PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 112, Number 1, May 1991

A UNIQUENESS RESULT FOR A SEMILINEAR REACTION-DIFFUSION SYSTEM

M. ESCOBEDO AND M. A. HERRERO

(Communicated by Barbara L. Keyfitz)

ABSTRACT. Let $(u(t, x), v(t, x))$ and $(\overline{u}(t, x), \overline{v}(t, x))$ be two nonnegative classical solutions of

$$\text{(S)} \qquad \begin{cases} u_t = \Delta u + v^p, & p > 0 \\ v_t = \Delta v + u^q, & q > 0 \end{cases}$$

in some strip $S_T = (0, T) \times \mathbb{R}^N$, where $0 < T \leq \infty$, and suppose that

$$u(0, x) = \overline{u}(0, x), \qquad v(0, x) = \overline{v}(0, x),$$

where $u(0, x)$ and $v(0, x)$ are continuous, nonnegative, and bounded real functions, one of which is not identically zero. Then one has

$$u(t, x) = \overline{u}(t, x), \qquad v(t, x) = \overline{v}(t, x) \quad \text{in } S_T.$$

If $pq \geq 1$, the result is also true if $u(0, x) = v(0, x) = 0$. On the other hand, when $0 < pq < 1$, the set of solutions of (S) with zero initial values is given by

$$u(t; s) = c_1(t - s)_+^{(p+1)/(1-pq)}, \qquad v(t; s) = c_2(t - s)_+^{(q+1)/(1-pq)},$$

where $0 \leq s \leq t$, c_1 and c_2 are two positive constants depending only on p and q, and $(\xi)_+ = \max\{\xi, 0\}$.

1. INTRODUCTION

In this article we shall concern ourselves with the following initial-value problem:

$$\text{(1.1a)} \qquad u_t = \Delta u + v^p; \qquad t > 0, \ x \in \mathbb{R}^N$$

$$\text{(1.1b)} \qquad v_t = \Delta v + u^q; \qquad t > 0, \ x \in \mathbb{R}^N,$$

with $N \geq 1$, $p > 0$, $q > 0$ and

$$\text{(1.2a)} \qquad u(0, x) = u_0(x), \qquad x \in \mathbb{R}^N$$

$$\text{(1.2b)} \qquad v(0, x) = v_0(x), \qquad x \in \mathbb{R}^N,$$

Received by the editors November 30, 1989 and, in revised form, March 1, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35K55, 35K57.
Key words and phrases. Reaction diffusion systems, uniqueness.
The first author's work was partially supported by Basque Government Grant G.V. 127310-1/87.
The second author's work was partially supported by CICYT Grant PB86-0112-C0202.

where $u_0(x)$ and $v_0(x)$ are nonnegative, continuous, and bounded real functions. Equations (1.1) represent a simple example of a reaction-diffusion system, and they can be viewed as a model to describe heat propagation in a two-component combustible mixture. System (1.1) has been analyzed by several authors in the case of bounded and unbounded domains (cf. [7], [8], [5], [4], [3], etc.). In particular, it has been shown in [3] that problem (1.1), (1.2) always has a classical solution in some strip $S_T = [0, T) \times \mathbb{R}^N$ with $0 < T \leq \infty$. By this we shall mean a pair of nonnegative functions, often denoted in the abridged way $(u(t), v(t))$, such that they belong to $C^{2,1}(S_T)$, satisfy (1.1) and (1.2), and remain bounded in any closed strip $\overline{S}_\tau = [0, \tau] \times \mathbb{R}^N$ with $0 < \tau < T$. A number of properties of solutions of (1.1), (1.2) were derived in [3]. In particular (cf. [3, Theorems 2 and 3]) if $pq > 1$ and

$$\frac{\gamma + 1}{pq - 1} \geq \frac{N}{2}, \tag{1.3}$$

where $\gamma = \max\{p, q\}$, every nontrivial solution blows up in finite time, in the sense that it becomes unbounded at some $t = T^* < +\infty$. On the other hand, if $pq > 1$ and (1.3) fails, solutions might be bounded in any strip S_T or have a finite blow-up time, according to the size of their initial values u_0, v_0. To complete this picture, let us point out that solutions are global in time if $0 < pq \leq 1$ (cf. [3, Theorem 1]).

For the scalar equation

$$u_t = \Delta u + u^p; \qquad t > 0, \qquad x \in \mathbb{R}^N, \tag{1.4}$$

it is well known that there is a finite-time blow-up of nontrivial solutions if

$$1 < p \leq 1 + \frac{2}{N}, \tag{1.5}$$

whereas global continuation and blow-up are both possible if $p > 1 + \frac{2}{N}$ (cf., for instance, [8], [2], [9]). Indeed, every solution is global if $0 < p \leq 1$. Notice that (1.3) coincides with (1.5) when $p = q$, in which case (1.1), (1.2) reduce to the Cauchy problem for (1.4) if $u_0 = v_0$.

As recalled in [3], local existence for problem (1.1), (1.2) is rather standard, as can be seen by, say, fixed-point arguments. Uniqueness, however, is not a priori clear except in the straightforward case $p \geq 1$ and $q \geq 1$, and was left open in that paper. For instance, one readily checks that, if $0 < pq < 1$, functions

(1.6a)
$$u_1(t) = c_1 t^\alpha \quad \text{with } \alpha = \frac{p+1}{1-pq}, \qquad c_1^{1-pq} = (1-pq)^{p+1}(p+1)^{-1}(q+1)^{-p}$$

and

$$v_1(t) = c_2 t^\beta \quad \text{with } \beta = \frac{q+1}{1-pq}, \qquad c_2\beta = c_1^q \tag{1.6b}$$

solve (1.1) and are such that $u_1(0) = v_1(0) = 0$. We show here the following result:

Theorem. *Assume that* p *and* q *are different from zero and* $p < 1$ *or* $q < 1$. *Then*

(a) *If* $0 < pq < 1$ *and* $(u_0, v_0) \neq (0, 0)$, *problem* (1.1), (1.2) *has a unique solution.*

(b) *If* $0 < pq < 1$ *and* $(u_0, v_0) = (0, 0)$, *the set of nontrivial nonnegative solutions of* (1.1), (1.2) *is given by*

$$u(t; s) = c_1(t-s)_+^{\alpha}, \qquad v(t; s) = c_2(t-s)_+^{\beta},$$

where $(r)_+ = \max\{r, 0\}$, s *is any nonnegative real constant, and* c_1, c_2, α, *and* β *are as in* (1.6).

(c) *If* $pq \geq 1$, *there is a unique solution of* (1.1), (1.2).

2. Proof of the theorem

We begin by recalling an auxiliary result already proved in [3, Lemma 2.4]:

Lemma 1. *Let* $(u_0, v_0) \neq (0, 0)$, *and let* $(u(t, x), v(t, x))$ *be a solution of* (1.1), (1.2). *Then for any* $\tau > 0$ *there exist constants* $c > 0$ *and* $\alpha > 0$ *such that*

$$u(\tau, x) \geq c \exp(-\alpha|x|^2), \tag{2.1a}$$

$$v(\tau, x) \geq c \exp(-\alpha|x|^2). \tag{2.1b}$$

Proof. We just sketch, for completeness, the case where $u_0 \neq 0$ and $q \geq 1$. Solutions of (1.1), (1.2) satisfy

$$u(t) = S(t)u_0 + \int_0^t S(t-s)v^p(s)\,ds \tag{2.2a}$$

$$v(t) = S(t)v_0 + \int_0^t S(t-s)u^q(s)\,ds, \tag{2.2b}$$

where

$$S(t)f \equiv S(t)f(x) = (4\pi t)^{-N/2} \int_{\mathbf{R}^N} \exp\left(-\frac{|x-\xi|^2}{4t}\right) f(\xi)\,d\xi. \tag{2.3}$$

We may assume without loss of generality that $u_0 > 0$ in some ball centered at the origin. Then there exists $R > 0$ such that $\nu = \inf\{u_0(\xi) : |\tau| \leq R\} > 0$ and, by (2.2a) and (2.3),

$$u(t, x) \geq S(t)u_0 \geq \nu \exp\left(-\frac{|x|^2}{2t}\right) (4\pi\tau)^{-N/2} \int_{|y|\leq R} \exp\left(-\frac{|y|^2}{2t}\right) dy,$$

whence (2.1a) with $\alpha = \frac{1}{2\tau}$ and $c = \nu(4\pi\tau)^{-N/2} \int_{|y|\leq R}(-\frac{|y|^2}{2\tau})dy$. On the other hand, using (2.2b) and Jensen's inequality,

$$\begin{aligned} v(t, x) &\geq \int_0^t S(t-s)(S(s)u_0)^q\,ds \geq \int_0^t (S(t-s)S(s)u_0)^q\,ds \\ &= \int_0^t (S(t)u_0)^q\,ds, \end{aligned}$$

whence

$$v(t, x) \geq t(S(t)u_0)^q . \tag{2.4}$$

From this (2.1b) follows, with perhaps a different choice of c and α.

We now specialize to the case $0 < pq < 1$.

Lemma 2. *Let* $(u(t, x), v(t, x))$ *be a nontrivial solution of* (1.1) *with* $0 < pq < 1$. *Then*

$$u(t, x) \geq c_1 t^{\alpha} \tag{2.5a}$$

$$v(t, x) \geq c_2 t^{\beta} , \tag{2.5b}$$

where c_1, c_2, α, *and* β *are given in* (1.6).

Proof. Assume first that $u_0(x) \geq c\exp(-\alpha|x|^2)$ for some $c > 0$ and $\alpha > 0$. Suppose, for definiteness, that $0 < p, q < 1$. Since

$$S(t)\exp(-\alpha|x|^2) = (1+4\alpha t)^{-N/2}\exp\left(-\frac{\alpha|x|^2}{1+4\alpha t}\right) , \tag{2.6}$$

it follows that

$$u(t, x) \geq S(t)u_0 \geq c(1+4\alpha t)^{-N/2}\exp\left(-\frac{\alpha|x|^2}{1+4\alpha t}\right)$$

and, by (2.2b),

$$v(t, x) \geq \int_0^t S(t-s)(u(s, x))^q\, ds ,$$

whence

$$\begin{aligned} v(t, x) \geq c^q \int_0^t &(1+4\alpha s)^{-N/2(q-1)}(1+4\alpha s+4\alpha q(t-s))^{-N/2} \\ &\times \exp\left(-\frac{\alpha q|x|^2}{1+4\alpha s+4\alpha q(t-s)}\right) ds \\ \geq{}& c^q(1+4\alpha t)^{-N/2}\exp\left(-\frac{\alpha q|x|^2}{1+4\alpha q t}\right) t . \end{aligned}$$

We substitute this inequality in (2.2a) and use (2.6) to get

$$\begin{aligned} u(t, x) \geq c^{pq} \int_0^t &(1+4\alpha s)^{-Np/2} s^p (1+4\alpha q s+4\alpha p q(t-s))^{-N/2}(1+4\alpha q s)^{N/2} \\ &\times \exp\left(-\frac{\alpha p q|x|^2}{1+4\alpha q s+4\alpha p q(t-s)}\right) ds , \end{aligned}$$

whence

$$u(t, x) \geq c^{pq}(1+4\alpha t)^{-Np/2}(1+4\alpha q t)^{-N/2}\exp\left(-\frac{\alpha p q|x|^2}{1+4\alpha p q t}\right)\frac{t^{p+1}}{p+1} .$$

By induction, we obtain, for $j = 0, 1, 2, \ldots,$

$$u(t, x) \geq c^{(pq)^{j+1}} f_j(t) \exp\left(-\frac{\alpha(pq)^{j+1}|x|^2}{1+4\alpha(pq)^{j+1}t}\right) B_{j+1} A_{j+1} t^{(p+1)} \times [(pq)^j + (pq)^{j-1} + \cdots + pq + 1], \tag{2.7}$$

where

(2.8a)

$$f_j(t) = (1+4\alpha t)^{-(N/2)p^{j+1}q^j}(1+4\alpha q t)^{-(N/2)p^j q^j}(1+4\alpha pqt)^{-(N/2)p^j q^{j-1}} \ldots (1+4\alpha p^j q^{j+1} t)^{-N/2}$$

$$B_{j+1} = \left(\frac{1}{p+1}\right)^{(pq)^j}\left(\frac{1}{(p+1)(pq+1)}\right)^{(pq)^{j-1}} \cdots \left(\frac{1}{(p+1)((pq)^j + (pq)^{j-1} + \cdots + 1)}\right) \tag{2.8b}$$

$$A_{j+1} = \left[\left(\frac{1}{(p+1)(q+1)}\right)^{q^{j-1}p^{j-1}}\left(\frac{1}{((p+1)(pq+1)q+1)}\right)^{q^{j-2}p^{j-2}} \cdots \left(\frac{1}{((p+1)((pq)^{j-1}+(pq)^{j-2}+\cdots+1)q+1)}\right)\right]^p. \tag{2.8c}$$

Notice that

(2.9)

$$\begin{aligned} B_{j+1} &= \left(\frac{1}{p+1}\right)^{(1-(pq)^{j+1})/(1-pq)}[(1+pq)^{-(pq)^{j-1}} \ldots ((pq)^j + (pq)^{j-1} + \cdots + 1)^{-1}] \\ &\geq \left(\frac{1}{p+1}\right)^{1/(1-pq)}(1-pq)^{1/(1-pq)}, \end{aligned}$$

whereas

$$\begin{aligned} A_{j+1} &= \left(\prod_{k=0}^{j-1}((p+1)((pq)^k + (pq)^{k-1} + \cdots + pq + 1)q + 1)^{-(pq)^{j-1-k}}\right)^p \\ &= \left(\prod_{k=0}^{j-1}\left((p+1)\left(\frac{1-(pq)^{k+1}}{1-pq}\right)q + 1\right)^{-(pq)^{j-1-k}}\right)^p \\ &\geq \left(\prod_{k=0}^{j-1}\left(\frac{(p+1)q}{1-pq} + 1\right)^{-(pq)^{j-1-k}}\right)^p = \left(\prod_{k=0}^{j-1}\left(\frac{q+1}{1-pq}\right)^{-(pq)^{j-1-k}}\right)^p, \end{aligned}$$

so that

$$A_{j+1} \geq \left(\frac{1-pq}{q+1}\right)^{p/(1-pq)}. \tag{2.10}$$

Finally,

$$f_j(t) = \left(\prod_{k=0}^{j}(1+4\alpha(pq)^k t)^{-(N/2)p(pq)^{j-k}}\right)\left(\prod_{k=0}^{j}(1+4\alpha p^k q^{k+1} t)^{-(N/2)(pq)^{j-k}}\right)$$
$$\equiv (f_j^1(t))(f_j^2(t))$$

and, since

$$4\alpha t(pq)^j(j+1) = \sum_{k=0}^{j}(pq)^{j-k}4\alpha(pq)^k t \geq \sum_{k=0}^{j}(pq)^{j-k}\log(1+4\alpha(pq)^k t),$$

we deduce that

$$\text{(2.11)} \qquad \lim_{j\to\infty} f_j^{(i)}(t) \geq 1 \quad \text{for } i = 1, 2.$$

Taking into account (2.8)–(2.11), we let $j \to \infty$ in (2.7) to obtain (2.5a) under our current assumptions. Estimate (2.5b) can be obtained in an analogous way. The case where p or q are larger than or equal to 1 is similarly dealt with.

As to the general case, we take $\varepsilon > 0$ but otherwise arbitrary, and set $u_\varepsilon(t) \equiv u(t+\varepsilon)$. One then has

$$u_\varepsilon(t) = S(t)u_\varepsilon(0) + \int_0^t S(t-s)v_\varepsilon^p(s)\,ds,$$

where, by Lemma 1, $u_\varepsilon(0) \geq c\exp(-\alpha|x|^2)$ for some c and α. Therefore $u_\varepsilon(t) \geq c_1 t^\alpha$, and accordingly

$$u(t) = u(\varepsilon + (t-\varepsilon)) \geq c_1(t-\varepsilon)^\alpha,$$

whence the result, since $\varepsilon > 0$ is arbitrary.

As a next step, we show

Lemma 3. *Assume that* $0 < pq < 1$, *and suppose that* $(u_0, v_0) \neq (0, 0)$. *Then there exists at most one solution of* (1.1), (1.2).

Proof. Suppose first that $0 < p < 1$ and $0 < q < 1$. We shall argue by contradiction, thus assuming that for some $(u_0, v_0) \neq (0, 0)$ there exist two different solutions $(u(t), v(t))$ and $(\overline{u}(t), \overline{v}(t))$ defined in some strip S_T. It then follows from (2.2), (2.5), and the mean value theorem that

$$\text{(2.12)}\,(u(t)-\overline{u}(t))_+ \leq \int_0^t S(t-s)(v^p(s)-\overline{v}^p(s))_+\,ds$$
$$\leq pc_2^{p-1}\int_0^t S(t-s)(v(s)-\overline{v}(s))_+ s^{(p-1)(q+1)/(1-pq)}\,ds.$$

In a similar way, we get

$$\text{(2.13)} \qquad (v(t)-\overline{v}(t))_+ \leq qc_1^{q-1}\int_0^t S(t-s)(u(s)-\overline{u}(s))_+ s^{(q-1)(p+1)/(1-pq)}\,ds.$$

From (2.12) and (2.13), it follows that, setting $\|f(t\,,.)\|_\infty = \sup_{x\in\mathbf{R}^N} |f(t\,,.)|$,

(2.14)

$$\begin{aligned}
&\|(u-\overline{u})_+(t)\|_\infty \\
&\le pqc_1^{q-1}c_2^{p-1}\int_0^t s^{(q+1)(p-1)/(1-pq)}\left(\int_0^s \tau^{(p+1)(q-1)/(1-pq)}\|(u-\overline{u})_+(\tau)\|_\infty\,d\tau\right)ds \\
&= pq(1-pq)^{-2}(p+1)(q+1) \\
&\quad\times\int_0^t s^{(q+1)(p-1)/(1-pq)}\left(\int_0^s \tau^{(p+1)(q-1)/(1-pq)}\|(u-\overline{u})_+(\tau)\|\,d\tau\right)ds.
\end{aligned}$$

We next show that the integrand above is indeed locally integrable. To this end, we first notice that, since p and q are less than one,

(2.15)

$$\|(u-\overline{u})_+(t)\| \le \int_0^t \|(v-\overline{v})_+(s)\|_\infty^p\,ds \le \int_0^t\left(\int_0^s \|(u-\overline{u})_+(\tau)\|_\infty^q\,d\tau\right)^p ds,$$

so that

$$\|(u-\overline{u})_+(t)\|_\infty \le (p+1)^{-(1/(1-pq))}t^{(p+1)/(1-pq)}. \tag{2.16}$$

This implies that the right-hand side in (2.14) is convergent. Moreover, substituting (2.16) in (2.14) yields

$$\|(u-\overline{u})_+(t)\|_\infty \le pq(p+1)^{-(1/(1-pq))}t^{(p+1)/(1-pq)}. \tag{2.17}$$

We may now use (2.17) to obtain a new bound for $\|(u-\overline{u})_+(t)\|_\infty$ via (2.14). Iterating this procedure k times, we obtain

$$\|(u-\overline{u})_+(t)\|_\infty \le (pq)^k(p+1)^{-(1/(1-pq))}t^{(p+1)/(1-pq)}. \tag{2.18}$$

Now letting $k\to\infty$, it follows that $u\equiv\overline{u}$, whence $v\equiv\overline{v}$.

It remains to consider yet the situation where one of the exponents $p\,,q$ is larger than or equal to one. Assume for instance that $q>1$, and set $f(s)=s^p$, $g(v) = S(s)v_0 + \int_0^s S(s-\tau)v^q(\tau)d\tau$. Using (2.2) as well as the mean value theorem for $(f\circ g)$, it follows that for some $w=\theta u+(1-\theta)\overline{u}$ with $\theta\equiv\theta(\tau)$ and $0<\theta<1$,

$$\begin{aligned}
u(t)-\overline{u}(t) \le pq\int_0^t S(t-s)&\left(\int_0^s S(s-\tau)w^q(\tau)\,d\tau\right)^{p-1} \\
&\times\left(\int_0^s S(s-\tau)w^{q-1}(\tau)(u-\overline{u})(\tau)\,d\tau\right)ds.
\end{aligned} \tag{2.19}$$

On the other hand, by the Hölder inequality,

$$\begin{aligned}
&\int_0^s S(s-\tau)w^{q-1}(\tau)(u-\overline{u})(\tau)\,d\tau \\
&\qquad\le\left(\int_0^s S(s-\tau)|w(\tau)|^q\,d\tau\right)^{(q-1)/q}\left(\int_0^s S(s-\tau)|(u-\overline{u})(\tau)|^q\,d\tau\right)^{1/q},
\end{aligned}$$

and therefore

$$\|(u-\overline{u})_+(t)\|_\infty \le pq\int_0^t \left(\int_0^s S(s-\tau)|w(\tau)|^q\,d\tau\right)^{p-1+(q-1)/q} \times \left(\int_0^s S(s-\tau)|(u-\overline{u})(\tau)|^q\,d\tau\right)^{1/q} ds.$$

Taking (2.5) into account again, we obtain

$$\|(u-\overline{u})_+(t)\| \le c\int_0^t s^{-(q+1)/q}\left(\int_0^s \|(u-\overline{u})_+(\tau)\|_\infty^q\,d\tau\right)^{1/q} ds, \tag{2.20}$$

where $c = pq(1-pq)^{-(q+1)/q}(q+1)^{1/q}(p+1)$. We now claim that

$$v^p(t) - \overline{v}^p(t) \le \left(\int_0^t S(t-s)(u-\overline{u})_+^q(s)\,ds\right)^p. \tag{2.21}$$

Assuming this inequality for the moment, the proof is complete, since (2.12) implies that

$$\|(u-\overline{u})_+(t)\|_\infty \le \int_0^t \|v^p(s)-\overline{v}^p(s)\|_\infty\,ds \le \int_0^t\left(\int_0^s \|(u-\overline{u})_+(\tau)\|_\infty^q\,d\tau\right)^p ds,$$

so that (2.16) holds, and the result follows at once from (2.20) and (2.16), as in the previous case. To show (2.21), let us write

$$K_t(x) = (4\pi t)^{-N/2}\exp\left(-\frac{|x|^2}{4t}\right).$$

For arbitrary $\alpha\in(0,1)$, one then has

$$\begin{aligned} v(t) &= S(t)v_0 + \int_0^t S(t-s)u^q(s)\,ds = S(t)v_0 + \int_0^t\int_{\mathbb{R}^N} K_{t-s}(x-y)u^q(s,y)\,dy\,ds \\ &= S(t)v_0 + \int_0^t\int_{\mathbb{R}^N} K_{t-s}^{(q-\alpha)/q}(x-y)u^{q-\alpha}(s,y)K_{t-s}^{\alpha/q}(x-y)u^\alpha(s,y)\,dy\,ds. \end{aligned}$$

Since $u = \overline{u} + (u-\overline{u}) \le \overline{u} + (u-\overline{u})_+$ and $u^\alpha \le \overline{u}^\alpha + (u-\overline{u})_+^\alpha$, we have

$$\begin{aligned} v(t) \le S(t)v_0 &+ \int_0^t\int_{\mathbb{R}^N} K_{t-s}^{(q-\alpha)/q}(x-y)u^{q-\alpha}(s,y)\cdot K_{t-s}^{\alpha/q}(x-y)\overline{u}^\alpha(s,y)\,dy\,ds \\ &+ \int_0^t\int_{\mathbb{R}^N} K_{t-s}^{(q-\alpha)/q}(x-y)u^{q-\alpha}(s,y)K_{t-s}^{\alpha/q}(x-y)(u-\overline{u})_+^\alpha(s,y)\,dy\,ds. \end{aligned}$$

Using Hölder's inequality, we obtain

$$\begin{aligned} v(t) &\le S(t)v_0 + \int_0^t \left(\int_{\mathbb{R}^N} K_{t-s}(x-y)u^q(s,y)dy\right)^{(q-\alpha)/q} \\ &\quad \times \left(\int_{\mathbb{R}^N} K_{t-s}(x-y)\overline{u}^q(s,y)\,dy\right)^{\alpha/q} ds \\ &\quad + \int_0^t \left(\int_{\mathbb{R}^N} K_{t-s}(x-y)u^q(s,y)\,dy\right)^{(q-\alpha)/q} \\ &\quad \times \left(\int_{\mathbb{R}^N} K_{t-s}(x-y)(u-\overline{u})_+^q(s,y)\,dy\right)^{\alpha/q} ds \\ &\le S(t)v_0 + \left(\int_0^t S(t-s)u^q(s)\,ds\right)^{(q-\alpha)/q} \\ &\quad \times \left(\int_0^t S(t-s)\overline{u}^q(s)\,ds\right)^{\alpha/q} \\ &\quad + \left(\int_0^t S(t-s)u^q(s)\,ds\right)^{(q-\alpha)/q} \left(\int_0^t S(t-s)(u-\overline{u})_+^q(s)\,ds\right)^{\alpha/q}, \end{aligned}$$

so that, using the fact that for any nonnegative a, b, and c and any $\theta \in (0, 1)$,

$$a + b^{1-\theta}c^\theta \le (a+b)^{1-\theta}(a+c)^\theta,$$

one arrives at

$$v(t) \le v(t)^{(q-\alpha)/q}\overline{v}(t)^{\alpha/q} + v(t)^{(q-\alpha)/q}\left(\int_0^t S(t-s)(u-\overline{u})_+^q(s)\,ds\right)^{\alpha/q}$$

and, setting $\alpha = pq$, (2.21) follows.

Concerning the case where $u_0 = v_0 = 0$, we have the following lemma:

Lemma 4. *Assume that* $0 < pq < 1$ *and* $u_0 = v_0 = 0$. *The set of nontrivial nonnegative solutions of* (1.1), (1.2) *is then given by the family*

$$u(t;s) = c_1(t-s)_+^{(p+1)/(1-pq)}, \qquad v(t;s) = c_2(t-s)_+^{(q+1)/(1-pq)},$$

where $0 \le s < t$ *and* c_1, c_2 *are the constants in* (1.6).

Proof. The proof consists of a slight modification of the corresponding result for the scalar equation (1.4) (cf. [1]). Let (u, v) be a nontrivial solution of (1.1), (1.2) under our current hypotheses. Then, by (2.2)

$$\|u(t)\|_\infty \le \int_0^t \left(\int_0^s \|u(\tau)\|_\infty^q\,d\tau\right)^p ds, \ \|v(t)\|_\infty \le \int_0^t \left(\int_0^s \|v(\tau)\|_\infty^p\,d\tau\right)^q ds,$$

so that

$$\|u(t)\|_\infty \le c_1 t^{(p+1)/(1-pq)}, \qquad \|v(t)\|_\infty \le c_2 t^{(q+1)/(1-pq)}, \tag{2.22}$$

with c_1, c_2 as in (1.6). By hypothesis, there exist $t > 0$ and $x \in \mathbb{R}^N$ such that $u(t, x) > 0$ or $v(t, x) > 0$. Assume, for definiteness, that $u(t, x) > 0$, and define τ as follows

$$\tau = \inf\{t > 0 : u(t, x) > 0\}.$$

By standard results, $u(t, x) > 0$ and $v(t, x) > 0$ for any $x \in \mathbb{R}^N$ and $t > \tau$. Now take $\bar{t} > \tau$ and set

$$\overline{u}(t, x) = u(t + \bar{t}, x), \qquad \overline{v}(t, x) = v(t + \bar{t}, x).$$

Then $(\overline{u}, \overline{v})$ solves (1.1) and $\overline{u}(0, x) > 0$, $\overline{v}(0, x) > 0$. Therefore, by Lemma 2,

$$u(t + \bar{t}, x) \geq c_1 t^{(p+1)/(1-pq)}, \qquad v(t + \bar{t}, x) \geq c_2 t^{(q+1)/(1-pq)}$$

for any $t \geq 0$. This implies that

$$u(t, x) \geq c_1 (t - \tau)_+^{(p+1)/(1-pq)}, \qquad v(t, x) \geq c_2 (t - \tau)_+^{(q+1)/(1-pq)} \tag{2.23}$$

for $x \in \mathbb{R}^N$, $t \geq 0$. Now choose $\underline{t} < \tau$ and define

$$\underline{u}(t, x) = u(t + \underline{t}, x), \qquad \underline{v}(t, x) = v(t + \underline{t}, x).$$

By our choice of τ, $(\underline{u}, \underline{v})$ solves (1.1), (1.2) with $\underline{u}(0) = \underline{v}(0) = 0$. Therefore, by (2.22)

$$u(t + \underline{t}, x) \leq c_1 t^{p+1/1-pq}, \qquad v(t + \underline{t}, x) \leq c_2 t^{(q+1)/(1-pq)}$$

for any $t \geq 0$ and $x \in \mathbb{R}^N$, and this implies

$$u(t, x) \leq c_1 (t - \tau)_+^{(p+1)/(1-pq)}, \qquad v(t, x) \leq c_2 (t - \tau)_+^{(q+1)/(1-pq)} \tag{2.23}$$

for $x \in \mathbb{R}^N$, $t \geq 0$, and the conclusion follows from (2.22) and (2.23).

Our last step consists of this lemma:

Lemma 5. *Let $p > 0$ and $q > 0$ be such that $pq \geq 1$. Then, if $(u(t), v(t))$ and $(\overline{u}(t), \overline{v}(t))$ are solutions of* (1.1), (1.2) *in some strip S_T, it follows that $u(t) = \overline{u}(t)$ and $v(t) = \overline{v}(t)$ in S_T.*

Proof. The result is rather classical if $p \geq 1$ and $q \geq 1$. Assume for instance that $0 < p < 1 \leq q$. As in the proof of Lemma 3, we set $f(s) = s^p$, $g(v)(s) = S(s)v_0 + \int_0^s S(s - \tau)v^q(\tau)d\tau$ and use the Mean Value Theorem to get (2.19). We now set

$$F(t) = \sup_{0 \leq s \leq t} \|(u - \overline{u})(s)\|_\infty, \qquad 0 < t < T.$$

Clearly

$$\begin{aligned}
&\int_0^s S(s-\tau)w^{q-1}(\tau)(u-\overline{u})(\tau)\,d\tau \\
&\quad\le F(t)\int_0^s S(s-\tau)w^{q-1}(\tau)\,d\tau \\
&\quad\le F(t)\left(\int_0^s S(s-\tau)w^q(\tau)\,d\tau\right)^{(q-1)/q} \\
&\quad\le F(t)\left\|\int_0^s S(s-\tau)w^q(\tau)\,d\tau\right\|_\infty^{(q-1)/q+p-1}\cdot\left(\int_0^s S(s-\tau)w^q(\tau)\,d\tau\right)^{1-p},
\end{aligned}$$

and substituting this into (2.19) yields

$$\|(u-\overline{u})(t)\|_\infty \le pqF(t)\int_0^t S(t-s)\left\|\int_0^s S(s-\tau)w^q(\tau)\right\|_\infty^{p-1/q} ds.$$

As $pq \ge 1$, one is thus led to

$$F(t) \le KtF(t) \quad \text{for } t \text{ small enough}, \tag{2.25}$$

for some constant K depending on p, q, T and the bounds on u and $\overline{u}$ in $\mathbb{R}^N \times [0, t]$. Since (2.25) implies $F(t) \equiv 0$ for t small enough, the result follows by a suitable iteration of the previous argument. The case $0 < q < 1 \le p$ is similar.

References

1. J. Aguirre and M. Escobedo, *A Cauchy problem for $u_t - \Delta u = u^p$ with $0 < p < 1$: Asymptotic behaviour of solutions*, Ann. Fac. Sci. Toulouse **8** (1986–87), 175–203.
2. D. G. Aronson and H. F. Weinberger, *Multidimensional nonlinear diffusion arising in population genetics*, Adv. in Math. **30** (1978), 33–76.
3. M. Escobedo and M. A. Herrero, *Boundedness and blow-up for a semilinear reaction diffusion system*, J. Differential Equations (to appear).
4. M. Floater and J. B. McLeod, in preparation.
5. A. Friedman and Y. Giga, *A single point blow up for solutions of semilinear parabolic systems*, J. Fac. Sci. Univ. of Tokyo, Sect. I **34** (1987), 65–79.
6. H. Fujita, *On the blowing up of solutions of the Cauchy problem for $u_t - \Delta u = u^{1+\alpha}$*, J. Fac. Sci. Univ. of Tokyo, Sect. I **13** (1960), 109–124.
7. V. A. Galaktionov, S. P. Kurdyumov and A. A. Samarskii, *A parabolic system of quasilinear equations* I, Differential Equations **19** (1983), 2123–2143.
8. ——, *A parabolic system of quasilinear equations* II, Differential Equations **21** (1985), 1544–1559.
9. F. B. Weissler, *Existence and nonexistence of global solutions for a semilinear heat equation*, Israel J. of Math. **38** (1981), 29–40.

Departamento de Matemáticas, Facultad de Ciencias, Universidad del Pais Vasco, Apartado 644, 48080 Bilbao, Spain

Departamento de Matemática Aplicada, Facultad de Matemáticas, Universidad Complutense, 28040 Madrid, Spain