B}_\sigma - \mathbf{X}$. The equality $\operatorname{dem}\mathbf{X} = \dim\mathbf{X} \leq n-3$ implies that $\mathbf{M}$ is simply connected. Since $\dim\mathbf{X}\times\mathbf{Y} < n$, for any open subset $\mathbf{V} \subset \mathbf{Y}$, the equality $\mathbf{H}^n_c((\operatorname{Int}\mathbf{B}_\sigma \cap \mathbf{X}) \times \mathbf{V}; \mathbf{Z}) = 0$ holds. By virtue of the Künnet formula, $\mathbf{H}^k_c(\mathbf{V}; \mathbf{H}^{n-k}_c(\operatorname{Int}\mathbf{B}_\sigma \cap \mathbf{X})) = 0$ for $k \leq n$. Apply the Alexander duality to obtain $\mathbf{H}^k_c(\mathbf{V}; \mathbf{H}_{k-1}(\mathbf{M})) = 0$ for $k \leq n$. It means [Ku] that $c\text{-}\dim_{\mathbf{H}_k(\mathbf{M})}\mathbf{Y} \leq k$ for all k. Apply Lemmas 1 and 2 to obtain the property $\mathbf{Y}\tau\mathbf{M}$. Hence there exists a map $q_\sigma : g^{-1}(\sigma) \to \operatorname{Int}\mathbf{B}_\sigma - \mathbf{X} = \mathbf{M}$ such that the restriction of q_σ onto $g^{-1}(\sigma^{(i)})$ coincides with restriction g_i onto $g^{-1}(\sigma^{(i)})$. The union of q_σ defines a map $g_{i+1} : g^{-1}(\mathbf{K}^{(i+1)}) \to \mathbf{E}^n - \mathbf{X}$ with the property $g_{i+1}(g^{-1}(\sigma)) \subset \operatorname{Int}\mathbf{B}_\sigma$ for each $\sigma \subset \mathbf{K}^{(i+1)}$. Note that $\mathbf{Y} = g_n^{-1}(\mathbf{K}^{(n)})$. The map $g_n : \mathbf{Y} \to \mathbf{E}^n - \mathbf{X}$ has the property $\rho(g_n, f) < \varepsilon$.

4. Proof of corollaries

We called [D-R-Š2] a map $f\colon \mathbf{X} \to \mathbf{E}^n$ regular branched if for every $k \geq 0$ $\dim\mathbf{B}_k(f) \leq k\dim\mathbf{X} - (k-1)n$, where $\mathbf{B}_k(f) = \{z \in \mathbf{E}^n : \# f^{-1}(z) \geq k\}$.

Denote by $\mathbf{R}(\mathbf{X}, \mathbf{E}^n)$ the set of all regular branched maps from $\mathbf{X}$ to $\mathbf{E}^n$.

Theorem 3 [D-R-Š2, Theorem 3.1]. *For every compactum* $\mathbf{X}$, *the set* $\mathbf{R}(\mathbf{X}, \mathbf{E}^n)$ *is of the second Baire category in* $\mathbf{C}(\mathbf{X}, \mathbf{E}^n)$.

Lemma 3. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be compacta such that* $\dim \mathbf{X} \leq \dim \mathbf{Y}$, $\dim(\mathbf{X}\times\mathbf{Y}) < n$ *and* $\dim \mathbf{X} + \dim \mathbf{Y} \leq (4/3)n - 1$. *Then* $\mathbf{N} = \{f \in \mathbf{C}(\mathbf{X}, \mathbf{E}^n) : \dim(\operatorname{Im} f \times \mathbf{Y}) < n$ *and* $\dim(\operatorname{Im} f) \leq \dim \mathbf{X}\}$ *is dense in* $\mathbf{C}(\mathbf{X}, \mathbf{E}^n)$.

Proof. The set $\mathbf{N}_2 = \{f \in \mathbf{C}(\mathbf{X}, \mathbf{E}^n) : \dim(\operatorname{Im} f) \leq \dim \mathbf{X}\}$ is of the second Baire category. This fact is well known and follows, for example, from [D-R-Š2, Corollary 3.3]. It suffices to prove that $\mathbf{R}(\mathbf{X}, \mathbf{E}^n) \subset \mathbf{N}_1 = \{f \in \mathbf{C}(\mathbf{X}, \mathbf{E}^n) : \dim(\operatorname{Im} f \times \mathbf{Y}) < n\}$. Then $\mathbf{N} = \mathbf{N}_1 \cap \mathbf{N}_2$ will be dense in $\mathbf{C}(\mathbf{X}, \mathbf{E}^n)$.

Since $\dim \mathbf{X} < \frac{2}{3}n$, it follows that $\mathbf{B}_k(f) = \varnothing$ for $k > 2$.

Consider the map $f \times \operatorname{id}_{\mathbf{Y}} : \mathbf{X} \times \mathbf{Y} \to \operatorname{Im} f \times \mathbf{Y}$. Then, for every $k > 2$, $\mathbf{B}_k(f \times \operatorname{id}_{\mathbf{Y}}) = \varnothing$. Since $\dim \mathbf{X} \leq \frac{2}{3}n - 1/2$, $\dim \mathbf{B}_2(f \times \operatorname{id}_{\mathbf{Y}}) = \dim(\mathbf{B}_2(f) \times \mathbf{Y}) \leq \dim \mathbf{B}_2(f) + \dim \mathbf{Y} \leq 2\dim \mathbf{X} - n + \dim \mathbf{Y} = \dim \mathbf{X} + \dim \mathbf{Y} + \dim \mathbf{X} - n \leq \frac{4}{3}n - 1 + \frac{2}{3}n - 1/2 - n = n - \frac{3}{2}$. Apply the Zarelua-Skordev two-to-one mapping theorem [Z], [Sk] to conclude that

$$\dim(\operatorname{Im} f \times \mathbf{Y}) \leq \max\{\dim(\mathbf{X} \times \mathbf{Y}),\ \dim \mathbf{B}_2(f \times \operatorname{id}_{\mathbf{Y}}) + 1\} < n.$$

Proof of Corollary 1. Suppose that $\dim \mathbf{X} < \dim \mathbf{Y}$. According to Lemma 3, there exists a map $f' : \mathbf{X} \to \mathbf{E}^n$ $1/2\varepsilon$-close to f and with $\dim(\operatorname{Im} f' \times \mathbf{Y}) < n$. Since $\dim \mathbf{X} \leq n - 3$, the Shtanko theorem [Sh] implies that there exists a reembedding $\varphi : \operatorname{Im} f' \to \mathbf{E}$ of the set $\operatorname{Im} f'$ which is $\varepsilon/2$-close to the initial embedding and has the tameness property. Theorem 1 applied to $\varphi(\operatorname{Im} f')$ and $\mathbf{Y}$ implies that there exists a map $g' : \mathbf{Y} \to \mathbf{E}^n - \varphi(\operatorname{Im} f')$ ε-close to a given map g. Thus, the maps $\varphi \circ f$ and g' are ε-close to f and g, respectively, and have disjoint images.

Proof of Corollary 2. Choose a number m so large that the inequality $\dim \mathbf{X} + \dim \mathbf{Y} + m \leq \frac{4}{3}(n + m) - 1$ holds, and then apply Corollary 1.

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Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996–1301

Current address: Steklov Mathematical Institute, Moscow, USSR 117966