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MAXIMAL COHEN-MACAULAY MODULES AND THE QUASIHOMOGENEITY OF ISOLATED COHEN-MACAULAY SINGULARITIES

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ABSTRACT. We conjecture that a complete isolated Cohen-Macaulay singularity of dimension ≥ 2 is graded if and only if sufficiently high syzygy modules of the residue field and of the transpose of the module of Kähler differentials are isomorphic. The "only if" part of the conjecture is proved for hypersurface singularities.

The purpose of this paper is to give module-theoretic conditions guaranteeing the quasihomogeneity of an isolated Cohen-Macaulay singularity. From a broader perspective, the existence of such conditions demonstrates, once again, the close ties between "representation" theory and "structure" theory. Our choice of maximal Cohen-Macaulay (or MCM, for short) modules ("representations") is not accidental. It is this category that has already proven itself to be closely connected with the geometry of the underlying ring (see introduction to [M1] for a brief description of the related results). A more conceptual explanation for our choice may possibly come from the theory of MCM approximations of Auslander and Buchweitz [AB].

The aforementioned conditions, besides being instructive and interesting in their own right, may also have a utilitarian purpose related to the classification problem of CM rings with only finitely many indecomposable MCM modules (nowadays called CM rings of finite (M)CM type). At the present moment all known examples of finite CM type are quasihomogeneous [AR]. Moreover, there can be no examples other than those known among homogeneous singularities [EH].

In §0 we put together a few basic facts about matrix factorizations and MCM approximation. Section 1 deals with the quasihomogeneity of isolated hypersurface singularities. We show that such a singularity is quasihomogeneous if and only if the MCM approximation of the corresponding moduli algebra has no

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free summands. In §2 we formulate an extended (to arbitrary positive dimension ≥ 2) version of author's Conjecture 2.4 from [M2] which characterizes the quasihomogeneity through a relation between the syzygetic behavior of the transpose of the module of Kähler differentials and the MCM approximation of either the maximal ideal or (depending on the parity of the dimension) the residue field. We then investigate this relation for quasihomogeneous hypersurface singularities.

Our rings are analytic k-algebras, understood here as quotients of formal power series rings over a field k. An isolated singularity is then defined through the Jacobian criterion, where the module of Kähler differentials is the universally finite module of differentials.

An analytic k-algebra $(R, \underline{m})$ is called *graded* if there exist a system of generators $x_1, \dots, x_n$ of the maximal ideal, positive integers $d_1, \dots, d_n$, and a k-derivation $\delta: R \to R$ such that $\delta x_i = d_i x_i$, $i = 1, \dots, n$. The elements of the eigenspace V_l are said to be homogeneous of degree l. (Unlike the case of affine algebras, the direct sum of V_l does not span the whole ring R.) The words "graded" and "quasihomogeneous" are synonymous. If $d_i = 1$, $i = 1, \dots, n$, the singularity is called *homogeneous.*

0. Background material

0.1. **Matrix factorizations** (see [E] for details). Let $(A, \underline{m})$ be a regular local ring and $x \in \underline{m}$. A matrix factorization of x is an ordered pair of homomorphisms $(\varphi: F \to G, \psi: G \to F)$ of free A-modules F and G such that $\varphi\psi = x \cdot \mathrm{Id}_G$ and $\psi\varphi = x \cdot \mathrm{Id}_F$. It turns out that $\mathrm{rk}\, F = \mathrm{rk}\, G$. Two matrix factorizations $(\varphi: F \to G, \psi: G \to F)$ and $(\varphi': F' \to G', \psi': G' \to F')$ are called equivalent if there exist isomorphisms $\alpha: F \to F'$ and $\beta: G \to G'$ such that $\varphi'\alpha = \beta\varphi$ and $\alpha\psi = \psi'\beta$. The factorization (φ, ψ) is called *reduced* if $\mathrm{Im}(\varphi) \subset \underline{m}G$ and $\mathrm{Im}(\psi) \subset \underline{m}F$. If $B := A/(x)$ and overbar denotes reduction modulo x then the periodic complex $\mathbb{F}(\varphi, \psi): \cdots \overline{F} \xrightarrow{\overline{\varphi}} \overline{G} \xrightarrow{\overline{\psi}} \overline{F} \xrightarrow{\overline{\varphi}} \overline{G}$ is a B-free resolution of the MCM B-module $\mathrm{Coker}\, \varphi$. Moreover the associations $(\varphi, \psi) \to \mathbb{F}(\varphi, \psi)$ and $(\varphi, \psi) \to \mathrm{Coker}\, \varphi$ induce bijections between the sets of

(1) equivalence classes of reduced matrix factorizations of x over A,
(2) isomorphism classes of nontrivial 2-periodic minimal free resolutions over B, and
(3) MCM B-modules without free summands.

0.2. **MCM approximations.** The results described here are due to Auslander and Buchweitz and can be found in the lecture notes [A] (unfortunately, not in the printed form yet) and [AB]. Throughout this subsection R is a complete local CM ring.

Definition 1. Let X be an R-module. A map $p: M \to X$ is called an MCM approximation of X if M is an MCM module and for any MCM module N

and any map $f\colon N \to X$ there exists a map $g\colon N \to M$ such that $f = pg$.

An MCM approximation $p\colon M \to X$ is called *minimal* if no direct summand M' of M is part of an MCM approximation $p'\colon M' \to X$. Taking $N \cong R$ we see that p is necessarily surjective. Minimal approximations always exist and are uniquely determined up to (nonnatural) isomorphisms. Henceforth we will only consider minimal MCM approximations.

This same concept can also be described by the equivalent

Definition 2. Let X be an R-module. An MCM approximation of X is a short exact sequence $0 \to V \to M \xrightarrow{p} X \to 0$ of R-modules where M is MCM and V has finite injective dimension.

This approximation is called *minimal* if V and M have no common direct summands.

Suppose now that R is a hypersurface ring of dimension $d-1$ and X an R-module. By [E] a minimal resolution of X becomes 2-periodic at the dth step. We thus have the diagram

$$\begin{array}{ccccccccc} & & & \nearrow & P_{d-1} & \to \cdots \to & P_1 \to P_0 & \to X & \to 0 \\ \cdots p_d \xrightarrow{\overline{\varphi}} P_d \xrightarrow{\overline{\psi}} P_d \xrightarrow{\overline{\varphi}} P_d & \overline{\psi} & & & & & & & \uparrow q \\ & & & \searrow & P_d & \xrightarrow{\overline{\varphi}} \cdots \to & P_d \to P_d & \to \Omega^{-d}(\Omega^d X) & \to 0 \end{array}$$

where Ω^n stands for the nth syzygy module and q is an iterated lifting of the identity map of $\Omega^d X$. It turns out that, in a slightly abused language, $\Omega^{-d}(\Omega^d X)$ is the nonprojective part of the MCM approximation of X, and if $q'\colon Q \to X/\operatorname{Im} q$ is a projective cover, then, upon identifying q' with its lifting to X, the map $q \coprod q'\colon \Omega^{-d}(\Omega^d X) \coprod Q \to X$ is a minimal MCM approximation of X. Notice that if Q is a nonzero module then q is not surjective.

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We begin with the results originally proved by the author (see [M1, §10]) for $n = 3$.

Proposition 1.1. *Let $P := k[|X_1, \ldots, X_n|]$ be the formal power series ring in n variables over a field k of characteristic 0, $f \in \underline{m}_P := (X_1, \ldots, X_n)$ a formal power series whose Jacobian ideal $j(f)$ is $\underline{m}_P$-primary and $R := P/(f)$. Then the following are equivalent.*

(1) *The moduli algebra $R/\overline{j(f)}$ (the overbar denotes the image in R) is Gorenstein.*
(2) *$f \in j(f)$.*
(3) *$f \in \underline{m}_P j(f)$.*
(4) *there exist an MCM module N without free summands and a surjection $\alpha\colon N \to R/\overline{j(f)}$.*
(5) *The MCM approximation of the moduli algebra $R/\overline{j(f)}$ has no free summands.*

Proof. (1) $\Leftrightarrow$ (2) is known. To prove the nontrivial implication (1) $\Rightarrow$ (2) we first remark that by [K] the Gorenstein ring $R/\overline{j(f)} \cong P/(f, j(f))$ cannot be an almost complete intersection. Thus either $f \in j(f)$ and we are done, or one of the partials of f is a linear combination of f and other partials: $\partial f/\partial x_i = \sum_{j\neq i} a_j \partial f/\partial x_j + bf$, $b, a_j \in P$. This relation gives rise to the k-derivation $\delta := \partial/\partial x_i - \sum_{j\neq i} a_j \partial/\partial x_j$ of P which preserves the ideal (f) and thus induces a k-derivation $\overline{\delta}$ of R which has a unit in its image (since $\overline{\delta}(\overline{X}_i) = 1$). This contradicts the result of Zariski asserting that the images of derivations of isolated singularities over a field of characteristic 0 are contained in the maximal ideal (see [Te, p. 586]).

(2) $\Leftrightarrow$ (3) is obvious since if $f \in j(f) - \underline{m}_P j(f)$ then one of the partials is a linear combination of f and other partials, the case just ruled out.

(4) $\Leftrightarrow$ (5) is true for an arbitrary R-module in place of $R/\overline{j(f)}$. Suppose that $\beta: M \coprod Q \to R/\overline{j(f)}$ is an MCM approximation with Q a nonzero projective. By the universal property of approximations the surjection $\alpha: N \to R/\overline{j(f)}$ factors through $\beta: \alpha = \beta\gamma$ for some $\gamma: N \to M \coprod Q$. Since α is surjective and $\beta|M$ is not (see the end of §0.2), the image of γ must contain an element from $Q - \underline{m}_R Q$. But then N has a free summand, a contradiction.

(3) $\Leftrightarrow$ (4) holds for an arbitrary ideal in place of $j(f)$ as was shown by Ding in his forthcoming thesis.

Corollary. *Suppose that, in addition to the assumptions of Proposition* 1, *the field* k *is algebraically closed. Then* R *is graded if and only if the moduli algebra* $R/\overline{j(f)}$ *is a homomorphic image of an MCM module without free summands.*

Proof. Under our assumptions, the fundamental result of K. Saito ([Sa], [SchW, Satz 5.7]) says that R is graded if and only if $f \in j(f)$. □

2. A GENERAL CONJECTURE

The results of the previous section proven for isolated hypersurface singularities suggest a possibility that there is a module-theoretic criterion for the gradedness of an arbitrary analytic k-algebra with an isolated singularity. In fact, in [M2] we conjectured that a two-dimensional integrally closed analytic k-algebra R is graded if and only if the Auslander module of R is isomorphic to the module of Zariski differentials of R, and also proved the conjecture in certain cases. Further supporting evidence was provided by Behnke [B]. The purpose of this section is to state an analog of the aforementioned conjecture for an arbitrary positive dimension and then consider the hypersurface case.

Conjecture 2.1. *Let* $(R, \underline{m})$ *be a complete isolated CM singularity of dimension* $d \geq 2$ *over a field* k *of characteristic* 0. *Then* R *is graded if and only if* $(\Omega^d(\mathrm{Tr}(D_k(R)))^*$ *is isomorphic to the MCM approximation of the maximal ideal when* d *is even and to the MCM approximation of the residue field when* d *is odd. Here "*" stands for* $\mathrm{Hom}_R(-, R)$, $\Omega^d(-)$ *is the* dth *syzygy module of "–,"* Tr *is the transpose of "–" (i.e., the cokernel of the transpose of a minimal* *presentation matrix of "−") and $D_k(R)$ is the module of Kähler differentials of R over k.*

Let $d = 2$. Dualizing a minimal presentation $P_1 \to P_0 \to D_k(R)$ we have the exact sequence

$$0 \to D_k(R)^* \to P_0^* \xrightarrow{\Delta^*} P_1^* \to \mathrm{Tr}(D_k(R)) \to 0$$

which shows that $\Omega^2(\mathrm{Tr}(D_k(R)))^* \cong D_k(R)^{**}$. Since the MCM approximation of the maximal ideal is the Auslander module of R, when $d = 2$ Conjecture 2.1 becomes Conjecture 2.4 of [M2].

Our goal is to prove the "only if" part of the conjecture when R is a hypersurface and k is algebraically closed.

Proposition 2.2. *Let $R = P/(f)$ where $P = k[|X_1, \dots, X_n|]$ is a formal power series ring over an algebraically closed field k of characteristic 0, and $f \in \underline{m}_P := (X_1, \dots, X_n)$ a formal power series with $\underline{m}_P$- primary Jacobian ideal $j(f)$. If R is graded then $\Omega^{n-1}(R/\overline{j(f)})^*$ is isomorphic to the MCM approximation of the maximal ideal $\underline{m}_R$ of R when $n-1$ is even, and to the MCM approximation of k when $n-1$ is odd.*

Proof. By Satz 5.7 of [SchW] there exists a grading of P with respect to which f becomes a homogeneous polynomial. Thus without loss of generality we may assume that each X_i is homogeneous of degree d_i, $i = 1, \dots, n$, and f is a homogeneous polynomial of degree d (in the chosen grading). Applying the Euler derivation we have

$$f = \sum_{i=1}^{n} y_i \partial f/\partial X_i, \tag{1}$$

where $y_i = d_i X_i/d \neq 0$. Notice that the elements y_i, $i = 1, \dots, n$, generate the maximal ideal of P. Therefore to obtain the MCM approximation of the residue field (and the maximal ideal) of R it suffices to construct a projective resolution over R of $R/(\overline{y}_1, \dots, \overline{y}_n)$. This can be accomplished with the aid of the Tate resolution (see [Ta, p. 20, Theorem 4]). But the partials of f, by hypothesis, also form a system of parameters and the same gadget gives a projective resolution of the moduli algebra $R/(\overline{\partial f/\partial X_i}, \dots, \overline{\partial f/\partial X_n})$. Thus our proposition is a statement about the properties of the Tate resolution. More precisely, in the notation of [Ta] the R-algebra $Y := R\langle T_1, \dots, T_n, S\rangle$ with each T_i of degree 1 and S of degree 2, and with differential d defined by

$$dT_i := \overline{y}_i, \qquad dS := \sum_{i=1}^{n} (\overline{\partial f/\partial X_i}) T_i \tag{2}$$

yields a projective resolution of the R-module $R/(\overline{y}_1, \dots, \overline{y}_u) = R/\underline{m}_R$. On the other hand if we define differential d' by

$$d' T_i := \overline{\partial f/\partial X_i}, \qquad d' S := \sum_{i=1}^{n} \overline{y}_i T_i, \tag{3}$$

we obtain a projective resolution of the R-module $R/\overline{j(f)}$. Notice that the elements y_i and $\partial f/\partial X_i$ can be viewed in (1) as either parameters or their coefficients. Utilizing this we introduce a symbolic involution $\mathcal{I}$ acting on those elements by

$$\mathcal{I}(\partial f/\partial x_i) := y_i, \qquad \mathcal{I}(y_i) := \partial f/\partial X_i, \qquad i = 1, \ldots, n.$$

We will also consider the obvious action of $\mathcal{I}$ on their images in R.

Let d_k (resp. d'_k) be the degree k homogeneous part $Y_{k+1} \to Y_k$ of d (resp. d'). Since the corresponding matrices have $\overline{y}_i$ and $\overline{\partial f/\partial X_i}$ as their nonzero entries, we can lift those matrices to P in the obvious way, where, without the danger of confusion, we denote them by the same letters. Involution $\mathcal{I}$ acts on their entries and we have the obvious identities

$$\mathcal{I}(d_k) = d'_k, \qquad \mathcal{I}(d'_k) = d_k. \tag{4}$$

In our new notation, $(\Omega^{n-1}(R/\overline{j(f)}))^* \cong (\operatorname{Coker} d'_{n-1})^*$. On the other hand, if n is odd the MCM approximation of the maximal ideal is isomorphic to $\operatorname{Coker} d_n$, which follows from the construction of approximations via negative syzygies; and if n is even the MCM approximation of the residue field is also isomorphic to $\operatorname{Coker} d_n$. Thus we want to show that $\operatorname{Coker} d_n \cong (\operatorname{Coker} d'_{n-1})^*$, or, utilizing (4),

$$\operatorname{Coker} d_n \cong (\operatorname{Coker} \mathcal{I}(d_{n-1}))^*.$$

Since $\operatorname{Coker} d'_{n-1}$ is MCM, we have $(\operatorname{Coker} \mathcal{I}(d_{n-1}))^* \cong (\operatorname{Coker}(\mathcal{I}(d_{n-1}))^t)$ where t denotes the transpose of a matrix. Thus we want to show that matrices d_n and $\mathcal{I}(d_{n-1})^t$ have isomorphic cokernels.

Since $\operatorname{rk} Y_{n-1} = \operatorname{rk} Y_n = \cdots = 2^{n-1}$, $\operatorname{Coker} d_{n-1}$ has no free summands and thus comes from a reduced matrix factorization, say, (d_{n-1}, Φ), where the matrices are viewed as matrices with entries in P and the R-modules $\operatorname{Coker} \Phi$ and $\operatorname{Coker} d_n$ are isomorphic. Since Φ is uniquely determined by d_{n-1}, it suffices to show that $(d_{n-1}, \mathcal{I}(d_{n-1})^t)$ is also a matrix factorization. In fact, it suffices to show that $\mathcal{I}(d_{n-1})^t \cdot d_{n-1}$ is the scalar matrix with f on the diagonal (the dot denotes the usual matrix product). In other words, for any two column vectors v_i and v_j of d_{n-1} we must have $\mathcal{I}(v_i)^t \cdot v_j = \delta_{ij} f$.

To prove this it is convenient to introduce new notation. Let I denote an increasing map from the set $[l]$ of integers $\{0, 1, \ldots, l\}$ to the set $[n] = \{0, 1, \ldots, n\}$ (an "l-subsimplex" of the "n-simplex"). Choosing the R-basis $T_{i_1}, T_{i_2}, \ldots, T_{i_l} S^{(p)}$, $l + 2p = m$ for Y_m, $m = 1, 2, \ldots$, we can use our new notation as shorthand: $T_I S^{(p)}$. Let I_j denote the map $[l-1] \to [n]$ defined by

$$I_j(q) = \begin{cases} I(q) & \text{if } q < j \\ I(q+1) & \text{if } q \geq j, \end{cases} \qquad j = 1, \ldots, l-1$$

(this is the "jth submaximal face of I"), and let CI denote the complement $[n] \setminus \operatorname{Im}(I)$ of the image of I in $[n]$ (the "vertices of $[n]$ not in I"). A trivial computation shows that

$$\begin{aligned} d_{m-1}(T_I S^{(p)}) &= \sum_{j\in[l]}(-1)^{j-1}\overline{y}_{I(j)}T_{I_j}S^{(p)} + (-1)^l T_I\Bigl(\sum_{k\in[n]}(\overline{\partial f/\partial x_k})T_k S^{(p-1)}\Bigr) \\ &= \sum_{j\in[l]}(-1)^{j-1}\overline{y}_{I(j)}T_{I_j}S^{(p)} + (-1)^l \sum_{k\in CI}(\overline{\partial f/\partial x_k})T_I T_k S^{(p-1)}. \end{aligned}$$

Now it is immediate that if v is the column-vector of d_{m-1} (with entries in P) corresponding to $d_{m-1}(T_I S^{(p)})$ with $p > 0$ or with $p = 0$ and $l = n$, then $\mathcal{I}(v)^t \cdot v = \sum_{i\in[n]} y_i(\partial f/\partial x_i) = f$ (by (1)). In particular

$$\mathcal{I}(v)^t \cdot v = f \quad \text{if } m = n.$$

Now we choose two different basis elements $T_I S^{(p)}$ and $T_J S^{(q)}$ from Y_n, where I is an increasing map $[l] \to [n]$ and J is an increasing map $[l'] \to [n]$ with $n = l + 2p = l' + 2q$. Then as before,

$$\begin{aligned} d_{n-1}(T_I S^{(p)}) &= \sum_{j\in[l]}(-1)^{j-1}\overline{y}_{I(j)}T_{I_j}S^{(p)} + (-1)^l \sum_{k\in CI}(\overline{\partial f/\partial x_k})T_I T_k S^{(p-1)}, \\ d_{n-1}(T_J S^{(q)}) &= \sum_{j\in[l']}(-1)^{i-1}\overline{y}_{J(i)}T_{J_i}S^{(q)} + (-1)^{l'} \sum_{h\in CJ}(\overline{\partial f/\partial x_h})T_J T_h S^{(q-1)}. \end{aligned}$$

If the corresponding column-vectors v_I and v_J are such that $\mathcal{I}(v_J)^t \cdot v_I = 0$ we have nothing to prove. Otherwise we can have the four cases:

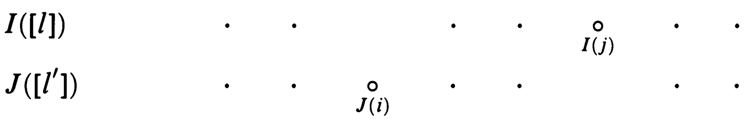
(1) $\exists j \in [l]$ and $\exists i \in [l']$ such that $T_{I_j}S^{(p)} = T_{J_i}S^{(q)}$;

(2) $\exists j \in [l]$ and $\exists h \in CJ$ such that $T_{I_j}S^{(p)} = (\pm)T_J T_h S^{(q-1)}$;

(3) $\exists i \in [l']$ and $\exists k \in CI$ such that $T_{J_i}S^{(q)} = (\pm)T_I T_k S^{(p-1)}$;

(4) $\exists k \in CJ$ and $\exists h \in CJ$ such that $T_I T_k S^{(p-1)} = (\pm)T_J T_h S^{(q-1)}$.

Case 1. It is clear that $I_j = J_i$, hence $l = l'$ and $p = q$. The corresponding contribution to the product is equal to $(-1)^{i+j}(\partial f/\partial x_{J(i)})y_{I(j)}$. Since the basis elements $T_I S^{(p)}$ and $T_J S^{(q)}$ are different we may assume, without loss of generality, that $J(i) < I(j)$ (and, therefore, $i \le j$). Symbolically the sets $I([l])$ and $J([l'])$ look as follows:



where the black dots indicate the elements in the image of $I_j = J_i$. It is obvious that $I([l]) \cup J(i) = J([l']) \cup I(j)$. We also notice that $p > 0$, otherwise $n = l = l'$ and $I = J$, contrary to the assumption. Therefore $T_I T_{J(i)} S^{(p-1)} = (\pm)T_J T_{I(j)} S^{(p-1)}$ is a basis element in Y_{n-1}. If we now define an increasing map $H\colon [l+1] \to [n]$ by

$$H(t) = \begin{cases} I(t) = J(t), & \text{if } t < i, \\ J(i), & \text{if } t = i, \\ I(t-1) = J(t), & \text{if } i < t \leq j, \\ I(j), & \text{if } t = j+1, \\ I(t-1) = J(t-1), & \text{if } t > j+1, \end{cases}$$

then $T_I T_{J(i)} S^{(p-1)} = (-1)^{l-i+1} T_H S^{(p-1)}$ and $T_J T_{I(j)} S^{(p-1)} = (-1)^{l-j} T_H S^{(p-1)}$ (remember $J(i) < I(j)$!). The contributions to $\mathscr{F}(v_J)^t \cdot v_I$ from the components corresponding to $T_H S^{(p-1)}$ will then be $(-1)^{1-i-j} y_{I(j)} (\partial f / \partial x_{J(i)})$ which cancels the previously computed product. Thus there can be no nonzero contribution to $\mathscr{F}(v_J)^t \cdot v_I$ in Case 1.

Case 2. Now $p = q - 1$ and $l' = l - 2$, and $\operatorname{Im}(I_j) = J([l-2]) \cup \{h\}$. In particular, there exists $j' \in [l]$ such that $h = I(j')$. First we assume that $j' < j$. Symbolically we are in the following situation:

$I([l])$ · · $\underset{h=I(j')}{\circ}$ · $\underset{h'=I(j)}{\circ}$ · ·

$J([l-2])$ · · · · ·

where the black dots indicate the elements of $J([l-2]) \subset I([l])$. The product of the components corresponding to the basis element

$$T_{I_j} S^{(p)} = (\pm) T_J T_h S^{(q-1)} \quad \text{is equal to} \quad (-1)^{j-1} y_{I(j)} ((-1)^{l-2+j'-1} y_{I(j')}).$$

But we also have $T_{I_j} S^{(p)} = (\pm) T_J T_{h'} S^{(q-1)}$, where $h' := I(j)$, and the corresponding contribution to $\mathscr{F}(v_J)^t \cdot v_I$ is $(-1)^{j'-1} Y_{I(j')} ((-1)^{l-2+j-2})$ (since $j' < j$!), which cancels the previously computed product.

If $j < j'$ the argument is identical to the one just given, and the case $j = j'$ can never be realized since $J([l-2]) \cup I(j') = \operatorname{Im} I_j$ and $I(j) \notin \operatorname{Im} I_j$.

Case 3. This case is completely analogous to Case 2. We should remark that examining the relations between p and q we see that no two of the three considered cases can happen simultaneously: we either have $p = q$ or $p = q - 1$ or $q = p - 1$. This remark actually legitimizes our argument. We now proceed to

Case 4. Here $I([l]) \cup k = J([l']) \cup h$ and $l = l'$. If $k = h$ then $I = J$, contrary to the assumption. If $k \neq h$ then $k = J(i)$ for some $i \in [l]$ and $h = I(j)$ for some $j \in [l]$. Symbolically:

$I([l])$ · · $\overset{I(j)=k}{\circ}$ · · ·

$J([l])$ · · · $\underset{J(i)=k}{\circ}$ · ·

where the black dots indicate the elements of $I([l]) \cap J([l])$. But then $I_j = J_i$, which is the possibility considered in Case 1. In fact $I_j = J_i$ if and only if $T_I T_k = (\pm) T_J T_h$, where $k = J(i)$ and $h = I(j)$. Thus Case 4 is exactly Case 1.

We now see that $\mathscr{F}(v_J)^t \cdot v_I = \delta_{IJ} f$, and Proposition 2.2 is proven.

Remarks. (1) We have actually proven more than intended. Namely, if v_I and v_J are the column-vectors of d_{m-1} corresponding to $d_{m-1}(T_I S^{(p)})$ and $d_{m-1}(T_J S^{(q)})$, where $I: [l] \to [n]$ and $J: [l'] \to [n]$ are increasing maps and $m = l + 2p = l' + 2q$, then:

(a) $\mathscr{F}(v_J)^t \cdot v_I = \delta_{IJ} f$ if $p > 0$ or if $p = 0$ and $l = n (= m)$.

(b) If $I \neq J$ then v_I and v_J either have no common nonzero components or they have exactly two.

It is not difficult to show that similar assertions about the row-vectors of the differentials d_m are true without any restrictions (in particular, for any row-vector r we have $r \cdot \mathscr{F}(r)^t = f$).

Clearly the assertions of this remark are true for any system of parameters $y_1, \dots, y_n$ of a regular local ring $(P, \underline{m})$ and any defining equation $f \in \underline{m}(y_1, \dots, y_n)$.

(2) Conjecture 2.1 is now proven in the following cases:

dim 2: quotient singularities, graded Gorenstein rings, hypersurfaces $z^n + f(x, y)$ (k is algebraically closed) (see [M2], as well as minimally elliptic singularities and rational singularities with reduced fundamental cycle (see [B]) (in the last two cases $k = \mathbb{C}$). In dimension 2 the "only if" part of the conjecture is always true if $k = \mathbb{C}$ (J. Wahl).

In every positive dimension the "only if" part is true for hypersurfaces (k is algebraically closed).

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