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SOME REMARKS ON THE HOMOLOGY OF MODULI SPACE OF INSTANTONS WITH INSTANTON NUMBER 2

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Dedicated to Professor Akio Hattori on his sixtieth birthday

ABSTRACT. Let M_2 be the framed moduli space of SU(2) instantons with instanton number 2. By combining the results of Boyer and Mann and the results of Hattori, we determine the structure of $H^*(M_2; \mathbf{Z}_2)$.

1. INTRODUCTION

We shall denote by M_k the framed moduli space of SU(2) instantons with instanton number $c_2 = -k$. Recently Boyer and Mann [1] constructed homology operations on M_k for all k and thus constructed new homology classes in $H_*(M_k; \mathbf{Z}_p)$. In the case $k = p = 2$, the result is as follows.

Theorem 1 [1]. *The elements of $H_*(M_2; \mathbf{Z}_2)$ constructed by Boyer and Mann are given by the following table*:

q	1	2	3
$H_q(M_2; \mathbf{Z}_2)$	$z_1 * [1]$	$z_1^2 \quad z_2 * [1]$	$Q_1(z_1) \quad z_2 * z_1 \quad z_3 * [1]$

4	5
$Q_2(z_1) \quad z_2^2 \quad z_3 * z_1$	$Q_1(z_2) \quad z_3 * z_2 \quad Q_3(z_1)$

6	7	8	9
$z_3^2 \quad Q_2(z_2)$	$Q_3(z_2) \quad Q_1(z_3)$	$Q_2(z_3)$	$Q_3(z_3)$

In another direction Hattori [4] completely determined the homotopy type of M_2 and as a result computed $H^*(M_2; \mathbf{Z})$ and $H^*(M_2; \mathbf{Z}_2)$. The results are as follows.

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Theorem 2 [4]. *The cohomology groups of* M_2 *with* $\mathbf{Z}$ *coefficients are given by the following table*:

q	1	2	3	4	5	6	7	8	9
$H^q(M_2;\mathbf{Z})$	0	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_3\oplus\mathbf{Z}_4$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}\oplus\mathbf{Z}_2$	0	$\mathbf{Z}_2$
generators		β	γ	p^*z δ	$\beta\gamma$	$\beta\delta$	ξ $\gamma\delta$		$\beta\gamma\delta$

Theorem 3 [4]. *The cohomology groups of* M_2 *with* $\mathbf{Z}_2$ *coefficients are given by the following table*:

q	1	2	3	4	5
$H^q(M_2;\mathbf{Z}_2)$	$\mathbf{Z}_2$	$\mathbf{Z}_2\oplus\mathbf{Z}_2$	$\mathbf{Z}_2\oplus\mathbf{Z}_2$	$\mathbf{Z}_2\oplus\mathbf{Z}_2$	$\mathbf{Z}_2\oplus\mathbf{Z}_2$
generators	u	u^2 v	u^3 uv	w u^2v	uw u^3v

q	6	7	8	9
$H^q(M_2;\mathbf{Z}_2)$	$\mathbf{Z}_2\oplus\mathbf{Z}_2$	$\mathbf{Z}_2\oplus\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$
generators	u^2w vw	u^3w uvw	u^2vw	u^3vw

The choice of the elements v and w will be specified later.

In this paper we combine these results to obtain further homological information about M_2.

2. Main results

We first study the following problem. Do the elements of Theorem 1 generate $H_*(M_2;\mathbf{Z}_2)$?

Proposition 1. *The elements of Theorem* 1 *generate* $H_*(M_2;\mathbf{Z}_2)$ *and the following relations hold*:

(1) $Q_1(z_1)+z_2*z_1+z_3*[1]=0$.
(2) $Q_2(z_1)=z_3*z_1$.
(3) $Q_1(z_2)+z_3*z_2+Q_3(z_1)=0$.

Proof. Let $\mathscr{C}_2$ be the orbit space of SU(2) connections with instanton number 2 by the action of the based gauge group and let $i:M_2\to\mathscr{C}_2$ be the inclusion.

Direct computations show that each element of Theorem 1 is nontrivial in $H_*(\mathscr{C}_2;\mathbf{Z}_2)$ and differs in $H_*(\mathscr{C}_2;\mathbf{Z}_2)$ except for

$$i_*Q_2(z_2)=i_*z_3*z_1.$$

Therefore by using Theorem 3 we see that the elements of Theorem 1 generate $H_*(M_2;\mathbf{Z}_2)$ and there must be one relation for $q=3,4,5$.

But [1, Proposition 9.10] shows that there are the following relations.

(i) $i_*(Q_1(z_1)+z_2*z_1+z_3*[1])=0$.
(ii) $i_*(Q_2(z_1)+z_3*z_1)=0$.

Using Cartan formula and Adem relation [2] we also see the following relation.

(iii) $i_*(Q_1(z_2) + z_3 * z_2 + Q_3(z_1)) = 0$.

Now by using Theorem 3 we see that the relations (i)–(iii) imply the relations (1)–(3) in Proposition 1.

Next we shall study the Kronecker products of elements of Theorems 1 and 3. On account of Proposition 1 we can take a basis of $H_q(M_2; \mathbf{Z}_2)$ for $q = 3, 4, 5$ as follows:

$$\begin{array}{lll} q=3 & Q_1(z_1) & z_2 * z_1 \\ q=4 & z_2^2 & z_3 * z_1 \\ q=5 & Q_1(z_2) & z_3 * z_2. \end{array}$$

Theorem 4. *The Kronecker products of elements of Theorems* 1 *and* 3 *are given by the following table*:

q	1	2	
Kronecker products	$\langle u, z_1 * [1]\rangle = 1$	$\langle u^2, z_1^2\rangle = 0$	$\langle v, z_1^2\rangle = 1$
		$\langle u^2, z_2 * [1]\rangle = 1$	$\langle v, z_2 * [1]\rangle = 0$

3		4	
$\langle u^3, Q_1(z_1)\rangle = 0$	$\langle uv, Q_1(z_1)\rangle = 1$	$\langle w, z_2^2\rangle = 1$	$\langle u^2v, z_2^2\rangle = 0$
$\langle u^3, z_2 * z_1\rangle = 1$	$\langle uv, z_2 * z_1\rangle = 1$	$\langle w, z_3 * z_1\rangle = 0$	$\langle u^2v, z_3 * z_1\rangle = 1$

5		6	
$\langle uw, Q_1(z_2)\rangle = 1$	$\langle u^3v, Q_1(z_2)\rangle = 0$	$\langle u^2w, z_3^2\rangle = 0$	$\langle vw, z_3^2\rangle = 1$
$\langle uw, z_3 * z_2\rangle = 1$	$\langle u^3v, z_3 * z_2\rangle = 1$	$\langle u^2w, Q_2(z_2)\rangle = 1$	$\langle vw, Q_2(z_2)\rangle = 0$

7		8
$\langle u^3w, Q_3(z_2)\rangle = 1$	$\langle uvw, Q_3(z_2)\rangle = 0$	$\langle u^2vw, Q_2(z_3)\rangle = 1$
$\langle u^3w, Q_1(z_3)\rangle = 0$	$\langle uvw, Q_1(z_3)\rangle = 1$	

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$\langle u^3vw, Q_3(z_3)\rangle = 1$

In the above table we define v by

$$\langle v, z_1^2\rangle = 1, \qquad \langle v, z_2 * [1]\rangle = 0.$$

Note that

$$\langle u^2, z_1^2\rangle = 0, \qquad \langle u^2, z_2 * [1]\rangle = 1.$$

We define w by

$$\langle w, z_2^2\rangle = 1, \qquad \langle w, z_3 * z_1\rangle = 0.$$

Note that

$$\langle u^2v, z_2^2\rangle = 0, \qquad \langle u^2v, z_3 * z_1\rangle = 1.$$

Proof. Let $\Delta : M_k \to M_k \times M_k$ be the diagonal. Then we can easily show the following relations.

$$\Delta_* z_1 = z_1 \otimes [1] + [1] \otimes z_1 ,$$
$$\Delta_* z_2 = z_2 \otimes [1] + z_1 \otimes z_1 + [1] \otimes z_2 ,$$
$$\Delta_* z_3 = z_3 \otimes [1] + z_2 \otimes z_1 + z_1 \otimes z_2 + [1] \otimes z_3 .$$

The following relation is known in [2].

$$\Delta_* Q_j(a) = \sum_{r,s} Q_{j-r}(a'_s) \otimes Q_r(a''_s),$$

where $\Delta_* a = \sum_s a'_s \otimes a''_s$. Theorem 4 easily follows from these results.

Next we shall study the integral classes. On account of Theorem 2 there exists an element σ that generates $\mathbf{Z}_4$ in $H_3(M_2; \mathbf{Z})$ and there exists an element τ that generates $\mathbf{Z}$ in $H_7(M_2; \mathbf{Z})$. Let

$$j_* : H_*(M_2; \mathbf{Z}) \to H_*(M_2; \mathbf{Z}_2)$$

be mod 2 reduction.

We shall study $j_*\sigma$ and $j_*\tau$.

Theorem 5. *The following relations hold.*

$$j_*\sigma = z_3 * [1],$$
$$j_*\tau = Q_3(z_2).$$

Proof. Let $\{E^r_*\}$ be the mod 2 homology Bockstein spectral sequence of M_2. The following Nishida relation is known in [2].

$$\beta Q^j(a) = (j-1)Q^{j-1}(a),$$

where β is the Bockstein operation.

By using the Nishida relation we compute E^2_* as follows.

q	1	2	3	4	5	6	7	8	9
E^2_q	0	0	$z_3 * [1]$	z_2^2	0	0	$Q_3(z_2)$	0	0

From this table Theorem 5 follows.

Next as an application of Proposition 1 and Theorem 4, we prove the following theorem.

Theorem 6. *The elements of Theorem* 2 *satisfy the following relations*:

(1) $\beta^2 = 2\delta$,
(2) $\delta^2 = 0$,
(3) $\gamma^2 = \beta\delta$.

Note that Theorem 6 completely determines the ring structure of $H^*(M_2; \mathbf{Z})$.

Proof. (1) is shown in [4].

As $H^8(M_2; \mathbf{Z}) = 0$ holds, (2) follows.

We shall prove (3). Let

$$j_* : H^*(M_2; \mathbf{Z}) \to H^*(M_2; \mathbf{Z}_2)$$

be mod 2 reduction.

All we have to show to prove (3) is $j_*\gamma^2 \neq 0$. Let u, v, w be elements in Theorem 3. Either $j_*\gamma = u^3$ or uv or $u^3 + uv$ holds. We shall show that $j_*\gamma = u^3$ cannot occur. Assertion 1. The following relations hold.

$$u^4 = 0, \qquad v^2 = w.$$

In fact, in the same way as the proof of Theorem 4, we see the following Kronecker products.

$$\langle u^4, z_2^2\rangle = 0, \qquad \langle u^4, z_3 * z_1\rangle = 0,$$
$$\langle v^2, z_2^2\rangle = 1, \qquad \langle v^2, z_3 * z_1\rangle = 0.$$

Assertion 2. The following relation holds.

$$j_*\beta = u^2.$$

In fact, the following holds.

$$j_*\beta = Sq^1 u = u^2.$$

Now suppose $j_*\gamma = u^3$. The table in Theorem 2 shows that

$$j_*(\beta\gamma) \neq 0.$$

But from Assertions 1 and 2 we have

$$j_*(\beta\gamma) = (j_*\beta)(j_*\gamma) = u^2 u^3 = 0.$$

This is a contradiction. Therefore either $j_*\gamma = uv$ or $u^3 + uv$ holds. Anyway

$$(j_*\gamma)^2 = u^2 v^2 = u^2 w \neq 0.$$

This completes the proof of (3).

Remark. In [4], whether $\gamma^2 = 0$ or not is left unknown.

Now by using the above results, we can completely determine $H^*(M_2; \mathbf{Z}_2)$.

Theorem 7. $H^*(M_2; \mathbf{Z}_2) = \mathbf{Z}_2[u, v]/(u^4, v^4)$ *and* $Sq^1 v = uv$ *hold. Note that the* $\mathscr{A}(2)$*-module structure of* $H^*(M_2; \mathbf{Z}_2)$ *is completely determined.*

Proof. The ring structure follows from Theorem 3 and Assertion 1 in Theorem 6. By using Theorem 4 and the following Kronecker products we can easily prove $Sq^1 v = uv$.

$$\langle Sq^1 v, Q_1(z_1)\rangle = 1, \qquad \langle Sq^1 v, z_2 * z_1\rangle = 1.$$

3. Appendix

The proof of Proposition 9.5 seems incomplete in [1]. By using Theorem 3, we shall give an explicit proof of this proposition.

Proposition 9.5 [1]. *$z_i * [1] = Q_i[1]$ for $i = 1, 2, 3$.*

Proof. The proof of $z_1 * [1] = Q_1[1]$ is given in [1].

(i) Proof of $z_2 * [1] = Q_2[1]$. Let $i : M_2 \to \Omega_2^3 S^3$ be the inclusion. Clearly $H_2(\Omega_2^3 S^3; \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and the basis of $Q_1[1]^2 * [-2]$ and $Q_2[1]$. By Theorem 3, $H_2(M_2; \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$. Note that z_1^2, $Q_2[1]$, $z_2 * [1]$ are elements of $H_2(M_2; \mathbf{Z}_2)$. But

$$\text{(A)} \qquad \begin{cases} i_* z_1 = Q_1[1] * [-1], \\ i_* z_2 = Q_2[1] * [-1] \end{cases}$$

are given in [1, Theorem 8.6]. Hence $i_* z_1^2 = (Q_1[1] * [-1])^2 = Q_1[1]^2 * [-2]$ and $i_* Q_2[1] = Q_2[1]$. Therefore $i_* : H_2(M_2; \mathbf{Z}_2) \to H_2(\Omega_2^3 S^3; \mathbf{Z}_2)$ is an isomorphism. But $i_*(z_2 * [1]) = (Q_2[1] * [-1]) * [1] = Q_2[1]$ by (A). Therefore $z *_2 [1] = Q_2[1]$ holds.

(ii) *Proof of* $z *_* 3[1] = Q_3[1]$. Let $f : \mathrm{SO}(3) \to M_2$ be the composite of $\mathrm{SO}(3) \to M_1 \times 1 \to M_1 \times M_1 \xrightarrow{*} M_2$ and let $g : \mathrm{SO}(3) \to M_2$ be the composite of

$$\mathrm{SO}(3) \to S^3 \times_{\mathbf{Z}_2} 1 \times 1 \to S^3 \times_{\mathbf{Z}_2} M_1 \times M_1 \xrightarrow{\theta} M_2 .$$

Clearly $f_* z_i = z_i * [1]$ and $g_* z_i = Q_i[1]$ hold for $i = 1, 2, 3$. But we have shown the following.

$$f_* z_1 = g_* z_1 , \qquad f_* z_2 = g_* z_2 .$$

By Theorem 3, all we need is to prove the following equalities.

$$\langle u^3 , f_* z_3 \rangle = \langle u^3 , g_* z_3 \rangle , \qquad \langle uv , f_* z_3 \rangle = \langle uv , g_* z_3 \rangle .$$

Let Δ be the diagonal; then we easily see the following.

$$\Delta_* z_3 = z_3 \otimes 1 + z_2 \otimes z_1 + z_1 \otimes z_2 + 1 \otimes z_3 .$$

Then $\langle u^3 , f_* z_3 \rangle = \langle u^2 , f_* z_2 \rangle \langle u , f_* z_1 \rangle = \langle u^2 , g_* z_2 \rangle \langle u , g_* z_1 \rangle = \langle u^3 , g_* z_3 \rangle$.

$\langle uv , f_* z_3 \rangle = \langle uv , g_* z_3 \rangle$ is similarly proved.

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