THE RADEMACHER COTYPE
OF OPERATORS FROM $l_\infty^N$

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Abstract. We show that for any operator $T: l_\infty^N \to Y$, where $Y$ is a Banach space, its cotype 2 constant, $K^{(2)}(T)$, is related to its $(2, 1)$-summing norm, $\pi_{2,1}(T)$, by

$$K^{(2)}(T) \leq c \log \log N \pi_{2,1}(T).$$

Thus, we can show that there is an operator $T: C(K) \to Y$ that has cotype 2, but is not 2-summing.

Introduction

The notation we use in this paper is loosely based on that given in [LT1, L-T2, P1].

We let $\varepsilon_1, \varepsilon_2, \ldots$ be independent Rademacher random variables, that is, $\Pr(\varepsilon_1 = 1) = \Pr(\varepsilon_1 = -1) = \frac{1}{2}$. A linear operator $T: X \to Y$ is said to have (Rademacher) cotype $p$ ($p > 2$) if there is a constant $C < \infty$ such that for all $x_1, x_2, \ldots, x_s$ in $X$ we have

$$\left( \sum_{s=1}^{S} \|T(x_s)\|^p \right)^{1/p} \leq C \mathbb{E} \left\| \sum_{s=1}^{S} \varepsilon_s x_s \right\|.$$

The smallest value of $C$ is called the (Rademacher) cotype $p$ constant of $T$, and is denoted by $K^{(p)}(T)$. These definitions extend to spaces in the obvious way; a space $X$ has cotype $p$ if its identity operator has cotype $p$.

We define the $(p, q)$-summing norm of a linear operator $T: X \to Y$, denoted by $\pi_{p,q}(T)$, to be the least number $C$ such that for all $x_1, x_2, \ldots, x_s$ in $X$ we have

$$\left( \sum_{s=1}^{S} \|T(x_s)\|^p \right)^{1/p} \leq C \sup \left\{ \left( \sum_{s=1}^{S} \langle x^*, x_s \rangle^q \right)^{1/q} \right\},$$

where the supremum is taken over all $x^*$ in the unit ball of $X^*$. We call a $(p, p)$-summing operator a $p$-summing operator, and write $\pi_p(T)$ for $\pi_{p,p}(T)$.
We say that the operator is \((p, q)\)-summing \((p\text{-summing})\) if \(\pi_{p,q}(T) < \infty\) (respectively \(\pi_p(T) < \infty\)).

If \(1 \leq p < \infty\) and \(1 \leq q \leq \infty\) then we let \(L_{p,q}(\mu)\) denote the Lorentz space on the measure \(\mu\). We refer the reader to [H] or [LT2] for details, but note that the \(L_{p,1}\) norm may be calculated using
\[
\|f\|_{p,1} = \int_0^\infty \mu(|f| > t)^{1/p} \, dt = \frac{1}{p} \int_0^\infty s^{1/p-1} f^*(s) \, ds,
\]
where \(f^*\) denotes the nondecreasing rearrangement of \(|f|\).

The basic motivation behind this paper is in classifying operators from \(C(K)\) that factor through a Hilbert space, where \(C(K)\) denotes the continuous functions on the compact Hausdorff topological space \(K\). The first result in this direction is due to Grothendieck, which states that any bounded linear operator \(C(K) \to L_1\) factors through Hilbert space. This was generalized by Maurey [Mai], allowing \(L_1\) to be replaced by any space of cotype 2, to give the following result (see also [P1]).

**Theorem 1.** Let \(T : C(K) \to Y\) be a linear operator, where \(Y\) is any Banach space. Then the following are equivalent:

(i) \(T\) is 2-summing;

(ii) \(T\) factors through Hilbert space;

(iii) \(T\) factors through a space of cotype 2.

However, we are still left with the following question: if the operator \(T : C(K) \to Y\) has cotype 2, does it follow that it factors through Hilbert space?

One way one might tackle this problem is to consider the \((2, 1)\)-summing norms of such operators. Jameson [J] showed that there is an operator \(T : l_\infty^N \to Y\) such that \(\pi_2(T) \geq c^{-1} \sqrt{\log N} \pi_{2,1}(T)\). Hence, if we can establish a strong relationship between the cotype 2 constants and the \((2, 1)\)-summing norms of such operators, then we can answer the above question in the negative. To this end, we have the following—the main result of this paper.

**Theorem 2.** There is a constant \(c\) such that for any operator \(T : l_\infty^N \to Y\), where \(Y\) is a Banach space, the cotype 2 constant is bounded according to the relation
\[
K^{(2)}(T) \leq c \log \log N \pi_{2,1}(T).
\]

**Corollary.** There is an operator \(T : C(K) \to Y\), where \(Y\) is a Banach space, that has cotype 2, but does not factor through Hilbert space.

Finally, before embarking on the proof of this result, we point out that for \(p > 2\), the above problems have been completely answered.

**Theorem 3.** Let \(T : C(K) \to Y\) be a bounded linear operator, where \(Y\) is a Banach space. Then for all \(p > 2\), the following are equivalent:

(i) \(T\) is \((p, 1)\)-summing;

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(ii) $T$ has cotype $p$;
(iii) $T$ factors through a space with cotype $p$.

The implication (i) $\iff$ (ii) is due to Maurey [Ma2]. The third equivalence follows from the fact that any $(p, 1)$-summing operator from $C(K)$ factors through $L_{p,1}$ (see [P2] or Theorem 5 below), and that $L_{p,1}$ has Rademacher cotype $p$ (see [C]).

**Theorem 4.** If $p > 2$, then there is a bounded linear operator $C(K) \to L_p$ that is not $p$-summing.

We refer the reader to [K].

**Proof of the main result**

To prove Theorem 2, we need the following two results. The first allows us to reduce questions about $(p, 1)$-summing operators from $C(K)$ to the canonical embedding $C(K) \to L_{2,1}(K, \mu)$ ($\mu$ a probability measure), and is due to Pisier (see [P2]).

**Theorem 5.** Let $T : C(K) \to Y$ be a $(p, 1)$-summing operator, where $Y$ is a Banach space and $p \geq 1$. Then there is a Radon probability measure $\mu$ on $K$ and a constant $C \leq p^{1/p} \pi_{p,1}(T)$ such that for all $x \in C(K)$ we have $\|Tx\| \leq C \|x\|_{L_{p,1}(K, \mu)}$.

The second result is about Rademacher processes, and is due to the second named author (for the proof, see [LdT]). First we establish some more notation. If $T$ is a bounded subset of $R^S$, we write

$$r(T) = E \sup_{t \in T} \left| \sum_{s=1}^S \epsilon_s t(s) \right|.$$ 

If $B$ is a subset of $R^S$, we write $\mathcal{N}(T, B)$ for the minimal number of translates of $B$ required to cover $D$. We write $B_1^S$ for the unit ball of $l_1^S$, and $B_2^S$ for the unit ball of $l_2^S$. From now on, we take all logarithms to base 2.

**Theorem 6.** There is a constant $c_1$ such that if $T$ is a bounded subset of $R^S$ and $\epsilon > 0$, then letting $D = c_1 r(T) B_1^S + \epsilon B_2^S$, we have

$$r(T) \geq c_1^{-1} \epsilon \sqrt{\log \mathcal{N}(T, D)}.$$ 

Now we will state the main result towards proving Theorem 2.

**Proposition 7.** There is a constant $c_2$ such that if $(\Omega, \mathcal{F}, \mu)$ is a probability space with $N$ atoms, and $x_1, x_2, \ldots, x_S \in L_\infty(\mu)$ are such that

$$E \left\| \sum_{s=1}^S \epsilon_s x_s \right\|_\infty \leq 1,$$
then

\[
\left( \sum_{s=1}^{S} \|x_s\|^2_{L^2,1(\mu)} \right)^{1/2} \leq c_2 \log \log N.
\]

Our first step in establishing this result is to restate Theorem 6 in a more suitable form.

**Lemma 8.** There is a constant \( c_1 \) (the same one as in Theorem 6) such that the following holds. Suppose that \((\Omega, \mathcal{F}, \mu)\) is a measure space, with \( \Omega \) finite, and \( x_1, x_2, \ldots, x_S \in L_\infty(\mu) \) with

\[
\sum_{s=1}^{S} x_s < c_2 \log \log N.
\]

Then for all integers \( k \), we may partition \( \Omega \) into at most \( 2^k \) measurable sets, find \( y_1, y_2, \ldots, y_S, z_1, z_2, \ldots, z_S \in L_\infty(\mu) \), and find \( x_1, x_2, \ldots, x_S \in L_\infty(\Omega, \mathcal{F}', \mu) \) (where \( \mathcal{F}' \) denotes the algebra generated by the partition), such that \( x_s = x_s + y_s + z_s \),

\[
E \left\| \sum_{s=1}^{S} \varepsilon_s x_s \right\|_{\infty} \leq 1.
\]

Proof. Let \( T = \{ (x_s(\omega))_{s=1}^{S} : \omega \in \Omega \} \), and let \( \varepsilon = c_1^{-1} 2^{-k/2} \). If we apply Theorem 6, we see that there are \( 2^{2^k} \) translates, \( t_l + c_1(B_1^S + 2^{-k/2}B_2^S) \) \((1 \leq l \leq 2^{2^k})\), that cover \( T \). We let the covering of \( \Omega \) be the sets

\[
A_l = \{ \omega : (x_s(\omega))_{s=1}^{S} \in t_l + c_1(B_1^S + 2^{-k/2}B_2^S) \},
\]

and if \( A_l \) is nonempty, we choose \( \omega_l \in A_l \). Define \( x_s(\omega) = x_s(\omega_l) \) if \( \omega \in A_l \). Now, if \( \omega \in A_l \), we know that \( (x_s(\omega) - x_s(\omega_l))_{s=1}^{S} \in c_1(B_1^S + 2^{-k/2}B_2^S) \), that is, there are \( (y_s(\omega))_{s=1}^{S} \in c_1B_1^S \) and \( (z_s(\omega))_{s=1}^{S} \in c_12^{-k/2}B_2^S \), with \( x_s(\omega) = x_s(\omega_l) + y_s(\omega_l) + z_s(\omega_l) \). \( \Box \)

**Lemma 9.** There is a constant \( c_3 \) such that if \((\Omega, \mathcal{F}, \mu)\) is a measure space with \( \Omega \) finite, then the following hold.

(i) If \( y \in L_\infty(\mu) \), then \( \|y\|_{2,1} \leq \|y\|_{1/2}^{1/2} \|y\|_{1/2}^{1/2} \).

(ii) If the smallest atom is of size \( a \) , then for all \( z \in L_\infty(\mu) \) we have

\[
\|z\|_{2,1} \leq c_3 \left( 1 + \sqrt{\log(\mu(\Omega)/a)} \right) \|z\|_2.
\]

(iii) If there are \( N \) atoms, then for all \( z \in L_\infty(\mu) \) we have \( \|z\|_{2,1} \leq \sqrt{N} \|z\|_2 \).
Proof. (i) We have that
\[
\|y\|_{2,1} = \int_0^\infty \frac{\sqrt{\mu(|y| > t)}}{t} dt \leq \left( \int_0^\infty dt \right)^{1/2} \left( \int_0^\infty \mu(|y| > t) dt \right)^{1/2} = \|y\|_\infty^{1/2} \|y\|_{1,2}^{1/2}.
\]

(ii) We have
\[
\|z\|_{2,1} = \frac{1}{2} \int_0^\infty \frac{z^*(s)}{\sqrt{s}} ds \leq \sqrt{\alpha} \|z\|_\infty + \frac{1}{2} \int_\alpha^\mu(\Omega) \frac{z^*(s)}{\sqrt{s}} ds \leq \|z\|_2 + \frac{1}{2} \left( \int_\alpha^\mu(\Omega) \frac{ds}{s} \right)^{1/2} \left( \int_\alpha^\mu(\Omega) \left( z^*(s) \right)^2 ds \right)^{1/2} \leq c_3 \left( 1 + \sqrt{\log(\mu(\Omega)/\alpha)} \right) \|z\|_2.
\]

(iii) Let \( B_1, B_2, \ldots, B_N \) be the atoms of \( \Omega \) arranged so that \( z^*(n) \), the value of \( |z| \) on \( B_n \), is in nonincreasing order. Also, let \( z^*(N + 1) = 0 \). Then
\[
\|z\|_{2,1} = \sum_{n=1}^N \left( \sum_{m=1}^n \mu(B_m) \right)^{1/2} \left( z^*(n) - z^*(n + 1) \right)^{1/2} \leq \sqrt{N} \left( \sum_{n=1}^N \sum_{m=1}^n \mu(B_m) \right) \left( z^*(n) - z^*(n + 1) \right)^2 \leq \sqrt{N} \left( \sum_{n=1}^N \mu(B_n) \right) \left( z^*(n) \right)^2 \leq \sqrt{N} \|z\|_2,
\]
as desired. \( \square \)

We remark that Lemma 9(i) is a well-known interpolation result, and is true for all measure spaces.

Lemma 10. If \( (\Omega, \mathcal{F}, \mu) \) is a probability space with \( \Omega \) finite, then
\[
\left( \sum_{s=1}^S \|y_s\|_{2,1}^2 \right)^{1/2} \leq \left\| \sum_{s=1}^S |y_s| \right\|_\infty.
\]

Proof. This follows straight away from Lemma 9(i). \( \square \)

Lemma 10 is also well known (and true for all probability spaces). In fact it is a reformulation of the statement that the canonical embedding \( C(\Omega) \to L_{2,1}(\mu) \) has \((2,1)\)-summing norm equal to 1.
Lemma 11. There is a constant $c_4$ such that if $(\Omega, \mathcal{F}, \mu)$ is a probability space with at most $N$ atoms then

$$
\left( \sum_{s=1}^{S} \|z_s\|_{2,1}^2 \right)^{1/2} \leq c_4 \sqrt{\log N} \left\| \left( \sum_{s=1}^{S} |z_s|^2 \right)^{1/2} \right\|_{\infty}.
$$

Proof. Let $A \subset \Omega$ be the union of those atoms of measure less than $1/N^2$, so that $\mu(A) \leq 1/N$. By Lemma 9(ii), we have $\left\| z_s \chi_A \right\|_{2,1} \leq c_3 \sqrt{\log N} \|z_s\|_2$, and by Lemma 9(iii), we have that $\left\| z_s \chi_A \right\|_{2,1} \leq \sqrt{N} \|z_s\chi_A\|_2$. Thus, we have that

$$
\left( \sum_{s=1}^{S} \|z_s\|_{2,1}^2 \right)^{1/2} \leq \left( \sum_{s=1}^{S} \|z_s \chi_{\Omega \setminus A}\|_{2,1}^2 \right)^{1/2} + \left( \sum_{s=1}^{S} \|z_s \chi_A\|_{2,1}^2 \right)^{1/2}
$$

$$
\leq c_3 \sqrt{\log N} \left( \sum_{s=1}^{S} \|z_s\|_{2,1}^2 \right)^{1/2} + \sqrt{N} \mu(A) \left\| \left( \sum_{s=1}^{S} |z_s|^2 \right)^{1/2} \right\|_{\infty}
$$

$$
\leq c_4 \sqrt{\log N} \left\| \left( \sum_{s=1}^{S} |z_s|^2 \right)^{1/2} \right\|_{\infty},
$$

as desired. $\square$

Proof of Proposition 7. Without loss of generality, we may suppose that $N = 2^{2^k}$. We prove the result by induction over $k$. Suppose that $\Omega$ has $2^{2^{k+1}}$ atoms. Apply Lemma 8 to cover $\Omega$ by $2^{2^k}$ subsets, and to give $x_1, x_2, \ldots, x_S$, $y_1, y_2, \ldots, y_S$, $z_1, z_2, \ldots, z_S$ as described in the lemma. Then, by the triangle inequality,

$$
\left( \sum_{s=1}^{S} \|x_s\|_{2,1}^2 \right)^{1/2} \leq \left( \sum_{s=1}^{S} \|x_s\|_{2,1}^2 \right)^{1/2} + \left( \sum_{s=1}^{S} \|y_s\|_{2,1}^2 \right)^{1/2} + \left( \sum_{s=1}^{S} \|z_s\|_{2,1}^2 \right)^{1/2}.
$$

By the induction hypothesis,

$$
\left( \sum_{s=1}^{S} \|x_s\|_{2,1}^2 \right)^{1/2} \leq c_2 k.
$$

By Lemmas 10 and 11 we have that

$$
\left( \sum_{s=1}^{S} \|y_s\|_{2,1}^2 \right)^{1/2} \leq c_1
$$

and

$$
\left( \sum_{s=1}^{S} \|z_s\|_{2,1}^2 \right)^{1/2} \leq c_1 c_4 2^{-k/2} \left( 1 + \sqrt{\log 2^{2^k}} \right) \leq 2c_1 c_4.
$$
Hence
\[
\left( \sum_{s=1}^{S} \| x_s \|_{2,1}^2 \right)^{1/2} \leq c_2 (k + 1),
\]
as required, taking \( c_2 = 1 + 2c_1 c_4 \). \( \square \)

The proof of the main result is now easy.

**Proof of Theorem 2.** By Theorem 5, it is sufficient to show that for any probability measure \( \mu \) on \( \{ 1, 2, \ldots, N \} \), the cotype 2 constant of the canonical embedding \( l^N_{\infty} \rightarrow L_{2,1}^1 (\mu) \) is bounded by some universal constant times \( \log \log N \). But this is precisely what Proposition 7 says. \( \square \)

**Final remarks**

There is a similar result for Gaussian cotype (see [Mo2]).

**Theorem 12.** There is a constant \( c \) such that, for any operator \( T : l^N_{\infty} \rightarrow Y \), where \( Y \) is a Banach space, the Gaussian cotype 2 constant, \( \beta^{(2)} (T) \), is bounded according to the relation:
\[
\beta^{(2)} (T) \leq c \sqrt{\log \log N} \pi_{2,1} (T).
\]

This result is the best possible, as is implicitly shown in [T].

**Theorem 13.** There is a constant \( c \) such that for any integer \( N \), there is an operator \( T : l^N_{\infty} \rightarrow Y \), where \( Y \) is a Banach space, such that
\[
\beta^{(2)} (T) \geq c^{-1} \sqrt{\log \log N} \pi_{2,1} (T).
\]

Since the Rademacher cotype 2 constant is greater than a constant times the Gaussian cotype 2 constant, we have the following corollary.

**Corollary.** There is a constant \( c \) such that for any integer \( N \), there is an operator \( T : l^N_{\infty} \rightarrow Y \), where \( Y \) is a Banach space, such that
\[
K^{(2)} (T) \geq c^{-1} \sqrt{\log \log N} \pi_{2,1} (T).
\]

We also have the following, the result originally stated in [T].

**Corollary.** There is an operator \( T : C(K) \rightarrow Y \), where \( Y \) is a Banach space, that is (2, 1)-summing, but does not have Rademacher cotype 2.

If we write \( R_N \) for the supremum of \( K^{(2)} (T) / \pi_{2,1} (T) \) over all \( T : l^N_{\infty} \rightarrow Y \), then we have shown that \( c^{-1} \sqrt{\log \log N} \leq R_N \leq c \log \log N \). Clearly, we are left with the following problem.

**Open question.** What is the asymptotic behavior of \( R_N \)?

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