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A CRITERION ON A SUBDOMAIN OF THE DISC FOR ITS HARMONIC MEASURE TO BE COMPARABLE WITH LEBESGUE MEASURE

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ABSTRACT. A subdomain $\mathscr{O}$ of the disc $\mathbb{D}$ is called a boundary layer if $\omega(\mathscr{O}, \cdot) \geq \alpha \cdot m$, where $\omega(\mathscr{O}, \cdot)$ is the harmonic measure of $\mathscr{O}$. The metric criterion in terms of $\partial\mathscr{O}$ is given for the case when α is near 1.

1. STATEMENTS

Much interesting analysis has resulted from attempts to understand the structure of $P^2(\mu)$, the closure in $L^2(\mu)$ of the set $\mathscr{P}_A$ of all analytic polynomials, where μ is a positive finite Borel measure with compact support in the complex plane $\mathbb{C}$. The recent achievement in this field due to J. Thomson [4] asserts that $P^2(\mu) \neq L^2(\mu)$ if and only if there is a point c such that

$$|p(c)| \leq K\|p\|_{L^2(\mu)} \qquad \forall p \in \mathscr{P}_A.$$

(Such points are called points of bounded evaluation.) In other words Thomson has solved a problem of Mergelyan–Brennan. He even has achieved more: a description of $P^2(\mu)$ in terms of points of bounded evaluations.

But unfortunately, this does not help much when one is interested in a description of $P^2(\mu)$ in terms of μ itself.

Perhaps the best example is a so-called splitting problem. In this problem,

$$(*) \qquad d\mu = w(1-r)r\,dr\,d\theta + h(\theta)\,dm(\theta) = \mu_{\mathbb{D}} + \mu_{\mathbb{T}}$$

is a sum of a radially symmetric measure $\mu_{\mathbb{D}}$ in the unit disc $\mathbb{D}$ and a measure $\mu_{\mathbb{T}}$ on $\mathbb{T} = \partial\mathbb{D}$. What are the conditions on the pair (w, h) necessary and sufficient for the splitting:

$$(**) \qquad P^2(\mu) = P^2(\mu_{\mathbb{D}}) \oplus L^2(\mu_{\mathbb{T}})?$$

This problem was solved in [5] and [6]. And, loosely speaking, the proof proceeds as follows. Assertion $(**)$ can be reduced to a uniqueness theorem for

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functions $f \in C^1(\mathbb{D})$ such that $(\overline{\partial} \overset{\text{def}}{=} \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}))$

$$|\overline{\partial} f(z)| \leq w(1 - |z|)$$

(w is from $(*)$). Introducing the set E_w of "singular values" of f; $E_w \overset{\text{def}}{=} \{z \in \mathbb{D}: |f(z)| \leq w(1 - |z|)\}$, one can see that for suitable bounded g the function $F \overset{\text{def}}{=} fe^g$ is holomorphic in $\Omega = \mathbb{D} \backslash E_w$.

Now the following problem arises naturally in the proof of the main result of [5] and [6]: How to characterize closed subsets E of the unit disc $\mathbb{D}$ for which the domain $\mathbb{D} \backslash E$ (we suppose that E does not split $\mathbb{D}$) is a *boundary layer*; i.e.,

$$\omega(\mathbb{D} \backslash E, I, \zeta) \geq c|I|, \qquad c > 0,$$

for all arcs I, $I \subset \mathbb{T}$? Here $|I| = mI$ is a normed Lebesgue measure of I.

Strictly speaking the E in [5] and [6] are of a special kind. So one can succeed there without answering this general question. But it is natural to try to answer this question. And this is done to some extent in the present work.

Let $\mathscr{O} = \mathbb{D} \backslash E$. We suppose that $0 \in \mathscr{O}$ and denote $\omega(\mathscr{O}, \cdot, 0)$ by $\omega(\cdot)$. In what follows, $\mathscr{O}$ is a regular with respect to the Dirichlet problem; $P(a, \zeta) = (1-|a|^2)/|1-\overline{a}\zeta|^2$ is the Poisson kernel, $a \in \mathbb{D}$, $\zeta \in \mathbb{T}$, $P^\mu(\zeta) \overset{\text{def}}{=} \int_{\mathbb{D}} P(a, \zeta) d\mu(a)$ for a measure μ in $\mathbb{D}$.

It is clear that $\mathscr{O}$ is a boundary layer if and only if

$$P^\omega(\xi) \leq q < 1, \qquad \xi \in \mathbb{T}. \tag{1.1}$$

When this case occurs we will say that $\mathscr{O}$ is a $(1-q)$-*boundary layer*. In fact, (1.1) means that $\omega(I) \geq (1-q) \cdot |I|$ for all I.

Surely $P^\omega \leq 1$ for an arbitrary $\mathscr{O}$, and $P^\omega < 1$ a.e. with respect to m if and only if the measure $\omega|\mathbb{T}$ and m are mutually absolutely continuous.

The metric criterion in terms of capacity for E to satisfy (1.1) will be given here in the following two cases:

(1) if q is less than a certain absolute constant;

(2) if E lies in a Stoltz domain;

$$\Gamma(e^{i\theta}) \overset{\text{def}}{=} \{a \in \mathbb{D}: |e^{i\theta} - a| \leq C(1 - |a|)\}, \qquad C < \infty.$$

To state the results we need some notation. Let $\mathscr{F} = \{I\}$ be the collection of all the dyadic arcs of T, Q_I be a square in $\mathbb{D}$ based on I, TQ_I be its top half, and C_I be the center of TQ_I. For a subset e of Q_I we denote by λe the image of e under the mapping $z \to \lambda(z - C_I) + C_I$.

1.1. **Proposition.** *Let* $1/2\mathbb{D} \subset \mathscr{O}$. *Then for an absolute* A_1

$$P^\omega(\xi) \leq A_1 \sum_{I \in \mathscr{F}} \left| \frac{1 - |C_I|}{1 - \overline{C}_I \xi} \right|^2 \operatorname{cap}\left(\frac{E \cap TQ_I}{2|I|}\right), \qquad \forall \xi \in T.$$

In particular,

$$\sup_{\xi \in T} \sum_{I \in \mathscr{F}} \left| \frac{1 - |C_I|}{1 - \overline{C}_I \xi} \right|^2 \operatorname{cap}\left(\frac{E \cap TQ_I}{2|I|}\right) \leq \frac{q}{A_1} \tag{1.2}$$

implies (1.1).

An estimate from below for $P^{\omega}(\xi)$ is replaced by the following proposition.

1.2. **Proposition.** *There exist absolute constants* A_2 *and* $q_0 > 0$ *such that if* (1.1) *holds with* $q \le q_0$, *then*

$$\sup_{\xi \in T} \sum_{I \in \mathscr{F}} \left| \frac{1 - |C_I|}{1 - \overline{C}_1 \xi} \right|^2 \operatorname{cap}\left(\frac{E \cap TQ_1}{2|I|} \right) \le A_2 q. \tag{1.3}$$

For E *in a Stoltz domain*

$$\sum_{I \in \mathscr{F}} \cdots \le A_2 P^{\omega}(\xi), \qquad \xi \in T. \tag{1.4}$$

The following result follows immediately.

1.3. **Theorem.** *If* E *is in a Stoltz domain, then the following assertions are equivalent*:

(1) $\mathscr{O}$ *is a boundary layer*;
(2) $\sup_{\xi \in T} \sum_{I \in \mathscr{F}} |\frac{1-|C_I|}{1-\overline{C}_1 \xi}|^2 \operatorname{cap}(\frac{E \cap TQ_I}{2|I|}) < \infty$; *and*
(3) $\sum_{I \in \mathscr{F}} \operatorname{cap}(\frac{E \cap TQ_I}{2|I|}) < \infty$.

For a domain $\mathscr{O} = \mathbb{D} \backslash E$, let us consider domains $\mathscr{O}_n = \mathscr{O} \cup (1 - 1/n)\mathbb{D}$, $\omega_n(\cdot) = \omega(\mathscr{O}_n, \cdot, 0)$. It is worthwhile to note that there is a criterion which guarantees that $\mathscr{O}_n$ is a $(1 - \varepsilon_n)$-boundary layer with $\lim \varepsilon_n = 0$. In this case, we call $\mathscr{O}$ a *good boundary layer*.

1.4. **Theorem.** $\mathscr{O}$ *is a good boundary layer if and only if the series*

$$\sum_{I \in \mathscr{F}} \left| \frac{1 - |C_I|}{1 - \overline{C}_I \xi} \right|^2 \operatorname{cap}\left(\frac{E \cap TQ_I}{2|I|} \right) \tag{1.5}$$

converges uniformly on T.

We conclude this section with one more notation:

$$\nu_E \overset{\text{def}}{=} \sum_{I \in \mathscr{F}} |I| \operatorname{cap}\left(\frac{E \cap TQ_I}{2|I|} \right) \delta_{C_I},$$

where δ_{C_I} is the point measure. The proof of Proposition 1.2 is rather amusing but we begin with the following instead.

2. The proof of Proposition 1.1

It is sufficient to verify that

$$\omega(TQ_I) \le A|I| \operatorname{cap}\left(\frac{E \cap TQ_I}{2|I|} \right). \tag{2.1}$$

We show that for large absolute constant B there is a rectangle R with base containing I, contained in $2I$, such that $\mathbb{D} \cap \partial R \subset Q_{2I} \backslash Q_I$, and

$$\omega(\mathscr{O}, E \cap TQ_I, z) \le B \operatorname{cap}\left(\frac{E \cap TQ_I}{2|I|} \right), \qquad \forall z \in \mathbb{D} \cap \partial R. \tag{2.2}$$

It follows from the maximum principle that it is sufficient to find a closed contour R_1 surrounding $E_I \stackrel{\text{def}}{=} E \cap TQ_I$ with property (2.2) and such that $R_1 \subset \frac{3}{2}TQ_I$. Let E_I^* be the image of E_I under the conformal mapping $\varphi: \mathbb{D} \to \frac{1}{4}\mathbb{D}$ such that $\varphi(C_I) = 0$. It is clear that

$$a_1 \operatorname{cap} E_I^* \le \operatorname{cap}(E_I/2|I|) \le a_2 \operatorname{cap} E_I^*,$$

where $0 < a_1, a_2 < \infty$ do not depend on I. Now we apply Lemma 6.1 (see §6) to $E_I^*, \frac{1}{4}\mathbb{D}$. Note that $E_I^* \subset \mathbb{D}/4\lambda$, $\varphi(\frac{3}{2}TQ_I) \subset \mathbb{D}/4\lambda'$ where $\lambda > \lambda' > 1$ and λ, λ' do not depend on I. Assertion (6.3) shows that $R_I = \varphi^{-1}(T/4\lambda'')$ suits us for some $\lambda'' \in (\lambda', \lambda)$.

An application of the Poisson formula in $\mathscr{O} \backslash \operatorname{clos} R$ to the function $\omega(\mathscr{O}, E_I, z)$, harmonic in this domain, gives us ($0 \in \mathscr{O} \backslash \operatorname{clos} R$ as $1/2\mathbb{D} \subset \mathscr{O}$):

$$\omega(\mathscr{O}, E_I, 0) \le 4B|I| \operatorname{cap}(E_I/2|I|).$$

3. The proof of Proposition 1.2

Let $\widetilde{TQ}_I$ denote the union of TQ_I and all TQ_J neighboring TQ_I.

We suppose that (1.1) is valid with q sufficiently small. We will show that under this assumption the inequality

$$\omega(\widetilde{TQ}_I) \ge a|I| \operatorname{cap}\left(\frac{E \cap TQ_I}{2|I|}\right) \tag{3.1}$$

holds. Inequality (3.1) is the reverse of (2.1). It is also clear that it may hold *only for sufficiently small* q.

It follows from (3.1) and Harnack's inequality that

$$P^{\nu_E}(\xi) \le Aa^{-1} P^{\omega}(\xi), \qquad \xi \in T, \tag{3.2}$$

which proves Proposition 1.2.

Inequality (3.1) is a stronger version of Lemma 6.2. To prove (3.1), we put $u_1(z) = \omega(\mathscr{O}, E \cap \widetilde{TQ}_I, z)$, $u_2(z) = \operatorname{cap}(\frac{E \cap TQ_I}{2|I|}) \cdot \omega(\mathscr{O}, \frac{1}{4}\Gamma, z)$ and compare these functions on a specially built contour γ, $\gamma \subset Q_I \backslash Q_{1/2I}$, surrounding $E_I = E \cap TQ_I$. From the left part of (6.2), it follows that

$$\begin{aligned} u_1(z) &\ge \omega(\widetilde{TQ}_I \backslash E, E, z) \\ &\ge \omega(\widetilde{TQ}_I \backslash E_I, E_I, z) \ge a u_2(z), \qquad z \in \partial TQ_I. \end{aligned} \tag{3.3}$$

We find two curves γ_1, γ_2 connecting ∂TQ_I with $\mathbb{T}$ and such that γ_1, γ_2 and a part of ∂TQ_I from a contour γ separating $1/2I$ from 0. We have to find γ_1, γ_2 in such a way that the left part of (3.3) is not less than the right part of (3.3) on γ. If it is the case, (3.1) will follow, as then

$$u_1(0) \ge a u_2(0) \ge a \cdot \operatorname{cap}\left(\frac{E_I}{2|I|}\right)|I|,$$

by virtue of the condition of our proposition.

Now let R_1, R_2 be rectangles which form $Q_I \backslash (TQ_I \cup Q_{1/2I})$. We will find $\gamma_i \subset R_i$, $i = 1, 2$.

It is obvious that

$$u_2(z) \le A \operatorname{cap}\left(\frac{E_I}{2|I|}\right) \cdot \frac{1-|z|}{|I|}, \qquad z \in R_1 \cup R_2. \tag{3.4}$$

The Green function of $\mathscr{O} = \mathbb{D}\backslash E$ will be denoted by $G_{\mathscr{O}}(a, b)$. Let $G(a, b) = \log|\frac{1-\bar{a}b}{a-b}|$. We will use the notation:

$$G_{\mathscr{O}}(\zeta) \stackrel{\text{def}}{=} G_{\mathscr{O}}(\zeta, 0), \qquad G(\zeta) \stackrel{\text{def}}{=} \log 1/|\zeta| = G(0, \zeta) = G(\zeta, 0).$$

Our first step will result in proving the implication

$$z \in R_1 \cup R_2, \qquad G_{\mathscr{O}}(z) \ge (1-\varepsilon_0)G(z) \Rightarrow u_1(z) \ge a \cdot \operatorname{cap}\left(\frac{E_I}{2|I|}\right) \cdot \frac{1-|z|}{|I|}, \tag{3.5}$$

where ε_0 is a certain positive absolute constant.

Combining (3.4) and (3.5) we see that it remains to be proven that for $i = 1, 2$ there is a curve γ_i in R_I connecting the top side of R_i with its bottom side such that

$$G_{\mathscr{O}}(z) \ge (1-\varepsilon_0)G(z) \tag{3.6}$$

on γ_i.

Step 1. Let R denote a rectangle in $\mathbb{D}$ based on $2I$ and $1/2|I|$ in height and let $u(z) \stackrel{\text{def}}{=} \omega(R\backslash E, l_I, z)$ where l_I is the middle half of the top side of R.

It follows from (3.3) that

$$u_1(z) \ge a \cdot u(z) \operatorname{cap}\left(\frac{E_I}{2|I|}\right), \qquad z \in R. \tag{3.7}$$

Let $u_0(z) \stackrel{\text{def}}{=} \omega(R, l_I, z)$. It is clear that

$$z \in R_1 \cup R_2 \Rightarrow b\frac{1-|z|}{|I|} \le u_0(z) \le B\frac{1-|z|}{|I|}, \tag{3.8}$$

with absolute b, B, $0 < b < B < \infty$. It follows from (3.7) and (3.8) and (3.5) is implied by the assertion

$$z \in R_1 \cup R_2, \qquad G_{\mathscr{O}}(z) \ge (1-\varepsilon_0)G(z) \Rightarrow u(z) \ge a u_0(z). \tag{3.9}$$

The Poisson formula in $R\backslash E$ gives:

$$\begin{aligned} u(z) &= u_0(z) - \int_{E\cap R} u_0(\zeta)\, d\omega(R\backslash E, \zeta, z) \\ &\ge b\frac{1-|z|}{|I|} - B\int_{E\cap R} \frac{1-|\zeta|}{|I|}\, d\omega(R\backslash E, \zeta, z) \\ &\ge b\frac{1-|z|}{|I|} - \frac{B}{|I|}\int_{E\cap R} \log\frac{1}{|\zeta|}\, d\omega(\mathscr{O}, \zeta, z) \\ &= b\frac{1-|z|}{|I|} - \frac{B}{|I|}\int_{E\cap R} G(\zeta)\, d\omega(\mathscr{O}, \zeta, z). \end{aligned} \tag{3.10}$$

Applying the Poisson formula in $\mathscr{O}$ to the positive harmonic function $G(\zeta) - G_{\mathscr{O}}(\zeta)$, we get

$$\int_E G(\zeta)d\omega(\mathscr{O}, \zeta, z) = G(z) - G_{\mathscr{O}}(z) \le \varepsilon_0 G(z) \le A\varepsilon_0(1 - |z|). \tag{3.11}$$

Combining this with (3.10), we get

$$u(z) \ge \frac{1 - |z|}{|I|}(b - \varepsilon_0 BA) \ge \frac{b}{2}B^{-1} \cdot u_0(z)$$

if $\varepsilon_0 \le \frac{1}{2}b/BA$. So (3.9) and at the same time (3.5) are proved. Note that $\varepsilon_0 = 1/50$ suits us well.

Step 2. The assertion that $P^{\omega}(\xi) \le q_0$ with small q_0 has not been used yet (we have used only that $q_0 < 1$). Here the smallness of q_0 will play its part. We will prove the following implication:

$$P_{\omega}(\xi) \le q_0, \tag{3.12}$$

$$\xi \in T \Rightarrow \forall I \in \mathscr{F} \operatorname{cap}\left(\frac{\{z \in TQ_I : G_{\mathscr{O}}(z) \le (1 - q_0^{1/4})G(z)\}}{2|I|}\right) \le Aq_0^{1/4}.$$

We see that (3.6) follows immediately from (3.12) and (6.1).

To prove (3.12), let us fix a point c on T, $\Gamma_{c,\delta} \overset{\text{def}}{=} \{z : z = c + \delta e^{i\theta}, 0 \le \theta < 2\pi\}$. The function $G_{\mathscr{O}}(z) \overset{\text{def}}{=} G_{\mathscr{O}}(z)(z \in \mathscr{O}) \overset{\text{def}}{=} 0(z \notin \mathscr{O})$ is subharmonic in $\operatorname{int}\Gamma_{c,\delta}$ for $\delta \le 1/2$ and $\Delta G_{\mathscr{O}} = \omega$ in the distribution theory sense. Let $\mu_{\mathscr{O}}(t) = \omega(\{z : |z - c| < t\})$. The application of the Jensen formula to $G_{\mathscr{O}}$ in $\operatorname{int}\Gamma_{c,\delta}$ gives

$$\int_0^{\delta} \frac{\mu_0(t)}{t}dt = \frac{1}{2\pi}\int_0^{2\pi} G_{\mathscr{O}}(c + \delta e^{i\theta})d\theta = \frac{1}{2\pi}\int_{\Gamma_{i,\delta}} G_{\mathscr{O}}(z)\frac{|dz|}{\delta}. \tag{3.13}$$

By the condition of Proposition 1.2 for all c, δ, $t < \delta$ we have

$$\mu_{\mathscr{O}}(t) \ge (1 - q_0)\mu(t),$$

where $\mu(t) \overset{\text{def}}{=} m(\{z : |z - c| < t\})$. The analogue of (3.13) with the subscript $\mathscr{O}$ skipped takes place. This and (3.13) show that

$$\frac{1 - q_0}{2\pi}\int_{\Gamma_{c,\delta}} G(z)\frac{|dz|}{\delta} \le \frac{1}{2\pi}\int_{\Gamma_{c,\delta}} G_{\mathscr{O}}(z)\frac{|dz|}{\delta},$$

i.e.,

$$\int_{\Gamma_{c,\delta}} (G(z) - G_{\mathscr{O}}(z))\frac{|dz|}{2\pi\delta} \le q_0\int_{\Gamma_{c,\delta}} G(z)\frac{|dz|}{2\pi\delta}. \tag{3.15}$$

Taking into account that $G(z) \le A\delta$ on $\Gamma_{c,\delta}$, $G(c + \delta e^{i\theta}) \ge \delta/2$ for $\theta \in (\pi/3, 2\pi/3)$, we get from (3.15)

$$m\left\{\theta \in \left(\frac{\pi}{3}, \frac{2\pi}{3}\right) : 1 - \frac{G_{\mathscr{O}}(c + \delta e^{i\theta})}{G(c + \delta e^{i\theta})} \ge q_0^{1/2}\right\} \le Aq_0^{1/2}. \tag{3.16}$$

Set $F_I \overset{\text{def}}{=} \{z \in TQ_I : G_{\mathscr{O}}(z) \le (1 - q_0^{1/4})G(z)\}$. This set is closed, as $G_{\mathscr{O}}$ is a continuous function. The function $G_{\mathscr{O}} - G$ is subharmonic in $Q_{2I} \backslash F_I$; $G_{\mathscr{O}} - G \le 0$ on $\partial(Q_{2I} \backslash F_I)$ and

$$G_{\mathscr{O}} - G \le -a \cdot q_0^{1/4}|I| \qquad \text{on } F_I. \tag{3.17}$$

Now we choose c to be the center of I, $\delta = 3/4|I|$, $z_0 = c + \delta e^{i\theta_0}$, $\theta_0 \in (\pi/3, 2\pi/3)$. Then it follows from (3.17) that

$$G_{\mathscr{O}}(z_0) \le -a \cdot q_0^{1/4}|I|\omega(Q_{2I} \backslash F_I, F_I, z_0). \tag{3.18}$$

At the same time point z_0 can be chosen in such a way (provided that q_0 is sufficiently small: see (3.16)) that

$$G_{\mathscr{O}}(z_0) - G(z_0) \ge -q_0^{1/2}G(z_0) \ge -2q_0^{1/2}|I|. \tag{3.19}$$

It follows from (3.18) and (3.19) that

$$\omega(Q_{2I} \backslash F_I, F_I, z_0) \le 2a^{-1} \cdot q_0^{1/4},$$

and (3.12) follows from this and the left part of (6.2).

4. The proof of Theorem 1.3

Let E lie in a Stoltz domain $\Gamma(\xi_0) = \{a \in \mathbb{D} : |\xi_0 - a| \le c_\Gamma(1 - |a|)\}$. The function P^{ν_E} is lower-semicontinuous, and so

$$\sup_{\mathbb{T}} P^{\nu_E}(\xi) = \operatorname{ess\,sup}_{\mathbb{T}} P^{\nu_E}(\xi) = P^{\nu_E}(\xi_0) = \sum_{I \in \mathscr{F}} \operatorname{cap}\left(\frac{E_I}{2|I|}\right).$$

Therefore, (2) is equivalent to (3).

To prove (1) $\Rightarrow$ (3), let us denote by I_n an arc of length 2^{-n} centered at ξ_0, $Q_n = Q_{I_n}$. Then $\sum_{n \ge 0} 2^n \omega(Q_{n+1} \backslash Q_n) \le A(c_\Gamma)P^\omega(\xi_0) < \infty$. So the series $\sum_{n \ge 0} 2^n \omega(Q_n)$ converges. It remains to apply Lemma 6.2 (or the stronger assertion (3.1)) and take into account that $\omega(I_n) \ge c \cdot 2^{-n}$.

On the other hand, if (3) takes place and $E \subset \Gamma(\xi_0)$, then (1.5) is valid and $\mathscr{O}$ is a good boundary layer (see Theorem 1.4).

5. The proof of Theorem 1.4

We put $E_{2^m} = \mathbb{D} \backslash \mathscr{O}_{2^m}$, $\nu_{2^m} = \sum_{I \in \mathscr{F}, |I| \le 2^{-m}} |I| \operatorname{cap}(E_I/2|I|)\delta_{c_I}$. Proposition 1.1 shows that

$$P^{\omega_{2^m}}(\xi) \le A_1 \cdot P^{\nu_{2^m}}(\xi), \qquad \xi \in T,$$

and so by virtue of (1.5), $\lim \|P^{\omega_{2^m}}\|_\infty = 0$ which means that $\mathscr{O}$ is a good boundary layer.

Let $\mathscr{O}$ be a good boundary layer. Then $\|P^{\omega_{2^m}}\|_\infty \to 0$. Now Proposition 1.2 shows that $\|P^{\nu_{2^m}}\|_\infty \to 0$.

Surely a good boundary layer is a boundary layer. The converse is true sometimes (e.g., if $E = \mathbb{D} \backslash \mathscr{O}$ lies in a Stoltz domain) but is not true in general.

V. I. Vasyunin called attention to the following:

5.1. **Example.** Let $D_n = D(r_n e^{i2^{-n}}, \delta(1-\Gamma_n))$, where δ is a fixed small constant, $E = \bigcup_{n\geq 0} D_n$. An inductive procedure permits us to choose $\{r_n\}$ converging to 1 so rapidly that $P^{\nu_E}(\xi) \leq q(\delta)$, $\xi \in T$, and $\lim_{\delta\to 0} q(\delta) = 0$. But (1.5) is obviously false.

5.2. *Remark.* Let $E \subset \Gamma(\xi_0)$. It is clear that the thinness of E (in the sense of Wiener) guarantees that $\mathbb{D} = \mathscr{O}\backslash E$ is a boundary layer. In fact, the Wiener criterion of thinness is $\sum n \operatorname{cap}(E \cap A_n(\xi_0)) < \infty$ where $A_n = \{z: 2^{-n-1} \leq |z-\xi_0| \leq 2^{-n}\}$. And (3) of Theorem 1.3 can be rewritten as $\sum \operatorname{cap}(\frac{E\cap A_n(\xi_0)}{2^{-n+1}}) < \infty$. It remains to use the fact that $\operatorname{cap}(E/\varepsilon) = \operatorname{cap} E/(1+\log\varepsilon\cdot\operatorname{cap} E)$. In the simplest case $E = \bigcup D_n$, $D_n = D(r_n, \rho_n)$, $\rho_n < 2^{-n}$, $1-r_n \sim 2^{-n}$, the domain $\mathscr{O}$ is a (good) boundary layer if and only if

$$\sum_n \left(\log\frac{1-r_n}{\rho_n}\right)^{-1} < \infty.$$

6. Auxiliary assertions

For the sake of completeness, we state in this section some standard and well-known assertions that were used above.

Recall that the capacity of a compact E is defined by the equality

$$1/\operatorname{cap} E = \lim_{z\to\infty}(g(z) - \log|z|),$$

where g is the Green function of $\mathbb{C}\backslash E$ with pole at ∞. We are always in a situation where g is contained in a disc of radius 3/4 in which case $\operatorname{cap} E \subset [0, (\log 4/3)^{-1}]$, and $\operatorname{cap} E$ may also be described as

$$\operatorname{cap} E = \sup\{\mu(E): \mu \text{ is a positive measure supported on } E \text{ with } U^\mu \leq 1\},$$

where $U^\mu(z) = \int \log\frac{1}{|z-\zeta|}\,d\mu(\zeta)$ is the logarithmic potential.

We define the 1-dimensional Hausdorff content of a set by

$$h_1(E) = \inf\left\{\sum r_i : E \subset \cup D(z_i, r_i),\ z_i \in \mathbb{C}\right\}.$$

It is well known [2] that

$$h_1(E) \leq A e^{-1/\operatorname{cap} E}. \tag{6.1}$$

We have used the following

6.1. **Lemma.** *Let Q be a square or a disc of diameter $d(Q)$ and $E \subset Q$ be a compact. If $\lambda > 1$ then*

$$a_\lambda \operatorname{cap}\left(\frac{E}{2d(Q)}\right) \leq \inf_{\zeta\in Q}\omega(\lambda Q\backslash E, E, \zeta) \leq a_\lambda^{-1}\operatorname{cap}\left(\frac{E}{2d(Q)}\right). \tag{6.2}$$

Also if $\zeta \in \frac{1+\lambda}{2}Q$ with $\operatorname{dist}(\zeta, E) \geq b\cdot d(Q)$ *then*

$$a_{\lambda,b}\operatorname{cap}\left(\frac{E}{2d(Q)}\right) \leq \omega(\lambda Q\backslash E, E, \zeta) \leq a_{\lambda,b}^{-1}\operatorname{cap}\left(\frac{E}{2d(Q)}\right). \tag{6.3}$$

The proof can be obtained by the comparison of the Green potential for Q and the logarithmic potential of the equilibrium measure of E.

6.2. **Lemma.** $\omega(\mathscr{O}, E \cap Q_{2I}, 0) \geq a \cdot \operatorname{cap}(E_I/2|I|)$.

Proof. Consider $u_1(z) = \omega(\mathscr{O}, E \cap Q_{2I}, z)$, $u_2(z) = \operatorname{cap}(\frac{E_I}{2|I|}) \cdot \omega(\mathscr{O}, 1/2I, z)$. These functions are harmonic in $\mathscr{O}$. Then we compare m on $\mathbb{D} \cap \partial Q_I$.

The left estimate in (6.2) implies that

$$u_1(z) \geq \omega(Q_{2I} \backslash E, E, z) \geq \omega(Q_{2I} \backslash E_I, E_I, z) \geq a \cdot u_2(z)$$

if $z \in \partial T Q_I$. If $z \in \partial Q_I \backslash \partial T Q_I$ then obviously

$$u_2(z) \leq A \cdot \operatorname{cap}\left(\frac{E_I}{2|I|}\right) \cdot \frac{1-|z|}{|I|}.$$

Our aim is to show that

$$u_1(z) \geq a \cdot \operatorname{cap}\left(\frac{E_I}{2|I|}\right) \cdot \frac{1-|z|}{|I|}$$

on $\partial Q_I \backslash (I \cup \partial T Q_I)$. Let R be a rectangle in $\mathbb{D}$ based on $2I$ and $\frac{1}{2}|I|$ in height. Let $u(z) \stackrel{\text{def}}{=} \omega(R, \partial R \cap \partial T Q I, z)$. Using the left part of (6.2) once again, we get for $z \in \partial R \cap \partial T Q_I$

$$u_1(z) \geq \omega(Q_{2I} \backslash E_I, e_I, z) \geq a \cdot \operatorname{cap}(E_I/2|I|) u(z).$$

This inequality is valid in R, as u_1 is superharmonic in R and positive. At the same time $u(z) \geq a \cdot \frac{1-|z|}{|I|}$ for $z \in R \cap \partial Q_I$. Thus $u_1(z) \geq a \cdot u_2(z)$ on $\mathbb{D} \cap \partial Q_I$. On $\partial \mathscr{O} \backslash Q_I$, $u_2 = 0$, $u_1 \geq 0$. So $u_1(0) \geq a \cdot u_2(0)$.

It is interesting to compare Lemma 6.2 with (3.1). Assertion (3.1) is stronger, but it is valid if $\mathscr{O}$ is a $(1-q)$-boundary layer with a small q. As to the inequality in Lemma 6.2, it is always true.

We conclude with two questions.

Question 1. Is any boundary layer the intersection of a finite collection of $(1-q)$-boundary layers with small q?

One can consider domains $\mathscr{O}$ of the sort $\mathscr{O} = \mathbb{R}^n_+ \backslash E$, E being a compact.

Question 2. What are the analogues of Propositions 1.1 and 1.2 for $n \geq 3$?

The diligent reader may suspect that the boundary layer problem has many points in common with problems concerning minimally thin and rarified sets. This is just the case. The article [3] treats these connections and contains references.

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