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A LOWER BOUND FOR THE SPECTRUM OF THE LAPLACIAN IN TERMS OF SECTIONAL AND RICCI CURVATURE

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ABSTRACT. Let M be an n-dimensional, complete, simply connected Riemannian manifold. In this paper we show that if the sectional curvature is bounded above by $-k \leq 0$ and the Ricci curvature is bounded above by $-\alpha \leq 0$, then the spectrum of the Laplacian on M is bounded below by $[\alpha + (n-1)(n-2)k]/4$. This improves a previous result due to H. P. McKean.

1. INTRODUCTION

McKean [4] has shown that if M is an n-dimensional complete, simply connected Riemannian manifold with sectional curvature satisfying $K \leq -k < 0$, then the bottom of the spectrum λ_0 of the Laplace operator Δ, defined as a positive selfadjoint operator on $L^2(M)$ (cf. [5]), is bounded below by $(n-1)^2 k/4$.

While this bound is sharp, as shown by the case of the Laplacian on the hyperbolic space [1], [4], it is easy to see that even in very simple cases McKean's inequality fails to provide good bounds for λ_0.

Consider for instance a product manifold $M = M_1 \times M_2$, where M_i is a complete, n_i-dimensional, simply connected manifold with constant sectional curvature $-k_i < 0$. Since the sectional curvature of the planes $\operatorname{span}\{X_1, X_2\} \subset TM$, where $X_i \in TM_i$ vanishes, McKean's inequality does not give any bound for λ_0.

On the other hand, denoting by Δ_i the Laplacian on M_i and by $\lambda_0^{(i)}$ the bottom of its spectrum, we have [1], [4] $\lambda_0^{(i)} = (n_i - 1)^2 k_i/4$ and, since spectrum(Δ) = spectrum(Δ_1) + spectrum(Δ_2), this implies $\lambda_0 = (n_1-1)^2 k_1/4 + (n_2-1)^2 k_2/4$.

The purpose of this article is to prove the following generalization of McKean's result:

Theorem 1. *Let M be an n-dimensional, complete, simply connected Riemannian manifold with sectional curvature K and Ricci curvature* Ric *satisfying $K \leq -k \leq 0$ and* $\mathrm{Ric} \leq -\alpha < 0$. *Then* $\lambda_0 \geq [\alpha + (n-1)(n-2)k]/4$.

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Observe that, if $K \le -k < 0$, then $\mathrm{Ric} \le -(n-1)k$, so $\lambda_0 \ge (n-1)^2 k/4$. Thus Theorem 1 implies McKean's inequality. Moreover, if M is a product manifold $M = M_1 \times M_2$, then $\mathrm{Ric}^{M_i} \le -\alpha_i$ implies $\mathrm{Ric}^M \le -\min\{\alpha_1, \alpha_2\}$, where Ric^{M_i}, Ric^M denote the Ricci curvature of M_i and of M, respectively. In the example above, where M_i is a space form of dimension n_i and sectional curvature $-k_i < 0$, Theorem 1 gives $\lambda_0 \ge \min\{(n_1-1)k_1, (n_2-1)k_2\}/4$. This bound, even though not optimal, shows that Theorem 1 is an improvement of McKean's result.

2. Notation

For the proof of Theorem 1 we will follow Chavel's notation [1, pp. 65–67]: Given $v \in T_pM$, $|v| = 1$, let $v^\perp \subseteq T_pM$ be the orthogonal complement of v, and let $\tau_t \colon T_pM \to T_{\exp tv}M$ be the parallel translation along the geodesic $\gamma_v(t) = \exp_p tv$. Denote by $\mathscr{A}(t, v) \colon v^\perp \to v^\perp$ the linear transformation defined by

$$\mathscr{A}(t, v)\eta = (\tau_t)^{-1} Y(t) \qquad \forall \eta \in v^\perp,$$

where $Y(t)$ is the orthogonal Jacobi vector field along γ_v determined by the initial conditions $Y(0) = 0$, $\nabla_t Y(0) = \eta$. Thus [1, p. 63] $\mathscr{A}(t, v)\eta = \tau_t^{-1}(T_{tv}\exp_p)(t\eta)$. Also let $\sqrt{g}(t, v) = \det \mathscr{A}(t, v)$. Using geodesic spherical coordinates at p, $(t, v) \in \mathbb{R}_+ \times ST_pM$, we can write the metric in the form (cf. [1])

$$ds^2 = (dt)^2 + |\mathscr{A}(t, v)\, dv|^2$$

and the volume element of M as

$$dV(\exp tv) = \sqrt{g}(t, v)\, dt\, d\mu_p(v),$$

where $d\mu_p$ denotes the $(n-1)$-dimensional measure on the unit sphere ST_pM of T_pM.

For $\eta \in v^\perp$, also let $\mathscr{R}(t, v)\eta = (\tau_t)^{-1} R(\tau_t v, \tau_t \eta) \cdot \tau_t v$, where R is the curvature tensor of M. Then (cf. [1, p. 66]) $\mathscr{R}(t, v)$ is a selfadjoint transformation of $v^\perp$ and $\mathscr{A}(t, v)$ satisfies the differential equation

$$\mathscr{A}''(t, v) + \mathscr{R}(t, v)\mathscr{A}(t, v) = 0$$

with initial conditions $\mathscr{A}(0, v) = 0$, $\mathscr{A}'(0, v) = \mathrm{Id}_{v^\perp}$.

3. The main geometric estimate

Theorem 2. *Let M be a complete, simply connected n-dimensional Riemannian manifold satisfying $K \le -k \le 0$ and $\mathrm{Ric} \le -\alpha \le 0$. Then*

$$\frac{1}{\sqrt{g}(t, v)} \frac{\partial}{\partial t} \sqrt{g}(t, v) \ge [\alpha + (n-1)(n-2)k]^{1/2}. \tag{1}$$

The proof follows from the two lemmas below:

Lemma 3. *With the same notations and hypotheses as in Theorem 2, we have*

$$\frac{1}{\sqrt{g}(t,r)}\frac{\partial^2}{\partial t^2}\sqrt{g}(t,v) \geq \alpha + (n-1)(n-2)k. \tag{2}$$

Proof of Lemma 3. Following Chavel, let $U(t,v) = \mathscr{A}'(t,v)\mathscr{A}^{-1}(t,v)\colon v^\perp \to v^\perp$. Observe that $\tau_t U(t,v)\tau_t^{-1}$ is the second fundamental form at $\exp_p tv$ of the geodesic sphere of radius t at p.

It is easy to see (cf. [1, p. 72]) that $U(t,v)$ is a selfadjoint map of $v^\perp$ and that U satisfies the matrix Riccati equation

$$U' + U^2 + \mathscr{R} = 0$$

so that, taking the trace, we have

$$(\operatorname{tr} U)' + \operatorname{tr}(U^2) + \operatorname{tr}\mathscr{R}(t,v) = 0.$$

Notice now that:

(i) $\operatorname{tr} U = \operatorname{tr}(\mathscr{A}'(t,v)\mathscr{A}^{-1}(t,v)) = (\ln(\det\mathscr{A}))' = \frac{1}{\sqrt{g}(t,v)}\frac{\partial}{\partial t}\sqrt{g}(t,v)$

(ii) $\operatorname{tr}\mathscr{R}(t,v) = \operatorname{Ric}(\dot\gamma_v(t), \dot\gamma_v(t))$ where $\gamma_v(t) = \exp_p tv$.

Then we have

$$\begin{aligned}
&\frac{1}{\sqrt{g}(t,v)}\frac{\partial^2}{\partial t^2}\sqrt{g}(t,v)\\
&= \frac{\partial}{\partial t}\left\{\frac{1}{\sqrt{g}(t,v)}\frac{\partial}{\partial t}\sqrt{g}(t,v)\right\} + \left\{\frac{1}{\sqrt{g}(t,v)}\frac{\partial}{\partial t}\sqrt{g}(t,v)\right\}^2\\
&= \frac{\partial}{\partial t}(\operatorname{tr} U) + (\operatorname{tr} U)^2\\
&= [(\operatorname{tr} U)' + \operatorname{tr}(U^2)] + [(\operatorname{tr} U)^2 - \operatorname{tr}(U^2)]\\
&= -\operatorname{Ric}(\dot\gamma_v(t), \dot\gamma_v(t)) + (\operatorname{tr} U)^2 - \operatorname{tr}(U^2)\\
&\geq \alpha + (\operatorname{tr} U)^2 - \operatorname{tr}(U^2).
\end{aligned}$$

Since U is selfadjoint, to complete the proof of the lemma it suffices to show that all the eigenvalues ρ_i of U are $\geq \sqrt{k}$, for then

$$\begin{aligned}
(\operatorname{tr} U)^2 - \operatorname{tr}(U^2) &= \left(\sum_{i=1}^{n-1}\rho_j\right)^2 - \sum_{j=1}^{n-1}\rho_j^2\\
&= \sum_{j\neq i}\rho_i\rho_j \geq (n-1)(n-2)k.
\end{aligned}$$

Therefore we need to show that

$$\langle U(t,v)\eta, \eta\rangle / \langle \eta, \eta\rangle \geq \sqrt{k} \qquad \forall \eta \in v^\perp, \ \forall t > 0.$$

For t_0 fixed, let $\eta = \mathscr{A}(t_0, v)\zeta$, so that $\eta = \tau_t^{-1}Y(t_0)$, where Y is the orthogonal Jacobi vector field along $\gamma_v(t) = \exp_p tv$ determined by the initial conditions $Y(0) = 0$, $\nabla_t Y(0) = \zeta$. Then we have

$$\begin{aligned}\langle U(t_0)\eta, \eta\rangle &= \langle \mathscr{A}'(t_0, v)\mathscr{A}(t_0, v)^{-1}\eta, \eta\rangle \\ &= \langle \mathscr{A}'(t_0, v)\zeta, \mathscr{A}(t_0, v)\zeta\rangle \\ &= \langle \tau_{t_0}\mathscr{A}'(t_0, v)\zeta, \tau_{t_0}\mathscr{A}(t_0, v)\zeta\rangle.\end{aligned}$$

By definition, $\tau_{t_0}\mathscr{A}(t_0, v)\zeta = Y(t_0)$, and it is easy to see that $\tau_{t_0}\mathscr{A}'(t_0, v)\zeta = \nabla_t Y(t_0)$, so that

(3) $$\langle U(t_0)\eta, \eta\rangle = \langle \nabla_t Y(t_0), Y(t_0)\rangle = |Y|(t_0)|Y|'(t_0).$$

Since the sectional curvature of M is bounded above by $-k$, we obtain, applying Rauch's Comparison Theorem (cf. [1, p. 67])

$$|Y|'(t_0)/|Y|(t_0) \geq \psi'(t_0)/\psi(t_0),$$

where

$$\begin{aligned}\psi(t) &= |Y(0)|\cosh(\sqrt{k}t) + |Y|'(0)\frac{1}{\sqrt{k}}\sinh(\sqrt{k}t) \\ &= |Y|'(0)\frac{1}{\sqrt{k}}\sinh(\sqrt{k}t).\end{aligned}$$

Hence

$$|Y|'(t_0)/|Y|(t_0) \geq \sqrt{k}\coth(\sqrt{k}t_0) \geq \sqrt{k}$$

and, substituting in (3),

$$\langle U(t_0)\eta, \eta\rangle/\langle \eta, \eta\rangle \geq \sqrt{k},$$

therefore completing the proof of Lemma 3.

Lemma 4. *Let M be a complete, simply connected Riemannian manifold with nonpositive sectional curvature. Assume that* $\frac{1}{\sqrt{g(t,v)}}\frac{\partial^2}{\partial t^2}\sqrt{g}(t, v) \geq \beta > 0$. *Then*

(4) $$\frac{1}{\sqrt{g(t, v)}}\frac{\partial}{\partial t}\sqrt{g}(t, v) \geq \sqrt{\beta}.$$

Proof of Lemma 4. Notice that, by Bishop Comparison Theorem I ([1, pp. 68–69]) the hypothesis on the sectional curvature implies that $\frac{1}{\sqrt{g(t,v)}}\frac{\partial}{\partial t}\sqrt{g}(t, v) \geq 0$. Moreover, since $\mathscr{A}(t, v)\eta = (\tau_t)^{-1}(T_{tv}\exp_p)(t\eta)$, and $(\tau_t)^{-1}(T_{tv}\exp_p) \to \mathrm{Id}_{v^\perp}$ as $t \downarrow 0$, we have $\sqrt{g}(t, v) = \det\mathscr{A}(t, v) \sim t^{n-1}$ as $t \downarrow 0$, so that $\frac{1}{\sqrt{g(t,v)}}\frac{\partial}{\partial t}\sqrt{g}(t, v) \sim \frac{n-1}{t}$ as $t \downarrow 0$ and $\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\sqrt{g}\right) < 0$ for t small enough.

Then we write

$$\frac{1}{\sqrt{g}}\frac{\partial^2}{\partial t^2}\sqrt{g} = \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\sqrt{g}\right) + \left(\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\sqrt{g}\right)^2.$$

Given t_0, if $\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\sqrt{g}\right)(t_0) \leq 0$, then $\left(\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\sqrt{g}\right)^2 \geq \frac{1}{\sqrt{g}}\frac{\partial^2}{\partial t^2}\sqrt{g} \geq \beta$ at t_0, and (4) holds at t_0.

If $\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{q}}\frac{\partial}{\partial t}\sqrt{g}\right) > 0$ at t_0 let $t_1 \le (0, t_0)$ be the point where $\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\sqrt{g}$ has its minimum in $(0, t_0]$. Then $\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\sqrt{g}\right) = 0$ at t_1 and

$$\left(\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\sqrt{g}\right)(t_0) \ge \left(\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\sqrt{g}\right)(t_1) = \left(\frac{1}{\sqrt{g}}\frac{\partial^2}{\partial t^2}\sqrt{g}(t_1)\right)^{1/2} \ge \sqrt{\beta}.$$

Thus (4) holds for every $t > 0$, and the proof of Lemma 4 is complete.

4. Proof of Theorem 1

The proof of Theorem 1 now follows familiar lines (cf. [1, p. 47], [5, pp. 67–69]). By Rayleigh's Theorem, it suffices to show that, $\forall f \in C_c^\infty(M)$,

$$\int_M |\nabla f|^2\, dV \ge [\alpha + (n-1)(n-2)]/4 \int_M f^2\, dV.$$

For convenience, set $\alpha + (n-1)(n-2) = \beta$. Using spherical geodesic coordinates $(t, v) \in \mathbb{R}_+ \times ST_pM$ and, denoting the local expression of $f \in C_c^\infty(M)$ with the same symbol, we have, $\forall v \in ST_pM$,

$$\begin{aligned}
\int_0^\infty f^2(t, v)\sqrt{g}(t, v)\, dt &\le 1/\sqrt{\beta}\int_0^\infty f^2(t, v)\frac{\partial}{\partial t}\sqrt{g}(t, v)\, dt \\
&= -2/\sqrt{\beta}\int_0^\infty f(t, v)\frac{\partial f}{\partial t}(t, v)\sqrt{g}(t, v)\, dt \\
&\le \frac{2}{\sqrt{\beta}}\left(\int_0^\infty f^2(t, v)\sqrt{g}(t, v)\, dt\right)^{1/2} \\
&\quad \times \left(\int_0^\infty \left[\frac{\partial f}{\partial t}(t, v)\right]^2 \sqrt{g}(t, v)\, dt\right)^{1/2},
\end{aligned}$$

where the first inequality follows from $\frac{\partial}{\partial t}\sqrt{g}(t, v) \ge \sqrt{\beta}\sqrt{g}(t, v)$ (Theorem 2), the equality on the second line is obtained by integrating by parts, and the last inequality is the Cauchy-Schwartz inequality.

Using the fact that $|\nabla f|^2(t, v) \ge \left[\frac{\partial}{\partial t}f(t, v)\right]^2$ and simplifying we find that

$$\int_0^\infty f^2(t, v)\sqrt{g}(t, v)\, dt \le 4/\beta \int_0^\infty |\nabla f|^2(t, v)\sqrt{g}(t, v)\, dt$$

whence, integrating over $v \in s^{n-1}$, we obtain

$$\begin{aligned}
\int_M f^2 dV &= \int_{S^{n-1}}\int_0^\infty f^2(t, v)\sqrt{g}(t, v)\, dt\, d\mu_p(v) \\
&\le \frac{4}{\beta}\int_{S^{n-1}}\int_0^\infty |\nabla f|^2(t, v)\sqrt{q}(t, v)\, dt\, d\mu_p(v) \\
&\le \frac{4}{\beta}\int_M |\nabla f|^2\, dV.
\end{aligned}$$

This completes the proof of Theorem 1. □

Remarks. As shown in [5], a lower bound for $\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\sqrt{g}$ also provides lower bounds for $\|\nabla f\|_p/\|f\|_p$ for every p, $1 \leq p < \infty$, and f compactly supported with f, $\nabla f \in L^p(M)$. Thus, in the hypotheses of Theorem 1 we have

$$\|f\|_p \leq \frac{p}{[\alpha + (n-1)(n-2)k]^{1/2}} \|\nabla f\|_p .$$

A lower bound to $\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\sqrt{g}$ can also be used to estimate the Cheeger's isoperimetric constant $h(M)$ of the manifold M (cf. [1], [2], [6]). An argument as in [1, pp. 95–96], shows that $\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\sqrt{g} \geq [\alpha + (n-1)(n-2)k]^{1/2}$ implies $h(M) \geq [\alpha + (n-1)(n-2)k]^{1/2}$. This together with Cheeger's Theorem yields another proof of Theorem 1.

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